DOCUMENT RESUME

ED 367 543 SE 054 049

AUTHOR	Atweh, Bill, Ed.; And Others
TITLE	Contexts in Mathematics Education. Proceedings of the Annual Conference of the Mathematics Education Research Group of Australasia (MERGA) (16th, Brisbane, Queensland, Australia, July 9-13, 1993).
INSTITUTION	Mathematics Education Research Group of Australasia, (Australia).
REPORT NO	ISBN-0-86856-902-X
PUB DATE	Jul 93
NOTE	579p.
PUB TYPE	Collected Works - Conference Proceedings (021)
EDRS PRICE	MF03 Plus Postage. PC Not Available from EDRS.
DESCRIPTORS	Constructivism (Learning); *Context Effect; Cultural Context; Elementary Secondary Education; Foreign Countries; Mathematical Concepts; Mathematics Achievement; Mathematics Curriculum; *Mathematics Education; Problem Solving; Social Influences
IDENTIFIERS	Australia; *Mathematics Education Research; *Mathematics Education Research Group of Australia

ABSTRACT

These proceedings contain 4 keynote addresses and 88 papers concerning the theme of the conference: Contexts in Mathematics Education. The four keynote addresses are: (1) "Contemplating Cultural Context" (Bill Barton); (2) "Conceptualising Cultural and Social Contexts in Mathematics Education" (Alan Bishop); (3) "Mathematical Learning in the Social Context" (George Booker); and (4) "Childrens' Construction of Mathematical Ideas" (Carolyn Maher). The 88 contributed papers deal with mathematical content ranging from prenumber to calculus, learners ranging from preschool to adult, teachers, technology, affect, assessment, metacognition, psycholinguistics, and a wide variety of social issues. (MDH)

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MERGA-16
BRISBANE '93

Sixteenth Annual Conference of
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(MERGA)

Contexts in Mathematics Education

Conference Proceedings

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Conference held at
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Contexts in Mathematics Education

Proceedings of the Sixteenth Annual Conference of the
Mathematics Education Research Group of Australasia (MERGA)
Brisbane, July 9-13 1993

Editors: Bill Atweh
Clive Kanes
Marjorie Carss
George Booker

Published by: The Mathematics Education Research Group of Australasia

ISBN 0 86856 902 X

Printed by: Queensland University of Technology-Kelvin Grove Campus
Published by the Mathematics Education Research Group of Australasia

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Printed on 100% recycled paper

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PREFACE

The Sixteenth Annual Conference of the Mathematics Education Research Group of Australasia was held in Brisbane, July 9-13, 1993. This was the second MERGA conference to be held in Brisbane (the previous in 1979) and the third in Queensland (the Townsville conference was in 1987). The conference was held under the theme: Contexts in Mathematics Education.

The conference was organised by a team representing four universities in Brisbane: The Queensland University of Technology, Griffith University, The University of Queensland, and the Australian Catholic University. The program included several innovations for MERGA conferences. One of the keynote speakers being from NEW Zealand and the increase number of papers from New Zealand and other South Pacific countries confirmed the increasing importance of the organisation in the region. For the first time, the program included a Poster Sessions as well as a Symposium organised by the presenters of three papers on a specific topic. The Policy session at the Conference held in Western Sydney in 1992 decided to re-introduce the practice of publishing conference proceedings. This year the proceedings were available to delegates art registration. Lastly, all papers submitted for presentation at the Conference were subject to a process of peer review..

The Guidelines For Presenting Papers at MERGA Conferences, adopted at the Hobart conference were used to select papers for this conference. For each paper, two independent reviewers were asked to comment on the suitability of the paper for presentation at the Conference. Where there was a disagreement, a third reviewer was appointed. Altogether 89 papers were accepted. A report about this experience will be presented to the Policy Session at the conferene with some recommendations.

Some papers could not be fully published in these proceedings either because of copyright concerns or because a disc copy of the paper was not available. Only abstracts of these papers appear in these proceedings. Lastly, it was necessary to reformat all papers for inclusion in these proceedings to assure uniformity of style between all papers. While all care was taken to minimise errors, it is inevitable that some may have occurred; and for these we apologise.

Editors
Brisbane, 1993

PRACTICAL IMPLICATIONS AWARD

This award, presented by the Key Centre for Teaching and Research in School Science and Mathematics (Especially for Women) at Curtin University in Perth, is design to stimulate the writing of papers that interpret theory and research to mathematics teaching. Papers submitted for the award should identify a persistent and recurring problem in the practice of mathematics teaching and describe a research activity related to this problem. The author(s) should develop strategies to resolve the problem based upon a comprehensive synthesis of the research evidence. Judges for the award are appointed from the Centre and from the MERGA membership.

This is the first year that the award was granted. The successful entrant was:

Margaret Taplin

Development of a model to enhance managerial strategies in problem solving

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KEYNOTE ADDRESSES

1

CONTEMPLATING CULTURAL CONSTRUCTS

BILL BARTON
Mathematics Education Unit
University of Auckland

PHILOSOPHIES OF MATHEMATICS

A Wide View

The philosophy of mathematics has, since Plato, been concerned with the objects of mathematics and the truth of its statements. Are numbers real or are they idealised thoughts? Are complex numbers images in our mind with no reality or are they valid mathematical realities? Does mathematics show us God's glory or is it a creation of human minds? Were the Mandelbrot pictures and the patterns of Chaos waiting to be discovered or were they created by our number representations and the coordinate system? Is <1+1=2> a true statement about the world or a logical tautology? Is it true that there is always another term in an infinite series? Is Goldbach's Conjecture either true or false? Is it reasonable to divide infinitely small quantities in calculus? How do we know that the Four Colour Theorem is true?

The debates between Platonists, logicists, formalists, intuitionists and others have tried to answer these questions definitively and without unpalatable consequences like logical paradoxes or circularities. Such debates have produced vast fields of mathematical endeavour such as Russell's propositional calculus and the field of recursive functions. It has been studded with breathtaking inventions and insights such as Turing Machines and Gödel's Theorem. But, despite 2000 years' work from the top minds of humankind, the fundamental issues of the nature of mathematical objects and the role of logic are far from being resolved (see Körner (1960) for a general review of these debates)

In the later half of this century there have been movements away from such questions. One of the starting points of this movement was the French philosopher Bachelard, writing in the 1930's, who proposed a philosophy of science which focused on the historical development of rationality. He suggested that what counts as 'objective' changes over time: thus a subjective element is introduced into the notions of proof and truth. These notions do, however, develop in a rational way (see Tiles (1984) or Smith (1982)).

The basic theme of the historical relativity of mathematical and scientific concepts is pursued in different ways by Kuhn and Popper. The latter sees the development of science as a process of humankind approaching closer and closer to some form of Platonic reality. Popper's idea that scientific ideas are posed to be criticised and thereby improved (sometimes called 'critical fallibilism') have been taken up, more recently, by Imre Lakatos'. Lakatos' contribution was to suggest that mathematics was like science in that mathematical statements are made to be refuted. They describe things as we see them, meaning 'how we think them' rather than 'how we perceive them with our senses'. He describes mathematics as 'quasi-empirical', and his interest is with the question: How do we criticise the statements of mathematics? What would have to happen for us to say that a mathematical truth was wrong? (Lakatos (1978))

At about the same time as Bachelard, Wittgenstein was writing about the way that meaning is negotiated. He suggests that any term, for example 'prime number', is in a constant process of definition as it is used in mathematical discourse. Mathematical concepts are defined by proofs and theorems, not the other way round. That is, there are no mathematical results "out there" waiting to be discovered, rather the construction, statement and proofs of theorems are part of the development of a mathematical concept (see Wright (1980) or Bloor (1983)). Paul Ernest (1991) is arguing a position which merges the social constructivist philosophies with the critical fallibilism of Lakatos.

Such a sketchy outline as the above cannot do justice to the sweep and depth of scientific and mathematical philosophy. However, it may be sufficient to make the point that mathematical philosophy, after 2000 years of concern over the nature of mathematical objects and the role of logic, has begun to shift focus onto our changing conceptions of these objects and their relationships. This can be seen as shift from objective concerns to subjective ones, (although there are many realists and intuitionists continuing to forcefully argue their views). Another way of describing this change is to say that instead of asking the questions "What are triangles?" and "How do I know that they contain 180°?", we are asking the questions "How do we talk about triangles?" and "What are the limits to the statement 'triangles contain 180°'?".

Ethnomathematics

What has this to do with ethnomathematics? Well, in the first place, it gives a context within which mathematics educators have begun to talk about a concept of the cultural relativity of mathematics. With the evolution of the idea that mathematics may be a changing subject, it is not surprising that teachers have begun to question what mathematics they are teaching. Teachers are familiar with the social perspective of cultural difference in educational ideas and practices, so it is natural for them to introduce the idea into their subject areas.

The difficulty is that, philosophically, neither classical theories, nor, it appears, those of Bachelard, Wittgenstein, Kuhn, Popper or Lakatos, offer any help in seeing how there might be culturally different mathematics. These theories either lead to a concept of a one true (ideal) mathematics, or they lead to a concept of a developing subject forever guided along a path towards a rational ideal of mathematics. The concept that there might be different mathematics' (of equal rational standing) sustained simultaneously cannot be accomodated in these theories. At least none of the proponents of these theories have tried to defend such a position (with the exception of David Bloor who argues for a social relativity, see Bloor (1976))

The result is that writers on ethnomathematics have developed a variety of conceptions as they try to reconcile cultural mathematics with the ideal, rational conception of mathematics which is the historical legacy. Some notions of ethnomathematics include: an elementary (pre-mathematical) stage of development (Ascher), a contributor to some universal and encompassing mathematics (Gerdes), an educational construct (Bishop), a political construct (Mellin-Olsen/Gerdes), or an historical development borne from the mathematical practices of groups of people (D'Ambrosio). Geraldo Pompeu has detailed the differences in six versions of ethnomathematics according to their educational implications. (Pompeu (1992)).

ANOTHER PERSPECTIVE

Mathematics As Process

The idea that I would like to pursue is that ethnomathematics is much more pervasive than that. I want to suggest that ethnomathematics is a constant lens through which we see the world, and, in particular, the mathematical activities of those around us. I am led to this view by taking the social, relativist philosophies of this century and applying them to the notion that mathematics is a process.

Rather than concentrating on the objects of mathematics, or the way we talk about mathematics, I'd like to ask philosophical questions which surround the action of mathematics. To use the examples above, I want to ask: "What do we think we are doing when we talk about triangles" and "Why do we investigate the angle properties of a triangle?".

I want to ask mathematicians whether the theorems of mathematics embody what they think they are doing, or whether theorems are just a record of where they have been? If the latter is the case, then what is the nature of the process of mathematics, and how do they know when they have done some worthwhile mathematics? What are the characteristics of successful mathematical action? Is it only possible to tell afterwards, or is the process itself valuable? Where do new mathematical ideas come from?

The philosophical programme I have in mind is to describe the workings of mathematicians (and metamathematicians), thereby opening up the processes of mathematics to examination so that possible limitations to the development of mathematics can be identified. Thus new ways of doing things, or new ways of thinking about things, might be explored. This is an active, creative programme, unlike those of classical mathematical philosophers which are post-hoc, descriptive programmes.

The Role of Culture

Asking philosophical questions such as this brings the actor into mathematics. It becomes impossible to separate the actor and their language, their preconceptions, their experience; mathematics is inextricably linked with the mathematician's images, concerns, metaphors, or their values, their perceptions and their view of the world. Now the sociology and anthropology of mathematics become concepts which we can understand and which we realise we know very little about. The developing literature on this aspect of mathematics is overdue and very welcome (Fang & Takayama (1975), Restivo (1992)).

The consequence of this view is that mathematics will be culturally determined to the extent to which a community of mathematicians share cultural characteristics (e.g. language, values, perceptions). It may be possible to "see" others' cultural influences, but it is never possible to escape some cultural influence or other. Mathematics as process, mathematics as action can only be relative in this sense.

It is possible to understand this view of ethnomathematics in relation to individual and social constructivist views. If we consider the processes of mathematics, and the actors in this process, these actors as operate at several levels simultaneously. They are individuals, members of an intimate group (e.g. a class or Maths Department), members of a wider community of those doing similar things, part of a cultural group and so on. These groups overlap and interact. Any one mathematical action is liable to be affected by any or all of these groups in some way - the action cannot be isolated from them. Similarly, it is difficult to separate the influence of any one group. So, how might an analysis be achieved? Under what conditions is it possible to isolate a cultural influence and call the resulting view ethnomathematical? Perhaps the answer lies in describing aspects of mathematical processes as ethnomathematical, rather trying to find ethnomathematics?

Another result of considering the processes of mathematics (and the actors involved) is that it opens up the question "What is mathematics". Instead of focussing on mathematics as it is already defined, it is possible to look at ways of behaving that have mathematical characteristics. The results of those processes then become mathematics. Taking a cultural view, this means that cultures which do not have a "mathematical" division of knowledge are not immediately dismissed from the sphere of mathematics, since some of the things members of that culture do are clearly mathematical in the sense of Bishop's six activities (Bishop (1988)).

SUMMARY

It is argued that classical philosophies of mathematics do not provide a basis from which it is possible to argue for different, simultaneous views of mathematics on cultural (or any other) grounds. A shift in the focus of philosophical attention from the objects and logic of mathematics to its processes provides a more active programme which includes a place for the actor in mathematics. Through the actor subjective influences are introduced at many levels, influences which it is not possible to escape. The influences that can, in some way, be identified as cultural, define the ethnomathematical component of the processes being considered.

It remains to illustrate these ideas using specific cultures and specific processes, and thus carry out the philosophical programme which is aimed at widening the mathematical creative process.

REFERENCES:

Bishop, A (1988) *Mathematical Enculturation*, Dordrecht: Kluwer Academic Publishers

Bloor, D (1976) *Knowledge and Social Imagery*, London: Routledge & Kegan Paul

Bloor, D (1983) *Wittgenstein. A Social Theory of Knowledge*, London: MacMillan Press

Ernest, P. (1991) *The Philosophy of Mathematics Education*, London: Falmer Press

Fang, J & Takayama, K (1975) *Sociology of Mathematics and Mathematicians. A Prolegomenon*, Hauppauge, N.Y.: Paideia Press

Körner, S (1960) *The Philosophy of Mathematics. An Introductory Essay*, London: Hutchinson University Library

Lakatos, I (1978) *Mathematics, Science and Epistemology. Philosophical Papers Vol.2.*, J. Worrall & G Currie (eds), Cambridge: Cambridge University Press

Pompeu, G (1992) Unpublished Doctoral Thesis

Restivo, S (1992) *Mathematics in Society and History: Sociological Enquiries*, Dordrecht: Kluwer Academic Publishers

Smith, R (1982) *Gaston Bachelard*, Boston: Twayne Publishers

Tiles, M (1984) *Bachelard: Science and Objectivity*, Cambridge: Cambridge University Press.

Wright, C (1980) *Wittgenstein on the Foundations of Mathematics*, London: Duckworth

CONCEPTUALISING CULTURAL AND SOCIAL CONTEXTS IN MATHEMATICS EDUCATION

ALAN J. BISHOP

1. INTRODUCTION

As I write this paper I have been in Australia now for nine hectic months and I can assure you that I have absolutely no regrets about my move! I am delighted to be part of the vibrant mathematics education community in Australia and it is a great honour for me to be invited to give the opening plenary address to MERGA 16.

The theme of the conference is "Contexts in Mathematics Education" and is in my mind, a thoroughly appropriate theme for MERGA at this time. The economic and political context is shaping much of our thinking, and indeed our work, in mathematics education at present and it is highly appropriate for us to be thinking about how contexts should, and could, be considered in relation to research in mathematics education.

Those of you who know my work will know that I have a particular research interest in social and cultural contexts, although I have been known to engage in shaping the political context as well. I hope however that what I say today will help to inform your thinking about contexts in general - so even if most of my examples will be from the cultural and social fields, I shall be choosing them in order to address issues in other contexts - economic, environmental, and indeed conceptual.

2. CONSTRUCTIVE ALTERNATIVISM

I chose that word `conceptualising' for my title because I believe that conceptualising is the key problem which needs to be addressed in considering contexts. I am a constructivist from way back, but not from the von Glasersfeld `camp'. My thinking was critically influenced by George Kelly's writings. He published two volumes called "The psychology of personal constructs" in 1955, but more accessible is "Inquiring Man" by D. Bannister and F. Fransella (Penguin, 1971). Kelly believed that if psychology is concerned with describing human learning and behaviour then one surely ought to be able to recognise oneself somewhere in the descriptions. As Bannister says (p.16) "unless you are the very modest kind of man who can see himself as the stimulus jerked puppet of learning theory, the primitive infant of psychoanalytic theory, or the perambulating telephone exchange of information theory" then "the model man of personal construct theory will look recognisably like you".

Kelly's core idea was "constructive alternativism" and as one might expect from a psychotherapist he was often professionally concerned with the situation which he graphically described as "hardening of the constructs". As educators and teachers we are of course also concerned with influencing that condition, but we must attend to Kelly's self-reflexive admonition if we are going to develop our research. We need to be loosening our own constructs and our own conceptualisations through, and with, our research. Kelly's person is also an experimenting person, creating constructs to give meaning to experience and testing out the predictive validity of those constructs.

Changing and developing our constructs can be a more or less emotional experience depending on how deep or how superficial they are for us. If a new experience can be assimilated relatively easily into one's existing schema (to use some Piagetian terminology) then it may only be a matter of adjusting one's surface constructs. If however a new experience requires accommodating one's schema then it is likely to involve some re-shaping of some of one's core constructs. Accommodation is always more painful than assimilation.

3. SOCIAL AND CULTURAL DEVELOPMENTS IN RESEARCH

Social and cultural developments in mathematics education are forcing the same kinds of accommodation and core construct adjustment on us and on our research. Up to ten years ago, Mathematics was generally assumed to be culture-free and value-free knowledge, explanations of `failure' and `difficulty' in relation to school mathematics were sought either in terms of the learners' cognitive attributes or in terms of the quality of the teaching they received; there were several attempts to make mathematics teaching more affectively satisfactory to the learners, with few long term benefits, and `social' and `cultural' issues in mathematics education research were rarely considered.

Within the last ten years, there has been an increasing move to make mathematics accessible to all learners, there has been an increasing questioning of the relevance of ex-colonial models of education in developing countries, and in countries with indigenous `minorities'; the social dimension has come into greater prominence in research in mathematics education and the cultural nature of mathematical knowledge has become clearer to many mathematics educators.

I chose the `ten year' time scale because 1983 was a time when I was lobbying for far greater attention to be paid to social aspects of mathematics education within the ICME framework. In the 1984 ICME there was an important plenary address by Ubi d'Ambrosio "socio-cultural bases for mathematics education" and an equally important Theme Group concerned with the theme of "Mathematics for All" but it was not until the 1988 ICME that it was accepted as a sufficiently important area to warrant a whole day being given over to the theme of "Mathematics, Education, and Society". It was a highly successful day with some 90 contributors from over 40 countries, and many congress participants attended. Interestingly, though, many others took the opportunity of having another day's excursion. There were some strong views expressed about 'the day' and its contents, ranging from those which held that these ideas were not appropriate in a conference on mathematics education, through to others which held that the organisers didn't go far enough and that they merely scratched the surface of the real issues.

What that day and its surrounding publications did do was to force some serious core construct adjustment within the mathematics education community. As Kelly would have said - "the world never looked the same again".

However, research practice, in my view, is only slowly coming to terms with the changing conceptualisations. The recent "Handbook of Research on Mathematics Teaching and Learning" (D. A. Grouws, MacMillan 1992) demonstrates well both the progress which has been made, but also the limitations which remain. There are chapters by Terezinha Nunes on "Ethnomathematics and everyday cognition", by Marilyn Nickson on "The culture of the mathematics classroom", and by Walter Secada on "Race, ethnicity, social class, language, and achievement in mathematics". There are also references to social and cultural issues in some of the other chapters.

If we were to use a social dimension like the following, as a kind of category system for current research activity

Cultural
Societal
Institutional
Pedagogical
Individual

I believe we would find a genuine interest in the cultural and pedagogical levels, but only a limited interest in the others. I would conjecture that this is largely to do with the fact that research has not yet come to terms with either the practical complications of researching social aspects, or the value-laden, even moral, issues of researching educational phenomena from a social perspective.

Our field of research has, I believe, developed from three principle traditions, and their interactions. These traditions I call the Pedagogue, the Empirical Scientist, and the Scholastic-Philosopher. Each of these has a history and each is alive and well at present. They are all research traditions because they all relate to the three key components of any research activity in our field -enquiry, evidence, and theory - but they relate to them in different ways as illustrated in this table:

	Goal of Enquiry	Role of Evidence	Role of Theory
Pedagogue tradition	Direct improvement of teaching	Providing selective and exemplary children's behaviour	Accumulated and shareable wisdom of expert teachers
Empirical-scientist tradition	Explanation of educational reality	Objective data, offering facts to be explained	Explanatory, tested against the data
Scholastic-philosopher tradition	Establishment of rigorously argued theoretical position	Assumed to be known. Otherwise remains to be developed	Idealised situation to which educational reality should aim.

It is clear that the constructs and conceptions which have come from our research traditions have helped develop the field of knowledge of mathematics education profoundly. Arguably our field is the most developed of any curriculum subject field.

But it is also clear to many mathematics educators that our research practices are no longer adequate to meet the challenges of today. In particular it is now clear that we have to begin to examine critically our research practices and our value-laden constructs if we are to make more progress in developing our social and cultural knowledge.

There is to be an ICMI study on "What is research in mathematics education, and what are its results?" and I hope that many of us here will play our part in shaping the debate and its outcomes. Already you can see what I mean about the need to question the constructs being used, when you reflect on the nature of the two questions used in that title!

Let me now indicate some ways in which we might address our constructs and our research practices, in relation to the theme of `context'.

4. RECONSTRUCTING CONTEXT (examples of research studies will be presented at the conference)

a) If we consider how context is constructed in research and theory writings, we can recognise ideas of `environment', of constraints, of imposed procedures, of applications, of sources of problems, of pressure groups, of boundaries, and others.

(b) We can recognise context as `external' in the sense of material environment, as well as `internal' in the sense of a learner's innate abilities.

(c) We can see a dimension of context such as the following:

]

Imposed control	Constraining influence	A neutral `Milieu'	Modifiable in certain aspects	Completely under one's control

which extends from a `negative' to a `positive' image, from being `trapped' to being `free'.

(d) We can recognise a fundamental difference between `context as constructed' and `context as determined'. You can choose to construct your context as either controllable or not.

(e) With the example of the social dimension as described earlier:

cultural
societal
institutional
pedagogical

individual

we see an image of concentric circles of increasing contextual influence centring on the individual learner. We must also recognise that there are different people involved in the different `circles of influence', and therefore that one person's situation may well be another person's context. As Kelly pointed out, I am part of your context and you are part of mine.

5. RECONSTRUCTING RESEARCH PRACTICES (examples of research studies will be presented at the conference)

Context enters research practices in various ways:

- as a source of research problem
- as an environment from which one samples materials, examples, actions, situations
- as people from whom one seeks data
- as a situation to which respondents refer
- as a source of research control
- as a source of research funding
- as a source of explanatory data
- as a constraint on research practice

We can thus distinguish:

(a) a context of a research study
(b) context in a research study

As interest in the latter grows, there is a need to relate practical decisions concerning context to the constructions of the previous section. Conceptualisations of context are becoming just as important as conceptualisations of the principal variables in a research study. There are many significant consequences which flow from these reconstructions.

From my own work on cultural and social aspects of mathematics education, I have found it increasingly necessary to attend to contextual constructs. For example, in considering cultural conflicts in mathematics learning I find it necessary to distinguish the following educational situations:

Approaches to culture conflict	Assumptions	Curriculum	Teaching	Language
Culture-free Traditional view	No culture conflict	Traditional Canonical	No particular modification	Official
Assimilation	Child's culture should be useful as examples	Some child's cultural contexts included	Caring approach perhaps with some pupils in groups	Official, plus relevant contrasts and remediation for second langauge learners
Accommodation	Child's culture should influence education	Curriculum restructured due to child's culture	Teaching style modified as preferred by children	Child's home language accepted in class, plus official language support

Amalgamation	Culture's adults should share significantly in education	Curriculum jointly organised by teachers and community	Shared or team teaching	Bi-lingual, bi-cultural teaching
Appropriation	Culture's community should take over education	Curriculum organised wholly by community	Teaching entirely by community's adults	Teaching in home community' s preferred langauge.

This kind of analysis makes us aware of critical questions such as:

- what should the research process be focussed on?
- who should decide on the research problems?
- who should carry out the research?
- where should it be situated?
- what data will be relevant?
- what use could be made of my ideas?

A moral dimension enters into all of these questions in a way which it normally doesn't if context is not taken seriously. This can be felt to be a drawback to some researchers. Also however, such considerations can significantly enrich the development of research agendas.

6. EXAMPLE RESEARCH AGENDA

As an example of how a research agenda can develop from these ideas, I offer the following related to my own area of focus - cultural conflicts.

1. Regarding the mathematical knowledge as represented in the intended curricula

(a) Formal mathematics education

How should the mathematics curriculum be culturalised?
What values are developed within the current mathematics curriculum?
What is an appropriate intended mathematics curriculum for a pluralistic society?
Who determines this mathematics curriculum?

(b) Non-formal mathematics education

What role are non-school alternatives fulfilling in relation to cultural conflicts?
Why are they on the increase?
Is their increase a measure of communities' satisfaction, or their dissatisfaction, with formal education?

(c) Informal mathematics education

In what sense are societal and community influences on mathematics learners educational?

2. Regarding the mathematical knowledge environment in classrooms

(a) Implementing a culture-blind intended mathematics curriculum.

To what extent can a culture-blind intended mathematics curriculum be made less of an obstacle to learning in the classroom?
Can mathematical learning activities be usefully characterised as more-or-less `open' in relation to their cultural framing?

(b) The mathematics teacher as social anthropologist

What outside-school mathematical knowledge do teachers recognise as legitimate inside the classroom?
What knowledge about the learners' cultures can help mathematics teachers with their classroom curriculum decision-making?

(c) Multi-lingual and multi-cultural mathematics classrooms

What teaching strategies can mathematics teachers usefully adopt in multi-lingual and multicultural classrooms to stimulate all learners? What kind of knowledge environment can mathematics teachers create in multi-cultural classrooms?

What language strategies are most effective in multi-lingual mathematics classrooms?

3. Regarding the mathematical knowledge attained by the learners

What outside-school mathematical knowledge do learners recognise as legitimate inside the classroom?

What cultural conflicts are actually experienced by mathematics learners and how do they cope with them?

How does the `cultural distance' of their home mathematical culture from the school mathematical culture relate to the quality of their mathematical learning in classrooms?

How does bi-cultural mathematical learning differ from bi-lingual mathematical learning?

What are the implications of all these questions for determining the appropriateness of any mathematical assessment procedures?

7. IN CONCLUSION

An academic conference is an excellent situation in which to compare, and to extend, one's constructs. The area of `context' is one in which construct extension is crucial at the present time, as mathematics education develops into a global field, with implications at all age levels, and in all spheres of human activity. Research needs to recognise this, and needs therefore to be consciously and critically appraising the ways in which all contexts are addressed.

11

MATHEMATICS LEARNING IN THE SOCIAL CONTEXT: RECONCILING INDIVIDUAL CONSTRUCTION WITH ENCULTURATION INTO THE MATHEMATICAL PRACTICES OF WIDER SOCIETY

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Abstract[*]

This presentation discusses observations and findings of classroom-based research focused on ways of developing children's mathematical understanding through authentic, situated activities. Instructional and heuristically structured activities were used to foster discussion, argumentation and resolution among alternative constructions, allowing children to engage in the social construction of notions compatible with ideas that had evolved historically, through social and cultural forces in the wider community.

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[*] Paper not available for publication

CHILDREN'S CONSTRUCTION OF MATHEMATICAL IDEAS[1]

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Our longitudinal research focuses on studying how children build meaningful interpretations of situations as they work in classrooms in which thoughtful mathematical explorations occur. Data come from videotape records of children who have been followed for several years doing mathematics. One segment of this work, children's invention of ideas related to mathematical proof, is presented in this report. The paper is organized into four sections: Part 1 gives a theoretical orientation; Part 2 suggests a framework for certain progressions of problem-solving behavior related to proof from earlier work; Part 3 extends this work to children's invention of proof to include problems that require building an abstract representation; and Part 4 provides a case study in which the earlier work on children's construction and use of mental representations in building a proof is related to the current work.

PART 1: THEORETICAL ORIENTATION

Over the last several years, our work has centered on studying how children build up their ideas when confronted with problematic tasks that promote thoughtfulness about mathematical situations. Tasks are deliberately chosen to challenge children to reorganize or to extend available existing knowledge.

The child, in the process of working out a solution to a problem task, retrieves from memory existing knowledge in the form of mental representations (Davis, 1984). If existing representations are an inadequate match to the problem, the child may find it necessary to reorganize and/or extend their available existing knowledge. The process of tackling the problem may trigger the construction of a more adequate representation and provide the incentive to reorganize or extend their available existing knowledge. This process can lead to the development of a new idea.

Heuristics

While pursuing a solution to a problematic situation, the child may call forth certain heuristics, such as "guess and check", "looking for patterns", "making a list", "building a model", "making a chart or diagram", "solving a simpler problem", and "special cases". In so doing, an effort is made by the child to assemble, clarify, organize, and extend original ideas. Children may work with physical objects that can be manipulated in various ways, or they may choose to represent the objects using a particular notation that can be manipulated and adjusted to test the new ideas.

Reorganization and Extension

Students, interacting with each other and with the instructor, provide the outside stimuli that can trigger reorganization and extension. In our classrooms, children are regularly asked to share their thinking and to explain their work. Hence, reorganization of one's initial representation of an idea can be encouraged by a challenge to explain one's solution. In so doing, the student may reorganize an original mental representation of the problem.

Students are also challenged to extend their thinking and reasoning from a particular problem to certain classes of problems. In responding to a request to justify a solution, the student often re-examines previously held locally organized representations of knowledge. This process can lead to an extension of an idea. These local organizations can then be re-assembled to fit together and form a new and coherent global mental representation.

Connections

Students are often asked whether the current problem task reminded them of others that they may have previously encountered. In responding, they are sometimes able to make connections between isomorphic problem tasks. As children reflect on earlier problem-solving experiences and re-examine them in light of their newly constructed knowledge, they are able sometimes to generalize certain methods. This process is illustrated in our work with student construction of methods of proof and is illustrated in this paper with the case of Brandon (Also see Maher, Martino, & Davis, in press).

Notation

Children, naturally, will often invent notation to represent their ideas. If the notation is unsuitable, it may be adjusted, replaced, or discarded. If modified, it may no longer match the initial mental representation. In an attempt to coordinate the two, the original representation may be altered. The process of coordinating notation with a mental representation could also result in a modification, reorganization, and/or refinement of the original idea.

PART 2: CHILDREN'S INVENTION OF PROOF

We have observed several sequential stages that children move through in their problem solving that may lead to providing a thoughtful justification for a solution. In each stage, new ideas appear and form the basis and motivation for the development of others. To illustrate this process, a summary of our analysis of children's thinking for a particular problem, The Tower Task, will be given (For more detail, see Maher, Martino, & Davis, in press; Davis, 1993; Davis, 1992).

The Tower Task:

The task required students to build as many towers as possible of a certain height (for example, all possible towers five cubes tall) when plastic cubes (Unifix cubes) in two colors were available, and then to convince other students that there were no duplicates and that none had been omitted.

Student Movement Through Stages

Children moved generally through a sequence of stages[2] in building a successful solution to the problem task.

Stage 1. Children working in the first stage made whatever towers they could think of that have not already been built. They began to notice relationships between two towers, such as a pair with opposite colors in corresponding positions. They monitored for duplicate arrangements by comparing a newly generated tower to ones that are already built.

Stage 2. A move from using random methods to using local systems of organization was characteristic of Stage 2. Students noticed patterns among sets of towers and found new ways to group them such as "inverting" a tower and its opposite to form another collection. During this stage, some children began to realize that their local organization schemes were inadequate to account for all possible towers.

Stage 3. As children were challenged to convince others that every possible tower had been built, they began to search for an overall scheme. In so doing, they recognized conflicts between different local organizations. As they attempted to resolve them, the notion of local special cases was extended to a global system that provided for a version of proof by cases, thus moving them into the third stage.

These three stages are outlined in Table 1.

	Finding New "Towers"	**Finding Duplicates**	**Proof of Completeness**
Stage 1	Random creativity "opposites" "cousins"	Look	"I can't think of any more!"
Stage 2	Further Development of local procedures; Some local organization; Towers with exactly one red cube ("staircase" and "elevator" patterns)	Local systems	Partial, by local organizations
Stage 3	Recognition of the need for an over-all organizational scheme; development of such a scheme	Global system	Proof by partial cases; proof by mathematical induction

Table 1. The stages of development in building methods of proof to the Tower Task.

PART 3: THE PRESENT STUDY

The mathematical ideas that are being considered here include the development of mathematical proof and combinatoric reasoning. The third, fourth, and fifth grade children who participated in the Towers Problem activity from the previous study were given a second task, the Pizza Problem. Both the Tower and the Pizza problems have an equivalent mathematical structure.

The fifth grade children from the former study were given an additional problem which we refer to as the Pizza Problem with Halves. It was chosen because it was structurally different from the Tower and Pizza problems.

For both problems, the children were asked to find all possible choices for ordering pizzas under the specified conditions and also to provide a convincing argument that they had accounted for all possibilities. Three questions guide the study.

Guiding Questions

1. In solving the Pizza Problem and Pizza Problem with Halves, do children go through a developmental stage sequence that is similar to the Tower Problem?

2. How do students go about solving the Pizza Problem with Halves?

3. Do students recognize any relationship between the Tower Problem and the Pizza Problem?

The Problems

The following Pizza Problem was given to all of the students in the study:

> Pizza Hut has asked us to help design a form to keep track of certain pizza choices.
>
> They offer a cheese pizza with tomato sauce. A customer can then select from the following toppings: peppers, sausage, mushrooms and pepperoni.
>
> 1. How many different choices for pizza does a customer have?
> 2. List all the possible choices.
> 3. Find a way to convince each other that you have accounted for all possible choices.

The following Pizza Problem with Halves was given to the fifth grade students:

> Capri Pizza has asked us to help them design a form to keep track of certain pizza sales. Their standard "plain" pizza contains cheese. On this cheese pizza, one or two toppings can be added to either half of the plain pie or the whole pie.
>
> How many choices do customers have if they can choose from two different toppings (sausage and pepperoni) that can be placed on either a whole cheese pizza or half of a cheese pizza?
>
> List all possibilities. Show your plan for determining these choices. Convince us that you have accounted for all possibilities and that there could be no more.

Problem Structure

The Tower Problem asks the children to determine the number of towers n cubes tall that can be built when there are two possible color choices for each of the cubes of the tower. The problem has an independence of selection structure in that the number of possible color choices for each of the cubes of the tower is not restricted in any way by the color choice made for any other cube of the tower. Thus there are two possible choices for each of the n cubes of the tower. Due to the independence of the selection process, there are $2 \times 2 \times .... \times 2$ or 2^n possible towers of n cubes tall that can be constructed.

Because we were interested in determining whether or not students' ideas develop in a parallel way when confronted with a problem with the same mathematical structure, we selected the Pizza Problem, which has a structure which is isomorphic to the Tower Problem. The Pizza Problem asks the children to determine the number of pizzas that can be made when there is the option of selecting from among n pizza toppings, where n is an arbitrary non-negative integer. The student can make two choices for each of the n pizza toppings, either to include it on the pizza or not. None of the individual topping choices are restricted by the choice made with regard to the choice of the other toppings. Since there are two possible choices for each of the n toppings and these choices are independent, there are $2 \times 2 \times ... \times 2$ or 2^n possible pizzas that can be made from n possible toppings.

In the Tower Problem, the actual objects (towers) can represent themselves. For the Pizza Problem, however, actual objects (pizzas) are not available and some kind of abstract representation of the problem must be constructed.

Both problems lend themselves to building a justification for the solution by either Mathematical Induction or a Proof by Cases. The number of possible towers of n blocks high that can be made from two colors can be generalized as 2^n and, similarly, the number of pizzas that can be made from n toppings as 2^n.

For the towers, focusing on one of the two colors in certain positions, one could partition the set of towers into $(n + 1)$ cases in which there are exactly none, one, two, ..., n of that color within a set, to develop a proof by cases. Similarly, one could have organized the set of pizzas according to the number of toppings.

The second pizza problem, Pizza with Halves, was selected because it was not isomorphic to either the Tower Problem or the Pizza Problem. Hence, we were interested in whether children could adapt their method of Proof by Cases to a problem with an unfamiliar structure.

The Pizza Problem with Halves does not have the independence of selection structure as does the Pizza Problem. This can be seen via the selection or not of the first two pizza ingredients on a toppings list. For the first topping on the toppings list, the child has three choices: 0, 1/2, or 1 (i.e., do not use the topping, put the topping on 1/2 of the pizza, or put the topping on the whole pizza.) For the second topping, the number of choices is dependent on the choice made for the first topping. If "0" or "1" were selected for the first topping, then the child could have any one of three choices for the second topping: 0, 1/2, or 1. However, if "1/2" were selected for the first topping, then the child would have four choices for the second topping: "0", "1", " 1/2 on the same side as the first topping", or "1/2 on the opposite side from the first topping".

The Pizza Problem with Halves can be solved by iterating the cases. If the problem is partitioned into two cases, the general problem can be solved using a conditional counting argument as follows:

Case 1: An ingredient, if selected, is put on the whole pizza. Since the child has two possible choices (yes or no) for each of n toppings, there are 2^n possible pizzas that satisfy Case 1.

Case 2: A combination of n ingredients, if selected for putting on 1/2 of the pizza, cannot be put on the other half of the pizza. For the first half of the pizza, there are 2^n possible combinations for selecting toppings from a list of n toppings (yes or no for each topping on the list). Since the choices for the other half of the pizza are restricted in that the choice can not match the pizza topping combination selected on the other half of the pizza, for each choice of one side of the pizza there are $2^n - 1$ possible choices for the other half of the pizza.

Thus for the two halves of the pizza there are $[2^n \times (2^n - 1)]/2$ possible choices for pizza toppings on the two halves of the pizza. The number of pizzas represented in the expression $2^n \times (2^n - 1)$ counts each choice combination twice. For example, the choice combination A, for the one half of the pizza, and the different choice B, for the other half of the pizza, are also counted again as the choice combination B and A. Since pizza combination A and B is the same as the pizza combination B and A, we must divide the expression $2^n \times (2^n - 1)$ by 2 in order to count each different pizza combination only once.

By finding the sum of the number of solutions from Case 1 and Case 2, the general solution to the Pizza with Halves problem can be expressed as

$2^{n-1} \times (2^n + 1)$.

It is important to emphasize that it is not our expectation that children find the general solution of the Pizza with Halves problem at this time. However, we are interested in knowing whether students recognize that 2^n is not a valid solution for the Pizza with Halves problem and whether students eventually can invent a method of Proof by Cases for determining that the solution for cases $n = 2$, 3 and 4 is 10, 36, and 136, respectively.

Methods and Procedures

Ongoing research about children's mathematical thinking is carried out in actual classrooms in three New Jersey school districts: Colts Neck, Kenilworth, and New Brunswick. Each of these districts has had an ongoing long-term partnership in mathematics teacher development with Rutgers University. The data for the study come from five classes distributed among the three project sites.

A class of 12 children in grade 5 come from the Harding School in Kenilworth, a blue collar community. Since grade 1, the students have been in classrooms in which their teachers, in partnership with Rutgers, have challenged them to do thoughtful mathematics. This is the site of a ten year partnership with the University.

Another fifth grade class of 18 students comes from the Redshaw School in New Brunswick, an urban school district with a five-year partnership with Rutgers. The children have been working for two years in classrooms that emphasize thoughtful mathematics.

The third site, Conover Road School in suburban Colts Neck, involves one third grade class of 23 students and two fourth-grade classes, one of 23 students and another of 26 students. Teachers in the school have participated in mathematics teacher development for three years. We have worked in classrooms of the fourth graders for two years and the third graders for one year.

The mathematical thinking of 26 of the 102 participants in the original study are described in this report. The data for this study consists of videotape transcripts[3] of students working on the Pizza Problem(s) with follow-up interviews, and the students written work from all three project sites. The data results are organized into three stages that were suggested in the analysis.

Results

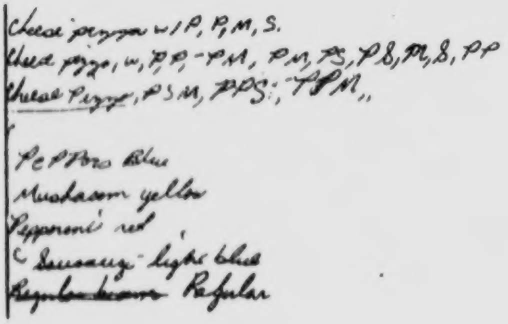
Results within each stage are organized by problem; First, the Pizza Problem and then, the Pizza Problem with Halves.

Stage I: The Pizza Problem

Invariably, the children, either individually or in dialogue with each other or an adult, first consider the meaning of the problem. They begin by making a creative but seemingly random listing, or by representing pizzas that meet the criteria as they understood them. Examples are recorded as the students think of them.

Stage I, Example I: Benny and Patrick, Grade 5

Benny and Patrick began the task by reading, and then discussing what the problem required. They initially used the heuristic of making a list recording their ideas as they emerged, using the first letter of each topping to represent that entry. Note that "P" was used to represent both pepperoni and peppers. Figure 1 is the boys' recording of their initial solution.



cheese pizza w/ P, P, M, S.
cheese pizza, w, PP, -PM, PM, PS, PS, M, S, PP
cheese Pizza, PSM, PPS; PPM,,

PePPero Blue
Mushroom yellow
Pepperoni red
Sausage - light blue
~~Regular~~ Regular

Figure 1. Benny and Patrick's initial list of pizza choices.

As Benny and Patrick attempt to identify possible pizza choices, we observe in the dialogue that follows that each boy makes reference to the need to convince the other that all possibilities had been found. The notion of convince, in this episode, appears to be interpreted by the boys to mean persuade.

Benny:	You could have put cheese pizza with pepperoni on top.
Patrick:	What? Say that again.
Benny:	Pepperoni pizza.
Patrick:	Yeah [pointing to the first entry on his list]... See pepperoni pizza. Cheese pizza with pepperoni ... then pepper... sausage...
Benny:	[reading from the problem] "Find a way to convince each other that you have found all possibilities." OK... Convince me. As you just said... there might be another one.
Patrick:	There might... but there can't ... there isn't.
Benny:	Why are you saying that?
Patrick:	Then you try to convince me with something. See... Here is a regular pizza, 'cause there's no topping. [points to the cheese pizza on the third row of his list] This is two, this is three ... four...

Patrick pointed to the first four letters in the first row of his diagram. Notice he used the letter "P" to represent pepperoni as well as peppers. His notation for pizzas with mushrooms alone and sausage alone were "M" and "S" respectively.

Benny: Can't you have two toppings on a pizza?
Patrick: You can have as many as you want and I already did that. Look. PM means pepperoni and mushrooms... peppers and mushrooms...

Now Patrick is pointing to the entries on the second line of his list. Note he is consistent in using the letter "P" to represent both peppers and pepperoni in combination with mushrooms. Although he does have the idea in his head, his failure to establish a one-to-one correspondence between object and notation to represent the object interferes with making further progress on the task.[4]

As Figure 1 indicates, they continue generating pizza combinations recording them until they can think of no others. Their notation continues to hamper their progress.

Benny; You've got PSM... now put PMS...
Patrick: That's the same thing...that's right here. PSS?
Benny: I know... I know... You've got PS...
Patrick: Put PSS...
Benny: OK... PSS ...
Patrick: [Erasing] No... It's got to be SPP... PPS [He includes this choice.]
Benny: Now convince me that it's all.

The boys noticed that two nearby classmates, Arteshia and Desiree, were using plastic cubes to model the problem. The boys then decided to use the heuristic of building a model with plastic cubes. They began by assigning four colors to represent each of the four toppings and made fifteen pizzas with toppings. Benny and Patrick then realized that they had no way to represent a plain cheese pizza so they designated a fifth color to represent a plain pizza. Together they modeled sets of "pizzas". This concrete representation appeared to be much more helpful to them and replaced their symbolic list as the basis for their solution to the problem.

Stage 1, Example 2: Benny and Patrick vs. Arteshia and Desiree, Grade 5

Benny and Patrick were sitting closeby to Arteshia and Desiree. Both pairs of children were building models of pizzas with plastic cubes. Arteshia looked across and noticed that her concrete representation was different from Benny and Patrick's. (Recall that the boys had chosen four colors to represent each of the four toppings and made fifteen pizzas with toppings.) A problem arose in that the boys had no way to represent a plain cheese pizza. Recognizing this, they designated a fifth color (brown) cube to represent a plain pizza. The girls had assigned a yellow cube to represent the plain cheese pizza which then formed the base of all sixteen of their stacks of plastic cubes.

The boys, unlike the girls, had been inconsistent in modeling their mental representation of crust, thereby causing difficulty in communication.[5] Their discussion was as follows:

Arteshia: What do the different colors equal?
Patrick: We can't tell you! [laughs] ... OK... See pepper stands for blue. Mushroom stands for yellow. Pepperoni stands...
Arteshia: How can you have peppers without a pizza [crust]?
Patrick: Huh?
Arteshia: How can you have peppers without a pizza [crust]?
Patrick: You don't even know our system yet!
Benny: See... Look... [He holds up a single yellow cube.]
Teacher: Let's hear their system.
Patrick: OK. This is our system. We're not using anything for the pizza ...
Benny: Won't you let me say something?
Patrick: OK ... [Benny holds up the cube and opens his mouth to speak, but, when Patrick continues, he shrugs and grins.] We're not using anything for the pizza [crust]... so... like if we put two toppings on it... that means two toppings ... and there's no signal [sic, symbol] for the pizza... you just know it's there.
Teacher: We're assuming though that there's a pizza under there? OK? I see one brown cube there... What's that?
Patrick: Just a regular pizza... one plain pizza.

The teacher then turned to the girls and asked them what they had used to represent a plain pizza.

Arteshia: [holding up a yellow cube] This is the pizza right here... but I was confused. Like they said [pointing to Benny and Patrick's brown cube] that was a pizza... but they don't have a pizza under the other toppings [the combinations with toppings].

Teacher: What could they do to make their's like yours?

Desiree: Add a pizza to each.

Stage 1, Example 3: Meredith, Grade 3

Meredith and her partner used the heuristic of building a chart. They initially interpreted the problem to include two sets of pizzas: (1) "whole pizzas" with a single topping and (2) pizzas with two toppings where one topping is on half of the pizza and the other topping is on the remaining half of the pizza. They then found a total of ten possible pizzas.

Teacher: Okay now you came up with ten [possible pizza choices].

Meredith: Yeah. Because first I took all the wholes [pizzas with a single topping]. That would be [a pizza with] peppers and [a pizza with] sausage and [a pizza with] mushrooms and [a pizza with] pepperoni. And then I did the ones where it could have two toppings... half peppers half sausage, half peppers half mushroom, half peppers half pepperoni, then you go to the next one with half sausage half mushrooms, then it went sausage with pepperoni, mushrooms with pepperoni.

When the teacher asked the children to consider whether customers could have any other choices when selecting from four toppings, Meredith explained that pizzas could not have more than two toppings.

Teacher: Is it possible to order a pizza with more than two toppings?

Meredith: No!

Teacher: Why?

Meredith: No cause pizza stores don't do that. They only do it with halves. Cause it would be too hard for them to do with quarters... [As she speaks, Meredith traces a pizza with her finger in the air which is divided into three pieces: two pieces are one quarter of the pie and the third piece is one half of the pie.] quarter uh, then another quarter with another topping and half with another topping.

Teacher: I see.

Meredith: They could do a four... say they had a four slice pizza... and they have a different thing on each slice.

Thus, Meredith was solving a different problem than was intended.[6] Later, when Meredith became aware that the problem required that she consider pizzas where toppings were mixed together on each slice, she was able to build pizzas that could also have three toppings or four toppings and, thus, was able to complete her organization by cases.

Stage 1, Example 4: Ebonie and Kersa, Grade 5

Ebonie and Kersa began by using the heuristic of making a list to generate collections of pizzas in a random manner. They then began to build a primitive system, in which there were cases of pizzas with a single topping and then pizzas with more than one topping together. They checked for duplicates by comparing a new pizza to the others that were listed. When they could think of no others and could only find duplicates, they volunteered to share their 16 different choices with the teacher (see Figure 2).

Teacher: And you're convinced about what?

Ebonie: That it's only how many that we've got.

Teacher: How many do you have?

Ebonie: Sixteen.

In response to the teacher's request to explain their solution, the girls read the list that was generated by them according to the following categories: plain pizza, pizza with one topping, and pizza with two toppings. For example:

Teacher: Can you explain it to me?

Kersa: [picks up her recording sheet] We have one plain cheese pizza, we have one pepper and cheese pizza and we have one sausage and cheese pizza, one mushroom and cheese pizza, one pepperoni and cheese pizza. Then over here we...

Ebonie: We combined, so now...

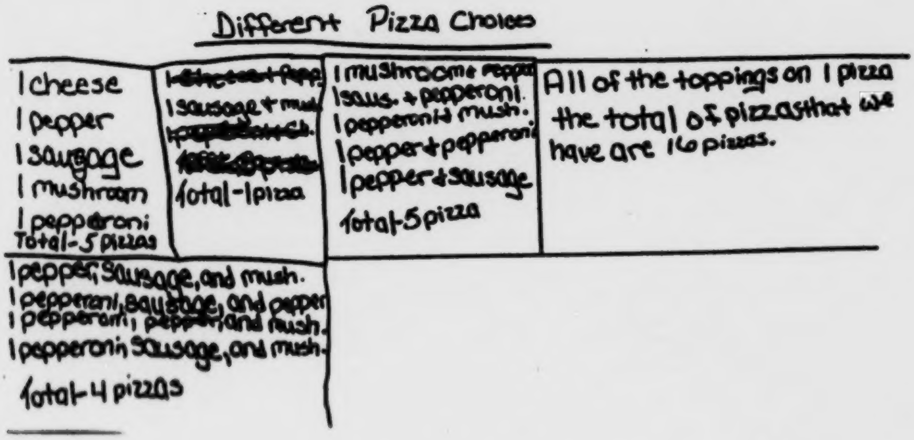
Kersa: We combined the different toppings.

Teacher: Two different toppings?

Girls: [reading together] Yea... one sausage and mushroom, one mushroom and pepper, one sausage and pepper, one pepperoni and mushroom, one pepper and pepperoni and one pepper and sausage.

Teacher: OK... So there's six of them altogether?

Kersa: Uh huh...



Different Pizza Choices

1 cheese
1 pepper
1 sausage
1 mushroom
1 pepperoni
Total-5 pizzas

~~1 cheese + pepp~~
1 sausage + mush
~~[illegible]~~
~~[illegible]~~
total-1 pizza

1 mushroom + pepper
1 saus. + pepperoni.
1 pepperoni + mush.
1 pepper + pepperoni
1 pepper + sausage
total-5 pizza

All of the toppings on 1 pizza the total of pizzas that we have are 16 pizzas.

1 pepper, sausage, and mush.
1 pepperoni, sausage, and pepper
1 pepperoni, pepper, and mush.
1 pepperoni, sausage, and mush.
total-4 pizzas

Figure 2. Ebonie and Kersa's chart with sixteen pizza choices.

The teacher then asked the girls whether they had found all possible pizzas with exactly two toppings. Their justification for finding all possibilities in that category indicated a Stage 1 level of development.

Teacher:	How can you... Do you have them all?
Ebonie:	We have them all with two toppings.
Teacher:	How can you be sure?
Ebonie:	Because every time we come up with toppings that we could mix together... it turns out to be one that we already have.

The girls then read the four possibilities for three-topping pizzas that they had listed. When asked to justify their answer, they returned to their list, trying various combinations only to find that each one was already there.

Stage 1: **The Pizza Problem with Halves**

Stage 1. Example: Michael and his group, Grade 5

A group of fifth-graders worked together and tried to make sense of the problem. They began by randomly drawing pictures of pizzas that fit their understanding of the problem requirement as depicted in Figure 3.

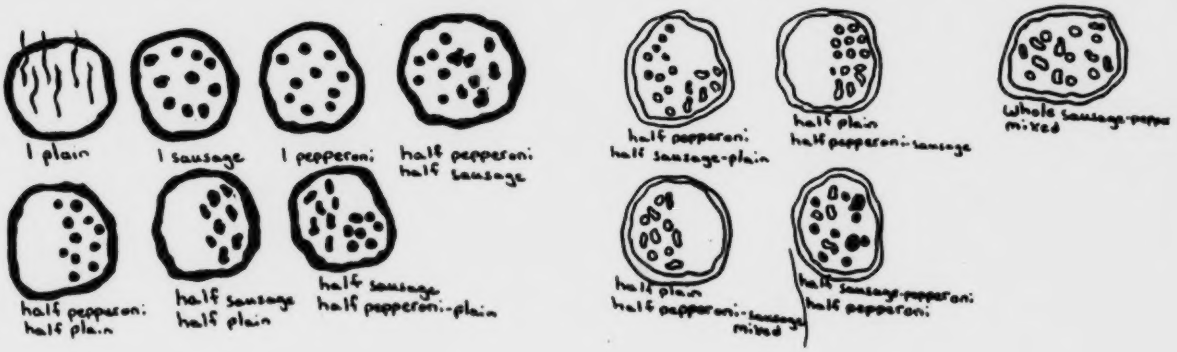

1 plain
1 sausage
1 pepperoni
half pepperoni half sausage
half pepperoni half plain
half sausage half plain
half sausage half pepperoni-plain
half pepperoni half sausage-plain
half plain half pepperoni-sausage
whole sausage-pepper mixed
half plain half pepperoni-sausage mixed
half sausage-pepperoni half pepperoni

Figure 3. Michael's realistic depiction of pizzas.

Their strategy was first to find "whole" pizzas. They defined the category of whole pizzas by noting that any slice from it was approximately the same as any other. They then represented half pizzas with toppings, and then made pizzas with quarters (See, for example, half sausage/half pepperoni - plain; and half pepperoni/half sausage - plain). Michael and his group used a picture to provide a careful depiction of their ideas. Michael recorded the entries of the group using small circles to represent pepperoni and irregular shapes for sausage. Notice the elaborate detail in producing a realistic drawing to match the group's representation of the problem.

Stage 2: The Pizza Problem

In Stage 2, we observe students attempting to justify their solutions. Their efforts indicate an organization of pizzas according to different characteristics. Some pizzas were grouped into topping categories, which accounted for some or all of the possibilities. These local organizations were sometimes accompanied by concrete representations or more elaborate notations to match them.

An important issue for the children was how to categorize a pizza with more than one topping, taking into account that the order of the toppings did not matter. This was particularly apparent for children who organized their solution around sequences of pizzas with each sequence building up from a base by adding another topping. The organizations and reorganizations that were developed made sense to the child, but often contained entries from sets that were not disjoint.

Stage 2, Example 1: Marcel and Frederick, Grade 5

Marcel and Frederick worked together to solve the Pizza Problem. Frederick worked on building a physical model of "pizzas" from stacks of plastic cubes while Marcel generated a list of pizzas symbolically. The boys checked the total number of stacks (not item by item), against the total number on Marcel's coded list. The two agreed that there were seventeen because each unknowingly had repeated an entry (Note both "SM" and "MS" are given as two-topping possibilities). Marcel's first drawing represented the way the boys initially were thinking about the pizzas (see Figure 4).

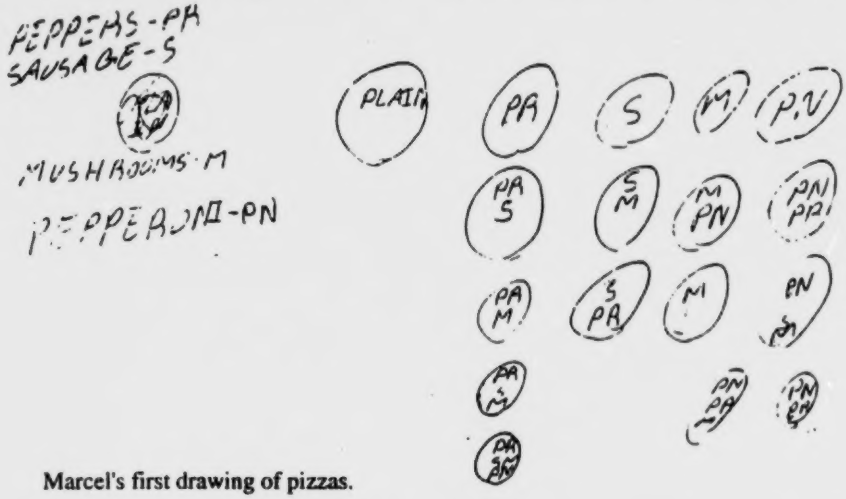


Figure 4. Marcel's first drawing of pizzas.

Marcel and Frederick's final drawings (see Figures 5 and 6), were produced as they worked together in preparation for later class sharing. In so doing, they noticed the duplicate entry and eliminated it. In preparation for convincing other classmates that all possibilities were considered, the boys changed their code so that only the pepperoni topping had a two letter code ("PR"). The boys also added a "C" to each combination, to indicate that cheese was included in each entry (Figure 5).

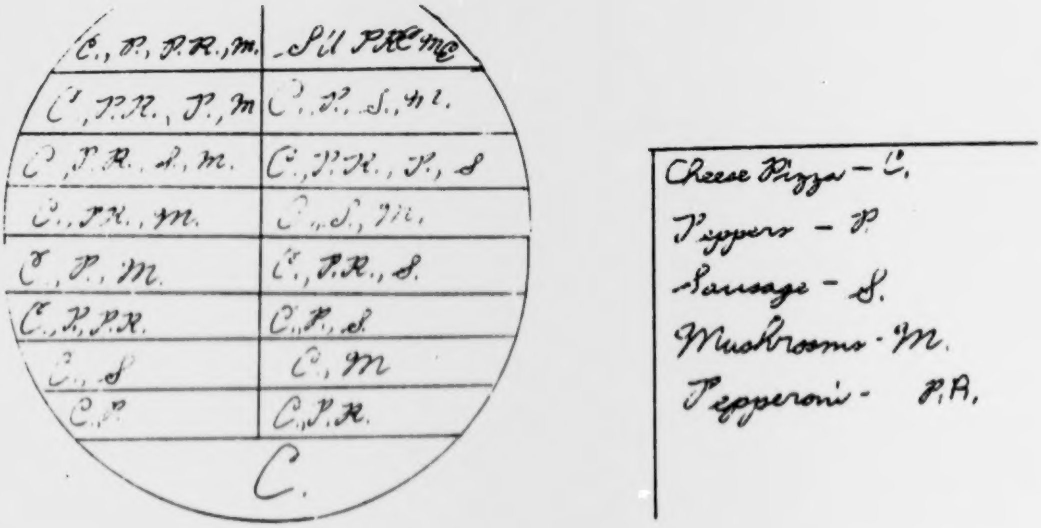

C., P., P.R., M.
SUPREME
C., P.R., P., M
C., P., S., M.
C., P.R., S., M.
C., P.R., P., S
C., P.R., M.
C., S., M.
C., P., M.
C., P.R., S.
C., P., P.R.
C., P., S
C., S
C., M
C., P
C., P.R.
C.
Cheese Pizza - C.
Peppers - P.
Sausage - S.
Mushrooms - M.
Pepperoni - P.R.

Figure 5. Marcel's final drawing of pizzas.

The final drawing of Frederick indicated a further revision of code in which the letter "B" was used to represent the pepperoni topping (see Figure 6).

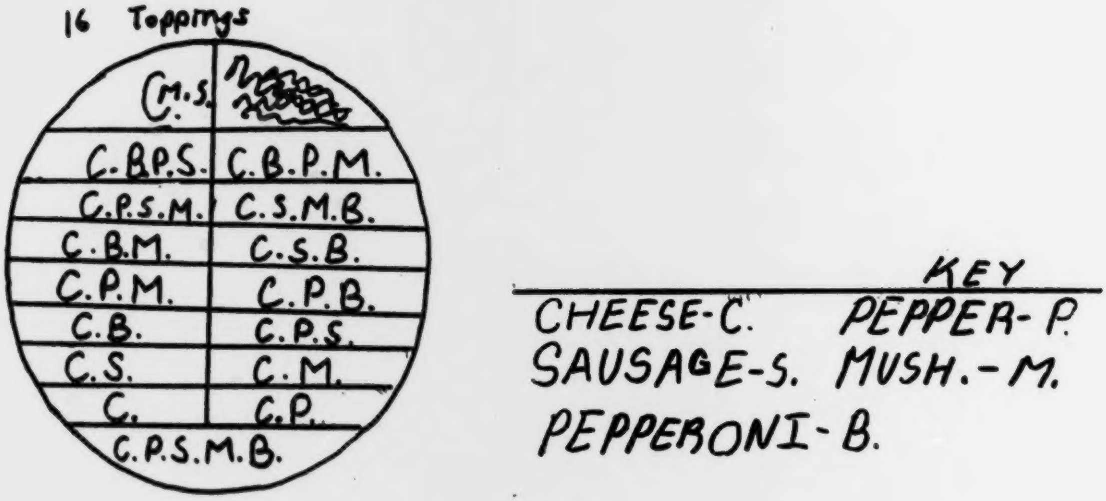

16 Toppings
C. M. S.
C. B. P. S.
C. B. P. M.
C. P. S. M.
C. S. M. B.
C. B. M.
C. S. B.
C. P. M.
C. P. B.
C. B.
C. P. S.
C. S.
C. M.
C.
C. P.
C. P. S. M. B.
KEY
CHEESE - C.
PEPPER - P.
SAUSAGE - S.
MUSH. - M.
PEPPERONI - B.

Figure 6. Frederick's final drawing of pizzas.

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Stage 2, Example 2: Arteshia and Desiree, Grade 5

Recall that Arteshia used the heuristic of building a model consisting of sixteen stacks of plastic cubes to represent pizzas with different toppings. Her partner, Desiree, used the heuristic of making a list. In order to compare their representations, the girls sorted the 16 individual pizzas into five cases according to the number of toppings. However, when the girls were asked to defend their solution, Arteshia reorganized the stacks of cubes by placing twelve of the "pizzas" into six groups (see Figure 7).

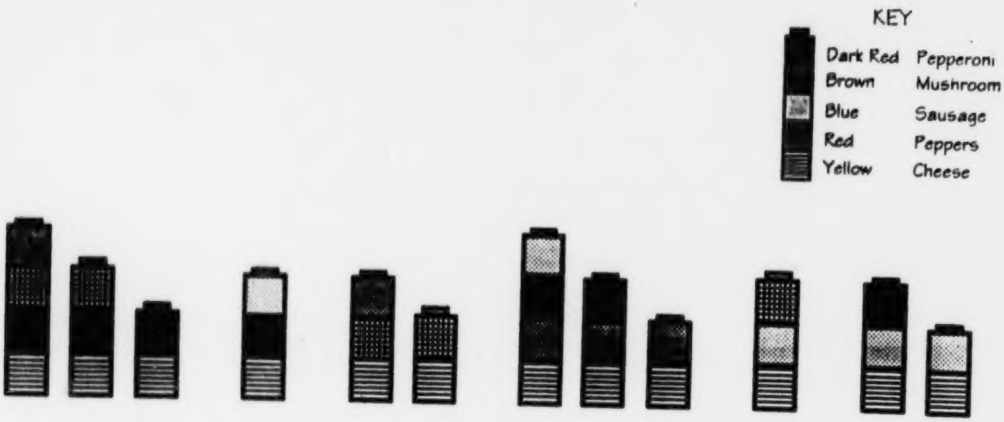


Figure 7. Arteshia's "family" organization for "pizzas".

Arteshia's plan was to make "families" of pizzas by taking pizzas with two or more toppings and removing one topping at a time to produce new combinations. Her first group contained three "pizzas"; each time Arteshia successively removed a plastic cube from the top of her three-topping pizza model, she made a new combination. The second group contained only one pizza with two toppings; when Arteshia removed a cube from the top of the fourth pizza, she obtained a duplicate (the third pizza) of the one-topping pizza in the first group. Her third group of "pizzas" contained a two-topping, one-topping sequence; her fourth group contained a three-topping, two-topping, one-topping sequence; her fifth group contained one pizza with two toppings and her sixth group contained a two-topping, one-topping sequence. She verbalizes her reorganization as trying to justify her solution.

Teacher: Why are you doing that?

Arteshia: So that I can prove that I have them all.

However, Arteshia was still unaware that an explanation of reorganization into groupings was not, in itself, a suitable justification. As she continued to try and explain her arrangement, she recognized that all of the groups did not contain the same number of "pizzas".

Arteshia: It's because we already used one of the toppings on each of them already and it [the "pizza" that would complete the group] will just be a duplicate.

Arteshia, with Desiree assisting by retrieving certain "pizzas", then turned to the "pizzas" that were not included with the other twelve. She rearranged these, beginning with the tallest, which represented the "pizza" with all four toppings and completed a staircase of five decreasing "pizzas" ending with the plain cheese pizza represented by a single yellow cube (Figure 8).

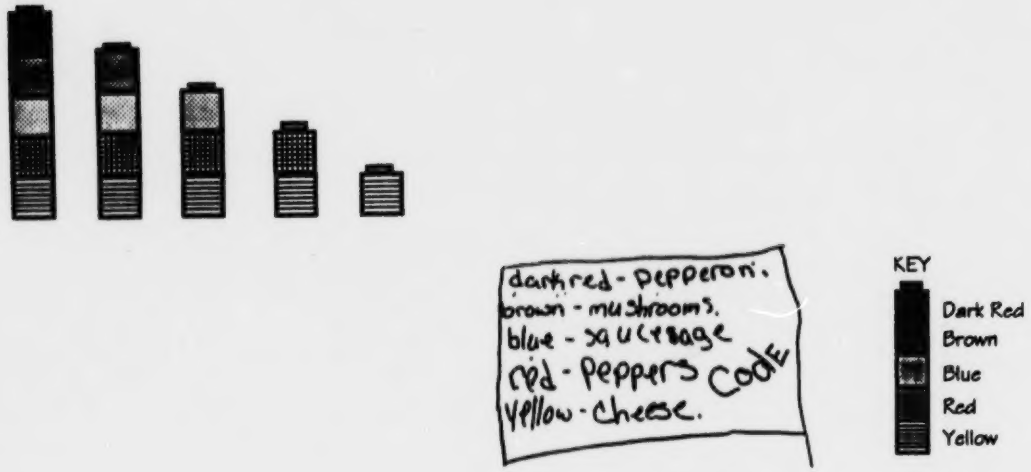

darkred - pepperoni.
brown - mushrooms.
blue - saucesage
red - peppers
yellow - cheese.
Code
KEY
Dark Red
Brown
Blue
Red
Yellow

Figure 8. Arteshia's "family" of five "pizzas" which successively decrease in size by one cube.

Artesia explained that this was the optimal pattern.

Arteshia: You can't make another pattern like this.
Teacher: Why not?
Arteshia: [holding up the single yellow cube] Because you've already used this one... so that will take one step out of the pattern and [holding up the five cube "pizza"] you've used all of the toppings plus a pizza ... you couldn't do that again ... So that will make patterns like these [pointing to the groups in the original line of "pizzas"].

Arteshia, in providing a justification for her work, formed these local organizations which took the form of "families".

Stage 2. Example 3: Alana and Jaime, Grade 4

When Alana worked to find all possible pizza choices when selecting from four toppings, she used the heuristic of drawing a diagram and developed a notation for representing each pizza choice. She represented each pizza by drawing a slice of pizza, and represented each topping with a symbol: "+" for sausage, "l" for mushroom, "." for peppers, and "o" for pepperoni, and placed the symbols inside each drawing of a slice of pizza (see Figure 9).

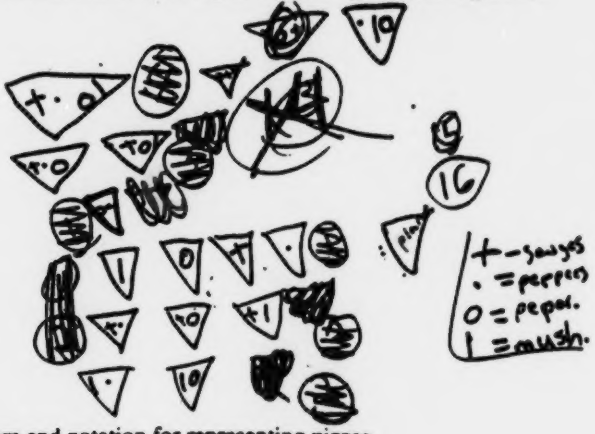

16
+ - sausges
. = peppers
O = pepol.
l = mush.

Figure 9. Alana's diagram and notation for representing pizzas.

Thus, a drawing of a slice of pizza with "+" and "I" inside the slice represented a pizza with sausage and mushrooms. Prior to developing this notation, both Alana and her partner Jaime decided to generate their combinations by considering which pizzas could be made with one topping, two toppings, and so on, a precursory stage to building a proof by cases. While Jaime used the heuristic of making a list by writing out each pizza choice in words, Alana developed her slice notation as a form of shorthand to avoid the tedious recording of each combination. In the following excerpt, Jaime attempts to understand Alana's form of notation.

Jaime: I don't know what you did cause you're drawing a picture and I don't get what, what's inside it. What's inside it?

Alana: [She points to a circle inside the slice of pizza.] That's pepperoni. I don't feel like writing pepperoni. That's pepperoni, sausage, mushrooms and peppers... I'm going to do a key [draws "+ = sauges", ". = peppers"] That would be peppers. How do you spell pepperoni? [draws "o = pepor"] What should I do for mushrooms?

Jaime: A square.

Alana: I'll just draw a line [draws "I = mush"].

For Alana, developing a form of notation was a way to reduce the amount of recording necessary to pursue her strategy rather than a way to further organize her development of a method of proof.

Stage 2: Pizza Problem with Halves

We see a parallel developmental sequence with the Pizza Problem with Halves as with the Pizza Problem. The following excerpt shows children trying to find appropriate language to describe their ideas.

Stage 2, Example 1: Amy-Lynn and her classmates, Grade 5

Amy-Lynn worked with several classmates on the Pizza Problem with Halves and produced a written description of her solution indicating: "When we made one (pizza) we made the opposite." In the discussion that followed during the next class, she was asked to explain what she meant.

Teacher: In your letter, you talked about opposites. What's an opposite in pizzas?

Amy-Lynn: If you make one pizza... sausage and pepperoni... I mean cheese and pepperoni [halves]... you'd just make the other ... pepperoni and cheese... no... I mean sausage and cheese.

Disagreement with Amy-Lynn' s meaning of "opposite" led to further discussion in which she was asked to clarify her meaning with specific examples.

Teacher: Does every pizza have an opposite?

Amy-Lynn: No. For cheese and sausage... the opposite is cheese and pepperoni... If you wanted to make the opposite of a plain... it would be sausage and pepperoni... and you had already made them.

As the children attempted to verbalize their concern with this use of the word "opposite", discussion continued.

Stephanie: It's like... the opposite of yes is no... because yes is not no.

Jeff: It's not different ... it is "the other"... the backwards....

Another student, Brian, tried to interpret Amy-Lynn's meaning and suggested that she might have been thinking about "opposite" from the perspective of the Towers Task, giving as an example the case when the opposite of all red cubes would have been all yellow cubes. Ankur, then, indicated that "opposite" can occur when two choices are possible.

Ankur: You can have an opposite if you have only two toppings... Like pepperoni and sausage... no plain.

The following example demonstrates the children's attempts to define categories for considering all possible cases for justifying their solution to the Pizza with Halves Problem. Several students in the class had formed three categories for pizzas: "whole", "half" and "mixed". For these children, a "whole" pizza was an entire pizza, and a "half" pizza was a pizza divided into two parts. Some of the children defined a "mixed" pizza as a pizza with at least one topping, while others defined it as a pizza with more than one topping. These conflicting definitions led to confusion. One student, Matt, expressed concern that the three categories of "whole", "half" and "mixed" pizzas proposed by several students were inappropriate. The students had not yet recognized that it mattered whether these categories were both mutually exclusive and exhaustive and so attempted to defend their position.

Stage 2, Example 2: Brian and Matt, Grade 5

Matt: Why are they having the "mixed" column? ... Why do they need it?

Teacher: What categories would you have?

Matt: Only whole and half.

Brian: But it's easier ...

Teacher:	Why?
Brian:	Because if you only have two rows [two categories of whole and half]... the half [category] is going to be a mile long with pizzas.
Teacher:	Does that matter?
Brian:	It will help the pizza parlor because if they make a menu and they show what stuff is on it... they'll have half like on Row 2 ... Row 3 ... Row 4... They're going to get mixed up! [He indicated that there would be too many entries on the menu for "half pizzas" if the mixed category was eliminated.]

Stage 3: The Pizza Problem

Stage 3, Example 1: Arteshia and Desiree, Grade 5

As Arteshia and Desiree established for themselves what appeared to be persuasive arguments within categories that were both consistent and exhaustive, they then began to expand the local organizations and develop a global one. We see this in their recording of possible "pizzas" in which they first began with the four-topping "pizza" and then successively removed one topping at a time to identify their cases. This "progressive elimination" strategy resulted in their identification of five categories. Desiree arranged the stacks of cubes into the following five groups in assisting Arteshia in preparation to share their solution to the class (see Figure 10).

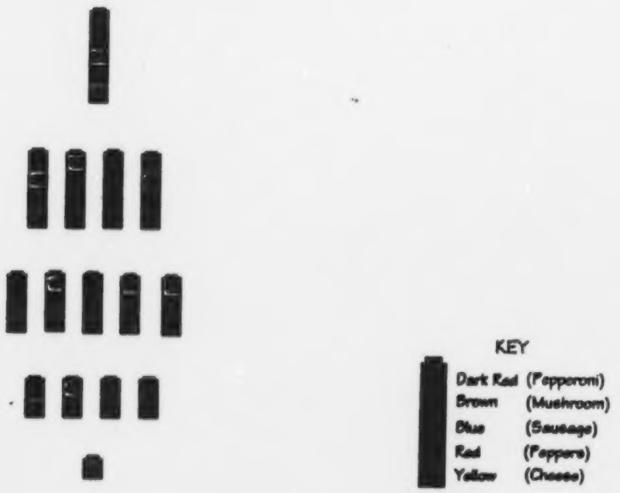
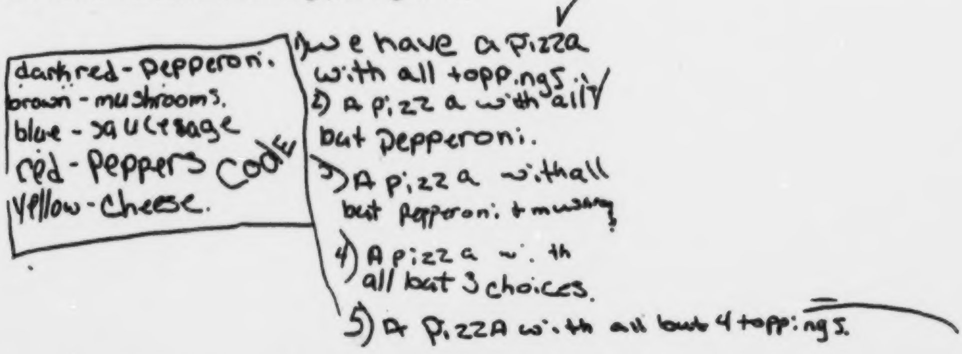


Figure 10. Arteshia and Desiree's groupings from zero to four toppings.

Arteshia used the stacks to model her thinking (See Figure 11).



darkred - pepperoni.
brown - mushrooms.
blue - saucesage
red - peppers
yellow - cheese.
Code

1) we have a pizza with all toppings.
2) A pizza with all but pepperoni.
3) A pizza with all but pepperoni + mushrooms
4) A pizza with all but 3 choices.
5) A pizza with all but 4 toppings.

Figure 11. The first page of Arteshia and Desiree's written work.

Arteshia: First I used the plain pizza by itself as one. [She points to the single yellow cube which was grouped alone.] ...and then I used the plain pizza as the bottom for each of the four toppings, so that would make four [pizzas] with one topping.

Teacher: So four [pizzas with one topping] plus the one plain?

Arteshia: Is five. Then I started putting two toppings by switching up... First I used this one pizza with one topping [She picks up the yellow and dark red stack]. Then I used this one pizza with one topping and then I added pepper to it [She removes the stack made of yellow, dark red and red from the two-topping group and places it beside her one-topping base of yellow and dark red.] ...Then I used [the same] one pizza with one topping and I added mushrooms to it [She removes the stack made of yellow, dark red and brown from the two-topping group and adds it to the new group around the yellow and dark red]. and then I used this one [same] pizza with one topping and added sausage to it.

At this point Arteshia paused and looked carefully at the remaining four two-topping "pizzas". She selected the stack made of yellow, blue and dark red and rebuilt it by reversing the blue and dark red cubes so it would be consistent with the other pair of two-topping stacks in her new group. She then included it in that group and indicated that she was done.

Teacher: OK... How many things could you add to this one... the one with dark red?

Arteshia was able to make three pizzas with two toppings from the one-topping pizza represented by the yellow (crust) and dark red cubes. She did this by adding a red cube, a brown cube and a blue cube to the top of three yellow and dark red stacks.

Arteshia: Three... and then I took another one-topping pizza [She picks up the one-topping stack with yellow and red.] ...I took this one with red ... and then I used a red one with mushrooms on it [She removes the two-topping stack with yellow, red and brown and places it in the new group that is forming around the second one-topping choice] ...and then I used a red one with sausage on it [She adds the yellow, red and blue stack.] ...and then ...instead of saying red with pepperoni, since I already said pepperoni with red... [She points to the yellow, dark red and red stack that she had placed in the first group.] ...It's like the same thing.

Teacher: You can't count it again?

Arteshia: No.

Teacher: How many now?

Arteshia: [She moves the last two-topping stack over with the others.] Six.

Teacher: You squeezed this one in... [the yellow, brown and blue stack] Where did it come from?

Arteshia: The brown one [indicating the one-topping stack of yellow and brown] ...plus sausage.

Teacher: Why are there no more?

Arteshia: I already used them... there were three with each, but if you make them again they'll be duplicates. There are 6 [different two-topping pizzas which could be made by building upon the one-topping pizzas].

Arteshia then turned to the four towers with three toppings and explained that she could generate these in the same way from the two-topping stacks. She indicated that this was more confusing because so many of them would be duplicates. Arteshia finally noted that there was only one pizza that could be made with all four toppings. Desiree agreed and checked Arteshia's statement by counting the stacks to find that there was a total of sixteen.

The girls were aware that the particular topping(s) that they held fixed could vary. They were able to demonstrate their reasoning either concretely or with notation. For example, they began with the case of four one-topping "pizzas"; they then built from it a justification for two-topping "pizzas" by holding one topping constant and varying the second for all possibilities. The argument for the case of three toppings was modified requiring that they hold two toppings constant and vary the third. In defending that they had found all possibilities, they were challenged to identify those pizzas that could have been built if another topping had been first chosen as the constant, and thus were able to generalize their strategy.

Stage 3, Example 2: Meredith, Grade 3

Meredith decided to tackle the problem by using the heuristic of constructing a chart to record the choices of pizzas. Her chart had four columns, one to represent each topping: peppers, sausage, mushrooms and pepperoni. She then constructed five cases: pizzas with one topping, pizzas with two toppings, pizzas with three toppings, pizzas with four toppings and pizzas with zero toppings, which she commonly referred to as the plain pizza.

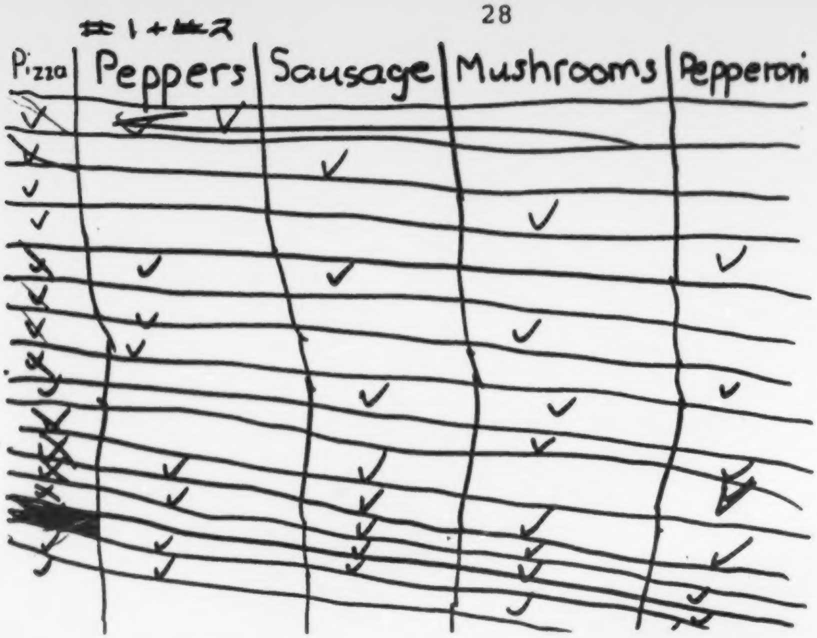


Figure 12. Meredith's chart organization by cases.

Meredith's method of organizing her data with a chart was helpful to her in building an exhaustive argument within cases as well as across them. An example of this was her consideration of the pizzas with exactly two toppings. Meredith began with an entry in the peppers column and then combined each of the other three toppings with peppers. Once she had exhausted all the possible two-topping pizzas with peppers, she moved to the sausage column and combined sausage with each of the two remaining toppings; mushrooms and pepperoni. Finally, she went to the mushroom column and combined mushroom with the one remaining topping, pepperoni, and thus accounted for all possible two-topping pizzas.

As Meredith recorded cases for pizzas she used pens with different colored ink to distinguish one case from another (red for pizzas with one topping, black for pizzas with two toppings, etc).

Meredith: Then the pizza with peppers and sausage, then you go to... you skip the sausage and say the pizza with peppers and mushrooms and then you skip the sausage and the mushrooms cause you already used that and you go to the pepperoni... so the pepperoni with peppers and then you go to the sausage cause there's no more toppings that you could use with pepperoni [sic, peppers]. You use the sausage with the mushrooms and then since you already used the mushrooms you skip the mushrooms and you go to the pepperoni and that's another possibility and then you can't use any more sausage cause there's no more toppings so you do the mushroom with the pepperoni and then you know that you have all of them cause you can't go this way [combine pepperoni with any topping to its left in the chart] or you'll have a duplicate so you went all the way that way and there's no more toppings.

Teacher: Interesting... so those are the "two's" [two topping pizzas] then.

Thus Meredith had developed a global organization for considering and exhausting all possible pizza choices. Her chart helped her to keep track of all possible pizzas within individual cases by providing a systematic format for the formation of combinations. She demonstrated this by first combining peppers with sausage; then by skipping the sausage column and combining peppers and mushrooms; then, by skipping the sausage and mushroom columns, and combining peppers with pepperoni.

Stage 3, Example 3: Steven, Grade 4

Steven used the heuristic of drawing a diagram and developed a notation which aided him in the formulation of a cyclic approach to exhausting all possible pizzas when selecting from four toppings. He used a single digit number to represent each topping, and devised a key to define the number designated for each topping. Steven used "0" to represent a pizza with a single topping, "1" to represent the pepper topping, "2" to represent the mushroom topping, "3" to represent the sausage topping, "4" to represent the pepperoni topping and "5" to represent the cheese or "plain pizza" (see Figure 13).

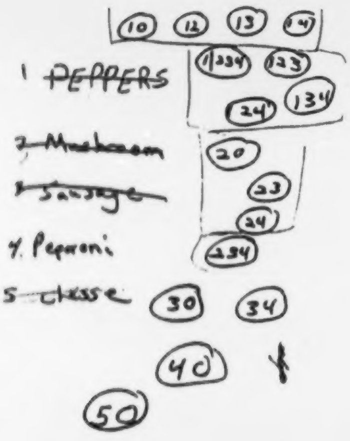


Figure 13. Steven's numerical code for representing pizzas.

He then used this coding system to generate all possible pizza choices. Steven began with number "1" for "peppers" and formed every possible combination of numbers which included "1" and the digits "2", "3" and "4". Thus, "1,0" represented a pizza with peppers and nothing else, "1,2" represented a pizza with peppers and mushrooms, and so on. With his numerical code, Steven was able to account for the pizza with pepper only using (1,0), all pizzas with two toppings where one topping was pepper [(1,2), (1,3) and (1,4)], all pizzas with three toppings where one was pepper [(1,2,3), (1,2,4), (1,3,4)] and the pizza with all four toppings [(1,2,3,4)]. Steven then crossed pepper off his list and moved on to combine mushroom with all the remaining toppings. He explained that by removing peppers from the possible options he avoided the production of duplicate pizzas.

Next he considered pizzas with mushrooms where the choice of pepper was removed. He made the plain mushroom pizza (2,0), mushroom with sausage (2,3), mushroom with pepperoni (2,4) and mushrooms with sausage and pepperoni (2,3,4).

Steven: Mushroom staying alone [He writes 2 0.], mushroom with sausage two, three [writes 2 3], mushroom and pepperoni, two and four [writes 2 4], then mushroom with all three of them, the other two [toppings] two, three, four [writes 2 3 4] so that would be four [combinations with mushroom, excluding pepper] and then... and then we're done with mushroom [Steven crosses out mushroom from this list.]... and so we go to sausage...

Steven then considered pizzas with sausage, with the choices of peppers and mushrooms eliminated. These were the plain sausage pizza (3,0), and the sausage and pepperoni pizza (3,4). The only remaining options were pizzas without peppers, mushrooms and sausage, which was a plain pepperoni pizza (4,0), and the pizza excluding all four toppings, the cheese or "plain pizza" (5,0).

Steven: Sausage alone... three, zero [writes 3 0] and sausage and pepperoni... three, four [writes 3 4] and then we're done with this and the only thing left is pepperoni with nothing... four, zero [writes 4 0]. Then cheese with nothing... five zero [writes 5 0].

Steven's use of notation provided him with a concise and simplified way to view the four elements in the topping set which facilitated his development of a cyclic approach to exhaust all possible choices.

Stage 3, Example 4: Brandon and Colin, Grade 4

Brandon, a fourth grader, used the heuristic of constructing a chart and developed a notation for representing pizzas when he was asked to make pizzas selecting from four toppings. He assigned the digit "zero" to represent the absence of a topping and the digit "one" to represent its presence. This notation, like Meredith's chart from the preceding example and Steven's numerical code, aided Brandon in developing a method of proof.

Teacher: What are you doing here, Brandon?

Brandon: Making a graph just like Colin. I put peppers, sausage, mushrooms and pepperoni down and have them like 1, 0, 1, 0 and put... and make a graph.

A portion of Brandon's initial graph is pictured below.

P	S	M	Peponi [sic]
1	0	1	0
1	1	1	1
1	1	0	1
1	1	1	0
0	1	1	1
1	0	0	1

Teacher: What does that mean 1, 0, 1, 0?
Brandon: Well instead of using [writing] .like pepper down or sausage... you just gotta put like a "1" for yes it's going in, and a "0" for no it's not.
Teacher: So then this first pizza has what on it?
Brandon: Peppers and mushrooms... and then you could have all [toppings] or zero [toppings]... just a plain.
Teacher: What would you write for an all plain one [a pizza with no toppings]?
Brandon: 0, 0, 0, 0.
Teacher: Are you doing something like that too, Colin?
Colin: Yeah.
Brandon: Yeah, he's doing it similar.

Colin had also created a chart, but was using check marks to represent the inclusion of a topping and leaving blank the column(s) in which a topping was excluded.

After Brandon had listed fourteen pizzas on his graph, he decided to draw another graph in which his pizzas were reorganized. His second drawing showed a more systematic arrangement and was useful in providing a convincing justification (proof by cases) for having exhausted all possibilities. The second drawing began with the symbols (1, 1, 0, 0); which Brandon quickly changed to (0, 0, 0, 0).

P	S	M	Peponi [sic]
0	0	0	0
1	0	0	0
0	1	0	0
0	0	1	0
0	0	0	1

Brandon: Colin, you can just have one of each. Colin... you could do this... [He records the four possible pizzas with exactly one topping.]
Colin: I have 15 so far... I think I have... I have 15.
Brandon: Fifteen?
Colin: Yeah, see look there's the half, the pepper alone, then the pepper and sausage, then the pepper and mushroom, then the pepper and pepperoni, and I put sausage...
Brandon: Oh Colin... Colin... I just noticed... ya know how you're going like that [points to the columns from peppers towards the right] then you could start from the middle and work your way down like that [points to the columns from mushroom towards the right]. That's how I did something like this...

Colin: Oh yeah, the middle...
Brandon: Wait... I'm gonna try... 0, 0, 1, 1, ya got 0, 0, 1, 1? Look... blank, blank, 1, 1.
Colin: Uh Hmm. I have every one.
Brandon: That's it.
Colin: Fifteen.
Brandon: Okay... fifteen?

Brandon counted the number of combinations he had represented in his new chart. These included the pizzas with no toppings, then exactly one topping, and then exactly two toppings.

Brandon: Eleven. And I didn't even get it three [He refers to including the pizzas with three toppings.]... Colin, I didn't even do three's and I'm onto 11 so there must be more than 15. Okay, watch...

Towards the conclusion of the activity, Brandon and Colin compared and checked their sixteen solutions. Colin, in trying to make sense of Brandon's representation, attempted to translate Brandon's code.

Colin: Peppers, sausage and mushroom?
Brandon: Peppers, sausage and mushroom?
Colin: You have two of the same [duplicate pizzas].
Brandon: Oh no, no I checked the wrong one...
Colin: Yeah, that's pepperoni and that's pepperoni and mushroom... [Colin points to Brandon's paper.] and sausage and mushroom and pepperoni... wait, do you have a double? [Colin searches Brandon's chart.]

Notice that in building these arguments, notational systems were refined and used as tools to justify that all combinations of toppings had been considered.

Stage 3: **Pizza Problem with Halves**

Stage 3, Example: Brian, Jeff and Matt, Grade 5
Resolution of the "whole", "half" and "mixed" discussion

In a continuation of the class discussion described earlier (in Stage 2, Example 2 of the Pizza Problem with Halves), the teacher confronted the students with a situation that required re-examination of the "half" and "mixed" categories. The fifth grade students, Brian, Jeff and Matt, responded.

Teacher: If I come up with a pizza that has sausage here [on one half] and sausage and pepperoni here [on the other half], where would it go?
Class: "Mixed".
Brian: Because half of it is mixed.
Matt: But that's confusing.
Brian: But part of it is mixed.
Matt: But so what? It's still half a pizza... it's still cut in half.

The teacher's request for students to classify a pizza that fell into both the "mixed" and "half" categories was ignored by Jeff who instead offered an alternative classification in which mixed pizzas were placed within the categories of whole and half pizzas, thereby indicating his regulation of the possibilities.

Jeff: Why can't you just do what I did which is... You can make a "whole" category. Then you have the plain [pizza] and the sausage [pizza] and the pepperoni [pizza] ... Then after that you put the [category] whole which is mixed... but you keep it in the same category for "wholes". Then [you do] the same [groupings] for the halves... you would have a category for halves that are mixed.
Matt: That is my point.

Matt provided an explanation to Jeff's point by making a chart of the cases which he showed to the other students who then agreed that particular examples could immediately fit into this scheme.

Summary of Part 3

Examples of children's problem-solving behavior for the two Pizza Problems across all three project sites are presented and organized according to common characteristics. Stages that seem to be normative in the construction of completely defensible solutions to the problem(s) are suggested to provide a classification scheme. Children's mathematical behavior on the Pizza Problem suggests that students move through stages similar to those that parallel the movement in the Tower Problem. These are summarized in Table 2.

	Finding New Pizzas	Finding Duplicates	Proof of Completeness
Stage 1	Random creativity Building and listing Finding Groups	Look	"I can't think of any more!"
Stage 2	Further Development of local procedures; Some local organization and reorganization [a]	Local systems	Partial, by local organizations; not necessarily mutually exclusive "opposites", "families"
Stage 3	Recognition of the need for an over-all organizational scheme; development of such a scheme	Global systems	Proof by Cases; Using a cyclic approach

[a] (1) Grouping by the number of toppings; (2) grouping by adding on another topping: a pizza with topping A, then a pizza with toppings A and B, then a pizza with toppings A, B and C.

Table 2. The stages of development in building methods of proof to the Pizza Problems.

DISCUSSION

In studying children' s constructions of solutions to problems that call for the creation of convincing arguments for their correctness, we have observed certain developmental patterns.

In Stage 1 children are earnestly engaged in trying to make sense of the problem. Their physical constructions and written work indicate efforts toward understanding what the problem task requires. We see in Meredith's work that her mental model of pizza resulted in an alternative interpretation of the problem, which in turn, resulted in her solving a different problem.

Also, in this stage, children work to build a representation of the problem using a "guess and check" strategy. They tend to be successful in building adequate mappings of their mental representations when using physical objects or carefully made drawings to depict authentic situations. Arteshia (with Desiree) used the heuristic of building a physical model and carefully mapped her mental representation to that model. Michael made carefully drawn pictures to depict the various kinds of pizzas. Kersa and Ebonie generated a list of verbal phrases to match the pizza types.

The use of notation in Stage 1 tends to be clumsy and serves to inhibit rather than to advance student progress. Benny and Patrick began by inventing a notation to represent their ideas. However their verbal statements sometimes did not match their recorded statements (recall they used P for peppers and P for pepperoni). Hence they were unable to establish a one-to-one correspondence between the idea and the symbol to represent the idea. Failure to monitor their thinking resulted in obstructing their progress. When the boys switched to using a concrete model, they had greater success, but not immediately. They again were not consistent in mapping their ideas with the physical model. Recall the boy's decision to use a single brown plastic cube to depict a plain pizza without any topping and, simultaneously, to build the remaining pizzas only with toppings. For a second time, Benny and Patrick were unable to build an adequate model of their idea and did not check to see if the problem representation matched the problem. The interaction of Artesia and Desiree, girls who were working nearby, provided the motivation for the boys to re-examine their model.

Children verify their work during this stage by comparing, often with a partner, their representations and checking them against those of others. Justification of a solution frequently meant that the child could not imagine other possibilities, as in the case of Michael and Ebonie. Justification is sometimes interpreted as persuasion that a solution is reasonable, as in the dialogue between Benny and Patrick.

Children who exhibit Stage 2 behavior have coordinated mental model and physical representation of that model. Notation at this stage serves to clarify and advance an original mental representation. This facilitates the generation of alternative groupings organized according to different schemes.

Arteshia, for example, manipulated models of groups of pizzas to form various local organizations. Alana used diagrams to facilitate the generation of groupings. Amy-Lynn and Jeff revisited their local organizations and extended their thinking by making reference to their earlier use of the meaning of "opposite" from the Tower Problem. Brian and Matt discussed particular local organizations of the Pizza Problem with Halves. With Jeff's help, the children reconsidered the conflicting local groupings of whole, half, and mixed, triggering a movement into Stage 3 thinking as Matt reorganized his solution according to cases. During this process, there was usually considerable disequilibrium as children began to recognize the inadequacy of certain local organization schemes.

All of the children who achieved Stage 3, produced a single global scheme that led to a Proof by Cases. Arteshia and Desiree, fifth graders, manipulated actual physical objects to build their global scheme. Meredith, a third grader, made a chart which was carefully color-coded to represent different numbers of toppings. Her chart facilitated the reorganization that produced an arrangement by cases. Steven's coding system enabled him to generate a cyclic scheme to account for all of his cases. Brandon and Colin, who were partners, each invented and used their own notation. Recall that Brandon worked with a binary code and Colin made use of a chart. Both boys articulated understanding of each other's representation. In each case, the notation facilitated the reorganization of data into a Proof by Cases.

Our findings suggest that there is a developmental pattern in building a solution to both of the Pizza Problems. The sequence parallels the development of stages that had been observed in our earlier work with methods of proof.

PART 4: BRANDON: A CASE STUDY

This final part of our analysis involves the case study of a fourth grade boy, Brandon, whose problem-solving behavior for the Pizza Problem characterizes the sequential stage development demonstrated with segments of children's problem solving in Part 3. Also, it provides an example that illustrates a student's recognition of an isomorphism between the Pizza Problem and the former Tower Problem.

Brandon; Grade 4, The Pizza Problem (March 11, 1993)

As described earlier (in Stage 3 Example 4), Brandon looked for all possible pizza choices when selecting from four pizza toppings, he developed a code to represent the presence of specific toppings on a pizza. He constructed a chart with four columns, one column for each of the four toppings, and used the following code to represent toppings: "P" for pepper, "M" for mushroom, "S" for sausage and "pepperoni" for pepperoni. To represent a pizza, Brandon decided to place either a 1 or a 0 in each of the topping columns to indicate whether or not a topping was present on the pizza under consideration.

Thus, when he placed a "0" in the peppers column it meant that the topping peppers was not included on the pizza under consideration, whereas a "1" in the sausage column meant that the topping sausage was included on the pizza.

Int: What does the zero represent?

Brandon: You have nothing on that... that's nothing. One means you have that [topping] on your pizza. I don't know why I chose to use zeros and ones... I don't know how I got it... It just popped into my head.

Using this system of coding, Brandon developed a proof by cases for all possible pizzas selecting from four toppings. His five cases were pizzas with: (1) zero toppings (a plain pizza with cheese and tomato sauce), (2) one topping, (3) two toppings, (4) three toppings, and (5) four toppings. [7]

Interview with Brandon (April 5, 1993)

When interviewed about his method of proof three weeks after the problem activity, Brandon immediately reconstructed his proof by cases to explain his solution. He was then able to prove that he had accounted for all possible cases, and was further able to demonstrate how he had accounted for all possible pizzas within each case (see Figure 14).

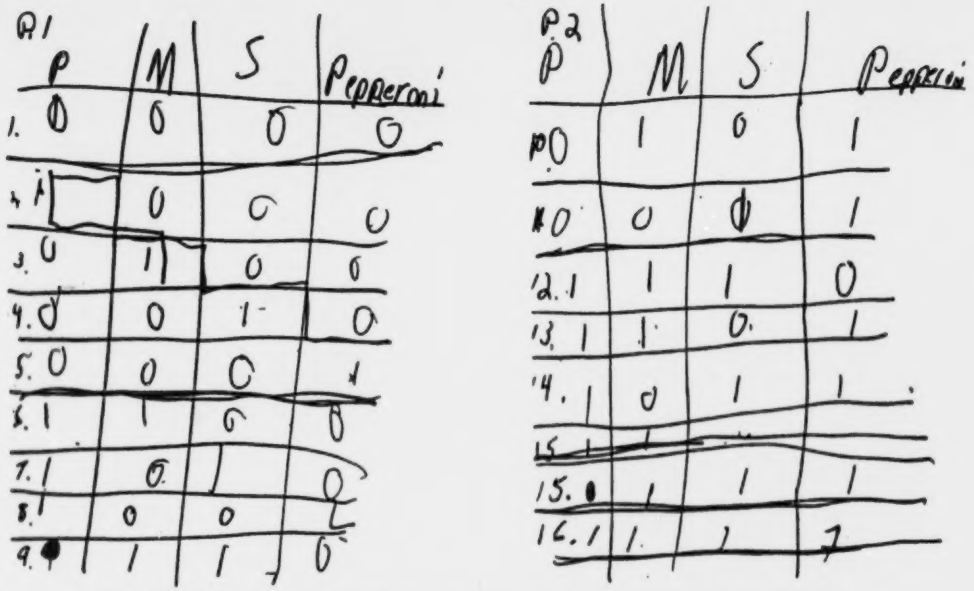

P1
P
M
S
Pepperoni
P2
P
M
S
Pepperoni

Figure 14. Brandon's proof by cases with pizzas represented in the form of a binary code.

Brandon: Nothing on a pizza... [places a "0" in each of the four topping columns of his chart to represent a pizza with no toppings] then you can have one pepper on a pizza with nothing else [places a "1" in the pepper column of his chart and zeros in the other three columns to represent a pizza with peppers only] one mushroom on a pizza with nothing else [places a "1" in the mushroom column and zeros in the other three columns] and you could have a couple of sausages on the pizza with nothing else [places a "1" in the sausage column and zeros in the other three columns] maybe a couple of pepperonis [places a "1" in the pepperoni column and zeros in the other three columns].

Brandon continued to organize his pizzas by cases and recorded each with a four digit sequence of zeros and ones. Here he considered the group of six pizzas which he called "Twos".

Brandon: And if you don't want any of that [pizzas with no toppings or one topping] you can start getting fancy, and start going to "two's", a pepperoni [sic, pepper] and mushroom and nothing else, then pepperoni [sic, pepper] and sausage and nothing else, then pepper and pepperoni and nothing else... you could have mushroom and sausage and nothing else.

Int: How did you know to go to mushroom now?

Brandon: Why didn't I use pepper anymore?

Int: Yeah.

Brandon: Cause I already used pepper there [with mushroom]... it's mushroom and pepper. And if I put a "1" down for mushroom and pepper that would be the same thing... So each time you go three, two, one...

Brandon explained that he could combine the first pizza topping (peppers, in this case) with the three remaining toppings, the second topping (mushrooms) with the two remaining toppings and the third topping (sausage) with the one remaining topping (pepperoni).

Brandon: ...actually 4... first you could use all the toppings then since you used already that... that's one less [topping to consider] so you could only use those three. You get three choices with the first one [the topping peppers], then with the second one... with mushrooms you only get two choices cause there's only sausage and pepperoni, then with sausage you can only do pepperoni.

The interviewer probed for affirmation of Brandon's strategy.

Int: What I don't understand is... when you move to [the] mushrooms [column] why can you put it with sausage and pepperoni, but you can't put it with peppers?

Brandon: Cause that would be a same thing... if I do that and I put a "1" there [in the peppers column] I've already got pepperoni [sic, peppers] sausage... that would be a same thing. It's just like saying you have an airplane and a car... saying you have a car and an airplane... it's still the same thing.

Int: I see.

Brandon: Now we go to "threes" [pizzas with three toppings] if you really like this stuff and you're really rich... One pepperoni [sic, peppers], a mushroom and sausage, nothing else...: you could have a pepperoni [sic, peppers], a mushroom, no sausage and pepperoni... this is where it gets real tricky... a pepperoni [sic,

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peppers] no mushroom, a sausage and a pepperoni... you could have no pepperoni, [sic, peppers] a mushroom, a sausage and a pepperoni and sixteen would be a pizza with everything!

Brandon had now provided an all-inclusive organization by cases for his data accounting for all possible pizzas. The interviewer asked him if this problem had reminded him of any others that he had worked on in the past.

Brandon: It kind of a little reminds me of the blocks [towers] cause my partner and I... did them in order like... yellow, red, yellow, red and then switch around and do the "opposite"... Red, yellow, red, yellow. It's kind of like what you do here [He referred to the case groupings for pizzas.] Just do it in groups... that's what we did with the blocks. How many ways can you make towers... how many ways can you make pizzas... the same problem, sort of.

Brandon took red and yellow cubes and quickly reproduced eight towers of height four, and organized them as four pairs of "opposites".[8]

Brandon was asked how the organization by "opposites" was an effective method to account for all possible towers. He did not respond, but appeared to be thinking. He was then challenged to consider if an organization such as the one used for pizzas would work. He responded that a chart would work and pursued constructing one organized by two columns, labeled red and yellow, to represent the colors of cubes. He then immediately realized that this organization was not going to be helpful, and rejected the idea to use a chart.

At this point in the interview, Brandon became preoccupied with building new arrangements of towers four cubes tall. His preoccupation was apparent by his lack of response to the interviewer's comments. He built one pair of "opposite" towers with three cubes of one color and one cube of the opposite color, and then built the remaining towers with three yellow cubes and one red cube. He did this by moving the one red cube up one position at a time starting with the tower with the red cube in the bottom position. He then built the "opposite" set of towers with one yellow cube and three red cubes by moving the yellow cube up one tower position at a time.

After organizing these eight towers into two staircases, each with four towers, Brandon verbalized his recognition of a connection between towers and pizzas. He said the following with enthusiasm:

Brandon: It's kind of like the pizza problem... like this [referring to the eight towers of height four with three cubes of one color and one cube of the opposite color] would be the one's group.

Int: Let's see...

Brandon: Oh, yeah... I see! this now... this is like the one's group [referred to eight towers: the four towers with exactly one yellow cube and the four with exactly one red cube] You only have one of the opposite color in these [towers with exactly one red or one yellow cube]... This isn't how I did it, but I just noticed this...

Int: This is very interesting.

Brandon: I just noticed it... then you would have... that would be the one's group, you only have one in there. Then... in pizzas [holds up the all red and all yellow towers] this would be like the "whole" group or "all" group... save that for last. Now you have the "two's" group... twos [He referred to the six towers of height four with two red cubes and two yellow cubes.] You have two of the opposite color in there.

In his new solution of towers, Brandon used a version of the grouping by cases to find all possible pizzas and organized his towers into three sets: (1) the two towers with cubes all the same color, (2) the eight towers with one cube of one color and three cubes of the "opposite" color, and (3) the six towers of height four with two cubes of each color. He eagerly talked with the interviewer as he worked.

Int: Okay, so you said these [six towers of height four with two red and two yellow cubes] all have two... two yellows or two reds?

Brandon: They had two of each color... they must to be in the "two's" group... if they had three of a color and one of the opposite color they'd be in the "one's" [group]... and you won't have any "three's" group [for towers]...

Int: Why not?

Brandon: Cause for the "three's" group you have three yellows and one red... that would be the same [pointed to the eight towers in his "one's" group]. A "three" group is like that group [pointed to the "one's" group] cause three of the opposite color... So a "three" group would be the same as a "one" group.

Int: Now if I wanted to call all this [the eight towers in the "one's" group] a three's group...

Brandon: Yeah, you could call it "one" or a "three" group.

Int: Why?

Brandon: Cause you could call it a "three" group cause it has three colors of one and one [cube of the] opposite [color], or you could call it a "one's" group because it has one of the opposite color. You can call that "three" or "one" it doesn't matter which.

We were interested in whether Brandon would recognize the structural isomorphism between finding all pizzas selecting from four toppings and finding all towers four cubes tall selecting from two colors. To direct Brandon's attention to this feature, the interviewer asked Brandon to consider one color within the towers as he reconsidered his groupings.

Brandon decided to place his focus on the yellow cubes within each tower, and reorganized his groupings of towers from three cases to five cases: (1) towers with no yellow cubes, (2) towers with one yellow cube, (3) towers with two yellow cubes, (4) towers with three yellow cubes and (5) towers with four yellow cubes.

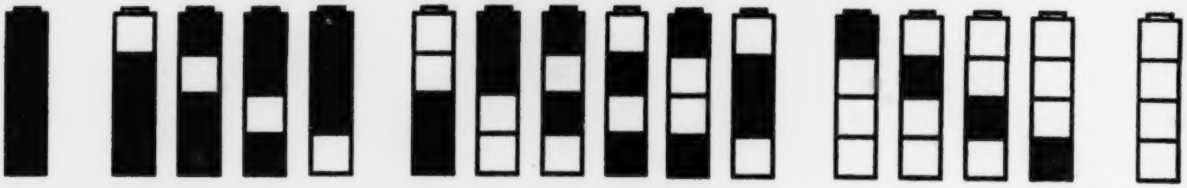

Figure 15. Brandon's newly constructed proof by cases for towers of height four when selecting from two colors.

At this point Brandon's insight moved him to exhilaration. He pointed out that the group of four towers with exactly one yellow cube was like the four pizzas with one topping in his chart. He carefully moved each tower and placed it on top of the corresponding pizza code on the chart thereby validating the relationship he had reorganized, he then explained how the red cubes in each tower corresponded to the "0"'s in his pizza chart and how the yellow cubes in each tower corresponded to the "1"'s on his chart. He then confidently began to match each of the sixteen towers to each of the sixteen pizzas represented on his chart as indicated.

Int: Now, what if we put our focus on say the yellow [cubes]... if we're looking at these eight towers here [Brandon's eight towers in the "one's" group]... and we're looking at yellow, and we're looking for the "one's", what would be a "one's" tower?

Brandon: A "one's" yellow tower?

Int: Yeah.

Brandon: [picks up a tower of height four with exactly one yellow cube in the second from the top position] That would be a "one's" yellow tower and a three red tower.

Int: What else would be a "one yellow" tower? [Brandon makes a staircase pattern with the four towers with exactly one yellow cube.]

Brandon: [As he arranges the staircase pattern with the four towers of height four with exactly one yellow cube] It's like the pizza problem... you work your way down. Like pepperoni, [He points to the tower of height four with exactly yellow cube in the top tower position.] mushroom, [points to the tower with exactly one yellow cube in the second from the top tower position] sausage, [points to the tower with exactly one yellow cube in the third from the top tower position] and pepper, [points to the tower with exactly one yellow cube in the bottom tower position]. [He picks up his pizza chart.] You start with zero...

Int: What would the "zero one" [pizza with no toppings] look like if we're looking at yellow [cubes]. [Brandon collapses all his towers pushing them to the side and clearing the table.] I don't understand.

We see how the interviewer's representation of an idea does not match that of the child. For Brandon, at this time a "zero tower" does not exist.

Brandon: There would be nothing.

Int: A "zero tower" if we're looking at yellow would be nothing? What was a "one yellow"? [Brandon realigns the four towers with exactly one yellow cube.] Tell me again how this is like the pizzas?

Brandon: You have the one pepperoni... since we're looking at yellows [cubes] the yellow cube would be "1" and the red would be "0". [He compares the red cubes in the towers to the digit "0" used in his pizza chart which represented the absence of a topping.] You could have one pepper... then it's like stairs if I draw a line down here like this... [He notes the similarity between the four towers of height four with exactly one yellow cube and entries two through five in his pizza chart and traces the diagonal line on the chart to show the similarity.] It's sort of like here you'd have one pepperoni, one mushroom, one sausage and one pepper [He points to the one yellow cube in each of the four towers.]

Int: Is what you're saying to me that a yellow cube is like a number "1" in you chart?

Brandon: Yeah... if you're focusing on red, then red would be number "1". Now you would work with the "two yellow group"...

Brandon then took each tower with exactly two yellow cubes and placed it above the corresponding series of "0"'s and "1"'s on the pizza chart, thereby establishing a one-to-one correspondence between towers of height four selecting from two colors of cubes and pizza choices when selecting from four toppings.

Brandon: If you lie them down and face them sideways... you're focusing on yellow [cubes]... [with the tower with yellow cubes in the bottom and second from the top tower positions] yellow stands for "1" so you'd have yellow "1", red "0", yellow "1" and red "0".

Brandon then compared the four possible pizzas with three toppings to the four towers with exactly three yellow cubes. Brandon took the tower with exactly four yellow cubes and called it the "pizza with everything" or four toppings, and concluded that "since the red (cubes) would stand for zero this (the tower with four red cubes) would be the "zero guy". It seems that Brandon had retrieved the interviewer's earlier question about the possible existence of a "zero yellow tower". Perhaps when he re-examined the question when it connected to something that made sense, Brandon was able to see that the tower with four red cubes corresponded to the series of four zeros making it the "zero yellow (cubes) tower".

Brandon: "I finally found out what the red would be... the red's the "zero guy"".

Brandon also articulated the symmetry of his argument and explained that if he had chosen to focus on the red cubes within each tower instead of the yellow cubes, the yellow cubes in each tower would correspond to the "0"'s on his pizza chart and the red cubes in each tower would correspond to the "1"'s. He then reorganized his proof by cases for towers to show towers with (1) no red cubes, (2) one red cube, (3) two red cubes, (4) three red cubes and (5) four red cubes.

Thus Brandon was able to recognize the isomorphism between the Tower Problem and the Pizza Problem, and was now able to apply a Proof by Cases method that was originally developed as he worked on the Pizza Problem to provide a convincing justification for the earlier Tower Problem.

CONCLUSION

Episodes from these case studies along with the detailed description of Brandon's problem-solving provide further insight into children's construction of methods of proof and supports earlier results. It also describes a process of recognition of certain patterns between isomorphic problem structures.

We observed in Brandon's follow-up interview (which occurred three weeks after his first encounter with the Pizza Problem on March 11, 1993) his re-construction of his original solution to the Pizza Problem using binary codes that he had assigned to represent the presence or absence of a topping. In responding to the interviewer about whether the problem reminded him of any he had previously encountered, he referred to the Tower Problem. In reconstructing a solution to the Tower Problem, he again used plastic cubes to build towers, and then organized them into pair of "opposites", a method he had used originally to find a solution (November 17, 1992). However, in the interview session, he re-organized the towers so that he eventually succeeded in mapping his new local organizations to the global organization that had produced a Proof by Cases for the Pizza Problem. In so doing, Brandon demonstrated his recognition of an isomorphism between the two problem tasks.

Brandon and the other children spent considerable time, in comparison to what one finds in most United States classrooms, working on these problems. They were encouraged to build up in their minds mental representations, and encouraged to pursue their own thinking. Careful observation, using videotapes, has helped us better understand the process of how children build their ideas, and has helped us reaffirm that children can indeed invent mathematical ideas.

REFERENCES

Davis, R.B. (1984). Learning mathematics: The cognitive science approach to mathematics education. New Jersey: Ablex Publishing.

Davis, R.B. & Maher C.A. (1990). What do we do when we learn mathematics? In R.B. Davis, C.A. Maher & N. Noddings (Eds.), Constructivist Views on the Teaching and Learning of Mathematics (pp. 53-69). Reston, Virginia: NCTM.

Davis, R. B. (1992). Understanding "Understanding". The Journal of Mathematical Behavior, 11, 225-242.

Davis, R.B., Maher, C.A. & Martino A. (1992). Using videotapes to study the construction of mathematical knowledge of individual children working in groups. Journal of Science Education and Technology, 1 (3), 177- 189.

Davis, R.B. (1993, March). Student discovery and teacher intervention. Paper presented at the NCTM Research Pre-session in Seattle, Washington.

Davis R. B. & Maher, C.A. (1993). Children's development of methods of proof. In Proceedings of the Seventeenth Annual Conference of the International Group for the Psychology of Mathematics Education. Tsukuba, Japan.

Maher, C. A. (1988). The teacher as designer, implementer, and evaluator of children's mathematical learning environments. The Journal of Mathematical Behavior, 6, 295-303.

Maher, C.A. & Martino, A. (1991a). The construction of mathematical knowledge by individual children working in groups. In F. Furinghetti (Ed.), Proceedings of the Fifteenth Annual Conference of the International Group for the Psychology of Mathematics Education (pp. 365-372). Assisi, Italy.

Maher, C.A., Davis, R.B. & Alston, A. (1991). Brian's representation and development of mathematical knowledge: A four year study. The Journal of Mathematical Behavior, 10, 163-210.

Maher, C.A., Davis, R.B. & Alston, A. (1992). Implementing a "Thinking Curriculum" in mathematics. The Journal of Mathematical Behavior, 10, 219-224.

Maher, C.A. (1992). Is dealing with mathematics as a thoughtful subject compatible with maintaining satisfactory test scores?: A nine-year study. The Journal of Mathematical Behavior, 10, 225-248.

Maher, C. A. & Martino, A. M. (1992b). Individual thinking and the integration of the ideas of others in problem solving situations. In W. Geeslin, J. Ferrini-Mundy & K. Graham (Eds.), Proceedings of the Sixteenth Annual Conference of the International Group for the Psychology of Mathematics Education (pp. 72-79). Durham , New Hampshire: University of New Hampshire.

Maher, C.A. & Martino, A. (1992a). Teachers building on children's thinking. The Arithmetic Teacher, 39(7), 32-37.

Maher C.A. & Martino, A. M. (in press). Conditions which contribute to conceptual change in ideas related to proof. In Proceedings of the Seventh International Congress for Mathematics Education. Laval: Quebec.

Maher, C.A. & Martino, A.M. (in press). Four case studies of the stability and durability of children's methods of proof.

Maher, C.A., Martino, A.M. & Davis, R.B. (in press). Young children's development of mathematical proofs.

Martino, A. & Maher, C.A. (1991b). An analysis of Stephanie's justifications of problem representations in small group interactions. In R. Underhill (Ed.), Proceedings of the Thirteenth Annual Meeting for the North American Chapter for the Psychology of Mathematics Education (pp. 70-76). Blacksburg, VA: Virginia Polytechnic University.

Martino, A. (1992). Elementary Students' Construction of Mathematical Knowledge: Analysis By Profile. Unpublished doctoral dissertation. Rutgers University.

Endnotes

1. This work was supported in part by grant number MDR 9053597 from the National Science Foundation. The opinions expressed are not necessarily those of the sponsoring agency, and no endorsement should be inferred.

2. For more detail see Davis and Maher, 1993; Maher, Martino and Davis, in press.

3. We are grateful to the following graduate students who assisted in the analysis of data: Carol Bellisio, Tracy Cozza, Joan Humenick, and Alice Miller. We would also like to thank Lynda Deming for her preparation of the artwork and figures for this paper and Robert B. Davis for his helpful comments in reading the paper.

4. The boys are unaware that their problem representation does not match the problem presented. Their talking aloud indicates that they do have the idea in their head. The difficulty, it seems, is failing to monitor their work. Davis (1984) makes reference to errors of this sort by students who are not in the habit of checking their work.

5. It is clear that Patrick has the idea in his head. As in the earlier episode in recording both peppers and pepperoni with "P", he is unaware that the problem representation does not match the problem.

6. As we see in the case of Meredith, her "real world" experience about pizzas influenced her problem representation. Reality outside of school will often fail to match some children's experience. If we interpret situated cognition in the context of providing children with something to think about (using plastic, cubes, rods, ets.) , it can indeed be abstract.

7. Initially when Brandon made pizzas he had organized them by starting with the pizza with no toppings then making a pizza with topping A, followed by a pizza with toppings A and B, followed by a pizza with toppings A, B, and C, etc... He reorganized his pizzas when he was challenged to develop an organization which would show that every possible pizza had been considered, for Brandon this was a proof by cases.

8 In November 1992 when Brandon and his partner were first challenged to solve the Tower Problem for towers four cubes tall, they used the strategy of building them as "upside down" pairs and "opposite" pairs. The boys recognized the confusion with using two different organizational systems, and instead organized their towers as eight pairs of "opposites", characteristic of Stage 1 reasoning. On several occasions during this session, Brandon noted the color distribution of cubes within towers (for example, a tower with three cubes of one color (A) and one cube of the opposite color (A') within a tower), but did not implement this as an organizing strategy for grouping his towers.

PAPER PRESENTATIONS

LEARNING ABOUT QUADRATICS IN CONTEXT

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This paper discusses one of three case studies to investigate students' conceptualizations of quadratics by solving quadratic contextual problems. The theoretical framework which guided the study was constructivism. The methodology was teaching interviews between a researcher and student solving problems. A multi-representational software, FUNCTION PROBE, was available as a tool to aid students in their problem solving. The results showed that using realistic situations as problem contexts invited multiplicity of interpretations and methods for defining quadratic functional relationships. By reasoning empirically from the problem context, students conceptualized quadratics relationships iteratively and in terms of summation in contrast to the most common view of quadratics as a product of two linear variations. Further, requiring students to verify and justify their strategies by cross-referencing between multiple representations of functional relationships and problem context led them to construct viable schemes to characterize quadratics in terms of rate of change, dimensionality, and symmetry.

Current trends in mathematics education today (NCTM, 1989) recommend that students explore patterns and functional relationships using multiple representational systems (tables, graphs, equations, and diagrams), and examine interconnections between representational systems. The constructivist perspective encourages mathematics educators to focus more on what students say and actively do when solving mathematics problems. (Confrey, (1990a). Researchers have found that a number of students experience great difficulties in understanding the essentials of the function concept (Dreyfus & Eisenburg, 1984; Malik, 1980) and in applying the formal set-theoretic definition to real life examples. (Monk, 1989; Lave, Smith & Butler, 1988). Because of the current practice of introducing the formal definition first before doing real life applications, many students only view functions as a rule of correspondence between domain and range. In contrast, functions as covariance between varying quantities which is more reflective of its historical development and more applicable in realistic situations is often neglected and de-emphasized except in the applied sciences.

THE STUDY

This study was conducted to investigate the use of contextual problems to encourage a developmental understanding of functional relationships in general and quadratics in particular. (Contextual problems refer to problems with context selected from realistic situations that students would encounter in their everyday experiences.) Three high school students and one university student were selected for the study using a pre-test which tested their traditional knowledge of linear and quadratic functions.

THEORETICAL FRAMEWORK

The constructivist perspective based on Piaget's theory of cognitive development was the guiding theoretical framework for this study. In particular, its use of the assimilation and accommodation processes to explain how an individual acquires knowledge. For example, an individual interacts dialectically with the situation or object through reflective abstraction until the individual has come to know the situation or object well and is able to internalize these actions as mental operations. (Kaufman, 1978; Steffe & Cobb, **).

The epistemology of constructivism is based on the philosophy that knowledge is NOT a mirror image of an ontological reality. Instead, knowledge is rooted in human experiences, and is constructed to make sense of human experiences. The starting point is the knower who builds his or her conceptual knowledge solely on the basis of his or her experiences and actions. Human constructions are in terms of schemes or systems of schemes which may be continually re-organized and modified or transformed through social interactions, negotiations of meanings, and communications with others. Hence, within this perspective, mathematics could be viewed as having more than its definitions, theorems, proofs, and its logical relationship, it would also include its form of representations, and its evolution of problems, methods of proof and standards of evidence. (Confrey, 1989).

METHODOLOGY

The methodology was teaching interviews between a researcher and a student solving a contextual problem. Students solved four contextual problems before doing a post-test and a final interview. The type and nature of questions during the teaching interview were guided by students' responses. Each student was given ample time to solve each problem. Students had available to them a multi-representational software: Function Probe (with a table, calculator and graph multi-linked windows) (Confrey et. al, 1989) on the Macintosh for their use a tool in problem solving.

Teaching interviews involved three two-hours sessions per week for six weeks. Each session was audio- and video-taped. The interviewer consistently noted down: (i) students' significant moments particularly when students formulated viable schemes; (ii) cycles of problematics and subsequent (tentative or otherwise) resolutions; (iii) students' rationales for their answers; and (iv) students' verbal and written representations of their ideas.

RESULTS

This paper discusses one of three case studies conducted to investigate students' conceptualizations of quadratic functional relationship by solving contextual problems. The first paper, Afamasaga-Fuatai (1992a), discusses Mary's iterative conceptualizations of quadratics and how she used rates of change (Δy and $\Delta(\Delta y)$) to highlight inadequacies of one algebraic representation to represent her preferred interpretation. In pursuing her preferred interpretation, she examined linear Δy and constant $\Delta(\Delta y)$ values which she later affirmed to be characteristic of quadratic functional relationships. Mary also constructed and further developed a scheme, by reasoning emprically from the context and subsequent numerical patterns, to predict coefficients A and B of a quadratic function: $f(x) = Ax^2 + Bx + C$ in terms of maximum point. (This is similar to the formal formula for calculating the axis of symmetry in terms of A and B.) The third paper (Afamasaga-Fuata'i, 1993) will discuss the case of Bob a senior high school student who conceptualized quadratics as covariations that maximize (or minimize) when the two sets of values: x and y are in a ratio equal to the ratio of their rates of change: Δx and Δy.

In contrast, this second paper will discuss Nan's strategies particularly in how she developed her iterative conceptualization of quadratic relationships to a more elegant and efficient representation as a product of two linear variations. While doing this, she developed a viable scheme for predicting the original quadratic function in terms of rates of change (Δy and $\Delta(\Delta y)$), and like Mary, also characterized quadratics as variations with linear Δy and constant $\Delta(\Delta y)$ ($\neq 0$).

Given the avocado problem as shown below:

> AVOCADO PROBLEM
>
> Farmer Joe has records showing that if 25 avocado trees are planted, then each tree yields 500 avocadoes (on the average). For each additional tree planted, the yield decreases by 10 avocadoes per tree. Determine the number of trees that would maximize total yield.

Nan initially interpreted the loss of fruits conceptually to be the same for all trees. However, in attempting to represent this algebraically, she used an iterative conceptualization that closely reflects her procedural actions of taking off 10 fruits from the newest tree and then 10 more off old trees as generalized by the following:

$s_n = s_{n-1} + (500-10) - 10(n-1)$; for n total trees..(1)

where s_n and s_{n-1} represented current and previous total yields. She explained that this is equivalent to taking 10 off every tree each time a new tree was added. For example:

> "it doesn't really matter if you take ... 30 off 1 tree ... and 20 of another or something ... on average it's all the same once you take care of that ... so that equation still works."

However, after a few more calculations, Nan realized that her iterative strategy was a long and tedious process. Thus, after expressing a preference for a more sophisticated method for predicting maximum total yield, Nan conjectured that perhaps she could utilize the numerical pattern of the differences (Δy) between her iteratively calculated s total yields to predict the original total yields. Nan, however, found that her difference equations were not much help either at this point. (It was not until much later on that these difference equations became significant in predicting the original quadratic function.)

By pursuing a different approach, Nan represented her second conceptualization of total yields as:

$TY = 12500 + 500(n-25) - 10n(n-25)$..(2)

in which loss is the same for all trees and is subtracted all at once instead of doing it differentially and iteratively as in her first interpretation represented by s_n. Nan appeared happier with this representation as it allowed her to predict yields for any value of n. Nonetheless, instead of using it to determine maximum yield immediately, Nan chose to reconcile her two interpretations: s_n and TY algebraically as she firmly believed that iteratively subtracting loss of fruits per additional tree had to be the same as taking them all off at the same time. Nan then spent some time manipulating numerical values and equation s_n to effect a reconciliation of the two sets of total yields until she eventually modified s_n to:

$s_n = 12500 + 500(n-25) - 10(n + (n-1) + (n-2) + ... + 26)$........................(3)

Nan also pointed out that the discrepancy between this new form of s_n and the formula TY is with the last term $10n(n-25)$ and $10(n+(n+(n-1)+...+26)$. Nan's problematic then was to represent the summation: $10(n+(n+(n-1)+...+26)$ as a formula. Nan tried but was unable to develop a summation formula. In addition, she found that total yields generated by the modified s_n and formula TY were not the same, instead, the discrepancies increased with increase in trees.

When asked how she would decide the better representation given the problem context, Nan generated average yield values by dividing total yields by trees. Subsequent values showed that average yields from the TY formula (b in Table 1) were more consistent with the loss of 10 in the context than those generated by the iterative s_n. (a in Table 1). Finally, she chose formula TY to be the better representation of the context as shown:

$s = 12500+500(n-25)-10n(n-25)$ ....(4)

Average Yields

When asked for an alternative method of generating average yields, Nan, reasoning empirically from initial conditions, predicted that: average yield = $(500-10(n-25))$. This new average yield formula was then extended to give an alternative formula for calculating total yields as:

Total Yield: $s = n(500-10(n-25))$...(5)

Nan immediately commented that this total yield equation was *"more elegant and efficient, and much easier to work with and involved less number of operations"* than her earlier formula TY in (2).

Table 1 - Comparison of Average Yields Using s &TY Strategies.

n	s	a=s/n	ty*	b=ty/n
tree number	total yield	average yield per	adjusted values for	average yield per
25.00	12500.00	500.00	12500.00	500.00
26.00	12740.00	490.00	12740.00	490.00
27.00	12970.00	480.37	12960.00	480.00
28.00	13190.00	471.07	13160.00	470.00
29.00	13400.00	462.07	13340.00	460.00
30.00	13600.00	453.33	13500.00	450.00
31.00	13790.00	444.84	13640.00	440.00
32.00	13970.00	436.56	13760.00	430.00

*ty had formula $ty = 12500+500(n-25)-10n(n-25)$

First Differences (Δy)

With her more elegant forms of total yield formulas and difference equations for various context variations, Nan was asked to revisit an earlier problematic of developing a scheme to predict total yields using differences. By comparing the new simplified total yield equations in Table 2 below, Nan found that the coefficient of (n-25) term in equation D was consistently twice the coefficient of the n^2 term in the total yield equation as shown:

$\Delta(\Delta y) = 2(\text{coefficient of } n^2)$ ....(6)

Table 2 - Nan's Table With Total Yields in Short Form

Function	Total Yield	Difference Equations	$\Delta(\Delta y)$
1	$TY = -10n^2 + 750n$	$D = 240-20(n-25)$.	-20
2	$TY = -6n^2 + 650n$	$D = 344-12(n-25)$	-12
3	$TY = -20n^2 + 1000n$	$D = -20-40(n-25)$	-40

However, while struggling to identify a similar relationship between the constant term in D and n term of TY, Nan noticed that Δy (or D differences) could be generated by using a formula such as:

Difference: $D = \Delta y_1 + \Delta(\Delta y)(n-25)$ ....(7)

where Δy_1 is the first Δy value in the sequence corresponding to the y values of x=0 and x=1, $\Delta(\Delta y)$ is the second difference (difference of Δy), and n represented total trees.

Simplifying her difference equations to: D_{10}[V] $= 740-20n$ compared to $TY_{10} = -10n^2 +750$ and $D_{20} = 980-40n$ compared to $TY_{20} = -20n^2+1000n$, Nan re-affirmed her relationship in (6), and further pointed out that the

---

[V] The subscript for D and TY, at this point, is to distinguish between the different loss rates. For example, D_{10} and TY_{10} refer to equations for loss of 10. This notation will be used for the loss of 20 and 6 also.

coefficient of the n^2 term would be half of $\Delta(\Delta y)$. After further reflections, Nan noticed that the sum of coefficients of the n^2 and n terms in TY equalled the constant term in equation D. This relationship was further verified with function 2 and its corresponding equations of: $TY_6 = -6n^2+650n$ and $D_6 = 644-12n$.

For discussion purposes, coefficients of n^2 and n terms in total yield equations were labelled A and B respectively to give the general form: $TY = An^2 + B$; and constant term in equation D was E. For example:

$$D = \Delta y_1 + \Delta(\Delta y)(n-25) = [\Delta y_1 - 25\Delta(\Delta y)] + \Delta(\Delta y)n = E + \Delta(\Delta y)n \text{ ....} (8)$$

Comparing the two general forms: $TY = An^2 + Bn$ and $D = E + \Delta(\Delta y)n$, and numerical patterns, Nan predicted that: $E = A + B$. where coefficient B would be:

$$B = E - A = E - \Delta(\Delta y)/2 \text{ ....} (9)$$

With persistent probing, Nan extended her pattern recognition strategies until she eventually generated two powerful ways of predicting the original total yield equation using difference equations as shown by her schemes in (6) to (9).

DISCUSSION OF RESULTS

Evidently, Nan's initial conceptualizations, developed empirically from the context, had evolved from an iterative sum into a more concise and elegant form as represented by the product of average yields and trees primarily as a result of her own evolving problem solving strategies. Her subsequent interpretations of the problem context and her subsequent representations of her procedural actions led to alternative but viable re-conceptualizations of quadratics as: (i) iterative sums, and (ii) sum of functions in contrast to the common view of quadratics as a product of two linear variations.

1. Quadratics As Iterative Sums. Nan 's iterative conceptualization of total yield was as:

Total Yield = Previous Yield + Yield from newest tree - Loss from old trees

$$s_n = s_{n-1} + (500-10) - 10(n-1) \text{ ....} (10)$$

or

$$s_n = s_{n-1} + 500 - 10n \text{ ....} (11)$$

In contrast, given the context and intended interpretation of losses from all trees at the same rate of 10 per additional tree per tree and her more elegant total yield formula, (denoted by c for discussion), total yield expressed as a product is: total yield: c = n * (500-10(n-25)).

Nan: $s_n = s_{n-1} + 500 - 10n = s_{n-1} + (500 - 10) - 10(n-1)$

26	12500 + 500 - 26(10) = (500 -10) - 25(10)	12740
27	12740 + 500 - 27(10) = (500 -10) - 26(10)	12970
28	12970 + 500 - 28(10) = (500 -10) - 27(10)	13190
29	13190 + 500 - 29(10) = (500 -10) - 28(10)	13400

Product: $c_n = c_{n-1} + [500 - 10(n-25)] - 10(n-1)$ $c = n(500-10(n-25)$

26	12500 + {500 - 10(1)} - 25(10) = 26(490)	12740
27	12740 + {500 - 10(2)} - 26(10) = 27(480)	12960
28	12960 + {500 - 10(3)} - 27(10) = 28(470)	13160
29	13160 + {500 - 10(4)} - 28(10) = 29(460)	13340

Figure 3 - A Comparison of Iterative Methods s and c.

To illustrate the legitimacy and viability of the iterative conceptualization of quadratics, Nan's strategy in (10) for the first 4 additional trees is compared to values generated by the re-conceptualized iterative form of equation c in Figure 3 above. Clearly, Nan's iterative s strategy represented a potentially viable alternative conceptualization. With a slight modification however in the interpretation of loss from newest tree, Nan's strategy could parallel that of the iterative c. For example, Nan needs to reflect the variable loss due to the addition of the new tree somewhere in her representation of average yield from newest tree. This could be done either in the loss as in (500-10(n-25)) of the iterative c version, or with the "expected" average yield from tree to tree as in (500 - 10((n-25) - 1) - 10). Since Nan insists on taking "10 off from the newest tree each time a tree was added," she would need to incorporate the variable loss effect on the "expected" average yield while keeping her loss for newest tree at a constant 10 as she preferred.

2. Quadratics As a Sum of Functions. Nan's TY formula suggested a second viable, alternative view of quadratics as a sum of a constant, linear function and a simple quadratic function[1] in contrast to the common product view. For example,

Total Yield = CONSTANT + LINEAR FUNCTION - QUADRATIC FUNCTION ....(13)

Nan's lengthy struggle to reconcile her two strategies (s and TY) was partly because of the arithmetic errors she was carrying throughout her subsequent calculations, and problematic of finding a summation formula.

3. Difference Equations. Nan's most significant construction was her difference-equation scheme which predicted the original quadratic function using rates of change [Δy and $\Delta(\Delta y)$] values. Nan was the only student of the four who represented differences (Δy) algebraically. For example, by expressing the linear Δy values of quadratic variations as a linear function, $D = Mx + E$, she could predict coefficients A and B of the original quadratic function: $y = Ax^2 + Bx + C$. Specifically, $A = M/2$ where $M = \Delta(\Delta y)$ and $B = E - A$. Coefficient C could be determined by finding the difference between predicted values (using only A and B of quadratic function: $y = Ax^2 + Bx$) and actual values as predicted from the context; or, by finding the y-intercept.

USE OF CONTEXT

Data from Nan's case analysis support the use of realistic situations as critical sites for students' mathematizing activities. Nan, given the realistic context and not being aware that it could be modelled by a quadratic function, did not immediately conceive of total yields as the product of average yields and trees. Instead she adopted an iterative strategy of generating total yields. In the course of her struggles with her various problematics and arithmetic errors, she investigated the concepts of rates of change, summations, and multiple representations to a much greater extent than the case would have been in a traditional presentation of quadratic functions. Evidently, solving problems with realistic contexts within a constructivistic perspective can provide students with the opportunity to conjecture and develop their own intuitive conceptualizations into more formalized schemes that are equally viable and even parallel those of formal mathematics. Unquestionably, well chosen realistic contexts is a powerful site for illustrating the inter-relationships of mathematics concepts that is missing from traditional, textbook context-free problems.

THEMES OF QUADRATIC FUNCTIONS

The interviewer initially organized her own conceptualizations of quadratics by using three themes as suggested by Confrey & Smith (1991): (1) rates of change, (2) symmetry, and (3) dimensionality. Briefly, (see Afamasaga-Fuata'i (1992) for a full discussion of this), quadratics functional relationships have linear Δy and constant ($\neq 0$) $\Delta(\Delta y)$ values, line symmetry, and dimensionality factor 2. Nan used all three themes in her characterizations of quadratic functions in contrast to linear ones, and in her various schemes for predicting maximum points, intercepts and original quadratic functions.

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[1] The simple quadratic function is in the form of a product of two linear functions: $y = 10m(m-25)$.

CONCLUSIONS

Nan's iterative conceptualization of quadratics indicated that students' so called errors in problem solving could easily be alternative, rather than erroneous ways of viewing a problem. Knowing and understanding the genesis of students' interpretations, and strategies at a deeper level would provide useful guidelines for the design and development of better contextual problems that: (1) facilitate diversity of interpretations, (2) are challenging enough to motivate students to want to solve them, (3) invite multiple solutions, and (4) embed mathematical concepts that are often difficult to understand by students. The formalization of mathematics concepts, definitions and formulas could develop after students have had a chance to construct their own schemes and conceptualizations of the mathematics embedded in a context .

The interviewer-student interaction eventually led to students' construction of more consistent resolutions and viable schemes. Although this kind of interaction is not viable in a normal class setting with a 1:30 teacher-student ratio, it is, nonetheless, important for identifying schemes students find useful in understanding mathematics concepts; and for considering how to make this kind of interaction occur more in educational settings.

REFERENCES

Afamasaga-Fuata'i, K. (1992a). Reconceptualizing Quadratics Through Contextual Problems. A Paper Presented at the South Pacific Conference on Mathematics and Mathematics Education, Port Moresby, Papua New Guinea.

Afamasaga-Fuata'i, K. (1993). Rates of Change and Maximum Points: The Case of Quadratics in Context. (in press)

Confrey, J., (1989) Learning to Listen: A Student's Understanding of Powers of Ten. In Glaserfeld, E.. Constructivism in Mathematics Education.

Confrey, J., (1990a, April) The Use of Contextual Problems and Multi-Representational Software To Teach Pre-Calculus Mathematics. A Paper Presented At the Annual Meeting of the American Educational Researcher Association, Boston, MA.

Confrey, J., Davis, P., Caroll, F., Smith, E., & Cato, S. (1989) FUNCTION PROBE © Software for the Apple Macintosh Computer.

Dreyfus, T., & Eisenburg, T. (1984) Intuitions of Functions. Journal of Experimental Education, 52, (2), p.77-85.

Kaufman, B.A., (1978) Piaget, Marx, and the Political Ideology of Schooling, in Curriculum Studies, 10(1), p.19-44.

Lave, J., Smith, S., & Butler, M. (1988) Problem Solving as Everyday Practice. In Charles, R.I., & Silver, E. (eds) The Teaching and Assessing of Mathematical Problem Solving. National Council of Teachers of Mathematics. Lawrence Erlbaum Associates, p. 61-81.

Malik, M.A. (1980) Historical and Pedagogical Aspects of the Definition of Function. International Journal of Mathematics Education in Science and Technology, 11, p.489-492.

Monk, G., (1989) A Framework For Describing Students' Understanding of Functions. A Paper Presented At The Annual Meeting of the American Educational Researcher Association, San Francisco, CA.

NCTM (1989) Curriculum and Evaluation Standards for School Mathematics. Reston, VA, National Council of Teachers of Mathematics, Inc.

Steffe, L.P., & Cobb, P., (**) Multiplicative and Divisional Schemes. -- . University of Georgia & Purdue University.

LINEAR AND QUADRATIC GRAPHS WITH THE AID OF TECHNOLOGY

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This paper describes the trial of a unit of work on linear and quadratic graphing with six year 10 classes. Two treatments were developed. The computer treatment made use of the ANUGraph software package, while the calculator treatment paralleled the computer treatment but used a combination of previously prepared graphs and graphs constructed by the student with the aid of a calculator. The emphasis in both treatments was on the interpretation of graphs related to real situations. Comparisons between pre-test and post-test results and interviews with twelve students showed that students learnt to handle the software proficiently, and that both groups improved on most of the topics taught. However, the calculator group seemed to be advantaged by practising plotting of points by hand. Implications for future work are discussed.

Most secondary schools are now equipped with computers that can run graphing and spreadsheet software, although they are still not frequently used for mathematics teaching. In Asp, Dowsey and Stacey (1992) we described a trial of a teaching unit using spreadsheets to assist students in constructing meaning for the important concepts of variable, expression, equation and solution. This teaching experiment gave an indication of which ideas about equations become easier with appropriate computer technology and what difficulties, both conceptually and in terms of classroom management, result from the use of this technology. In this paper we describe results from a trial of a teaching unit on linear and quadratic functions and their graphs, a topic which seems similarly suited to the use of computers.

Frequently all aspects of a complex mathematical idea cannot be expressed with a single representational system. The idea may require multiple, linked representations for its full expression and these different representations may aid the learner's understanding of the idea. Kaput (1992) sees the ability to make translations from one representation of a function to another as a particularly important aspect of mathematical thinking which may be enhanced by technology. The convenient access provided by graphing software to numerical and graphical representations of a variety of functions may assist students to develop a broader and deeper understanding of the function concept. Graphing software enables rapid and automatic translation between algebraic, graphical and numerical (tabular) representations, whereas translation by hand is generally a slow and laborious process for students.

Teaching linear and quadratic graphs using the new technology may bring students to insights that enable them to construct richer and more coherent graphing concepts and develop important techniques needed to reason with mathematical ideas. On the other hand, the use of such technology may encourage students to use ad hoc, intuitive procedures with limited application in the future study of mathematics. In addition, it is possible that the use of computer software for drawing graphs may result in students having insufficient practice on skills such as plotting which may be essential for the development of a good understanding of functions and their graphs. For calculator use in primary schools, these questions seem to have been resolved (see, for example, Hembree and Dessart, 1986) but they now need to be resolved as computing and calculator technology begin to find a place in the algebra program.

To explore aspects of the use of technology in learning secondary mathematics a study based on a unit of work on functions and graphing was conducted with all the Year 10 students and their teachers from a northern Melbourne secondary college. In keeping with what we feel will be the aspects of graphing that remain important when all users of mathematics have free access to graphing software, the unit emphasised using graphs to describe real situations and interpretation of graphs. The study was designed to investigate students' understanding of linear and quadratic functions and their graphs and in particular, sought information on the following questions:

1. What effect does doing less graphing of functions by hand from calculated values have on students' understanding of functions?
2. Given access to graphing software, can students learn to swap between algebraic and graphical representations to solve problems?

In addition, we were interested in questions related to the classroom use of computers such as:

3. What is the 'lead-time' for students to learn how to use graphing software for mathematical purposes?
4. Can students remember how to use software between units of work or must they relearn?
5. What is the most appropriate form of teaching material to guide students' work at the computer?

The 1992 experiment was the first trial for the unit on graphing. In this short report, we will summarise the results which are now informing our further development of the material. In particular, we discuss the results of the pre-test, post-test and interviews which have highlighted skills and concepts which are difficult for students and will need greater attention in future versions of the unit.

PROCEDURE

There were six classes of Year 10 at the school, taught by four teachers. Two parallel treatments for the unit on linear and quadratic functions and their graphs were developed. Each unit was designed to take about ten lessons and replace a textbook unit. The treatments were randomly assigned, subject to the requirements that there were three classes per treatment and the two teachers with two Year 10 classes should have one class assigned to each treatment. Each student was provided with an individual work booklet (two versions) and notes were given to their teachers on appropriate class discussion points. At meetings with the researchers, teachers advised on the directions taken in the booklets and were given an introduction to the computers and the software.

One treatment (the "computer classes") made constant use of ANUGraph in the school's new laboratory of Macintosh computers to create the graphs required. A description of this treatment is given in Asp et al (1992b). For the other treatment, the needed graphs were either plotted by hand using a calculator or were given. As one example, the computer classes found solutions to simultaneous equations by using ANUGraph to draw the graphs of the two functions. The co-ordinates of the intersection points are given automatically when the mouse is appropriately positioned: the scale on the axes is not used. In contrast, the calculator classes were supplied with axes on which the graph of one function was provided and students plotted the other (usually easier) function by hand to find the solutions. This required reading the co-ordinates using the scale. Copies of the booklets used in both treatments are available from the authors.

Written pre- and post-tests for determining students' understanding were administered to all classes. A total of 136 students sat for the pre-test and 132 for the post-test. In addition, at the end of the unit, interviews were conducted with 12 students who were randomly chosen subject to the constraint that there were 2 from each class, one scoring near the 80th and one near the 40th percentile on the pre-test. The six computer treatment students used the computers during their interviews. The purposes of the interviews included examining student thinking, knowledge and facility in using the computers.

RESULTS AND DISCUSSION

A brief discussion of classroom use of the computers and software follows, and then the results are presented under the headings of reading and plotting points and then according to type of translation between representation (graphical, algebraic, numerical, situation). Because of shortage of space, not all questions or all parts of questions on the pre-and post-tests are discussed and only the main findings are illustrated.

Classroom use of computers and software.

Both teachers and students were able to learn how to use the computers confidently in a few lessons. During these early lessons, it was valuable to have the presence of a second adult (the research assistant observing the lessons). When working in the computer laboratory the teachers tended to adopt a management role and seemed to miss important teaching opportunities. This was also observed in our teaching experiment on equation solving with computers (Asp et al, 1992a). This continued despite the provision of notes for teachers on important points for class discussion and urging by the experimenters to spend some time in the computer room not on the machines. ANUGraph performed very satisfactorily. Initially we found that not being able to name the axes was a serious source of confusion. For example, the first lessons were based around a profit-loss context. Students had to work with graphs (profit in dollars vs litres sold) which had axes that were not named. The teachers and research assistant eventually came to believe that having to sort out the confusion made students consider very carefully what quantities were being represented on each axis. Future versions of the unit will suggest that students draw a set of axes on paper and name them before going to the computer to start work on the problem.

Reading and plotting points.

The first question on the pre-and post-tests required students to plot points on a cartesian plane, which we regard as a essential pre-requisite skill to using graphs. The scales on the x-axis and y-axis were deliberately chosen to be different, as frequently happens with graphing software. On the x-axis, labels (numbers) were written at intervals of 0.5 and a grid line was shown at twice this frequency. On the y-axis, labels were written at intervals of 2.0 and a grid line was shown at twice this frequency. The points which students were asked to plot could therefore be on or off a grid point and with labelled or unlabelled x and y co-ordinates. On both the pre-test and post-test, there was a facility of over 85% for plotting labelled grid points in the positive quadrant. For other points, the most common errors were interchanging x and y values, problems with plotting negative values and problems related to scale. Scale problems were linked both to points not being labelled and not being on the grid. For example, the point (2.5,1.5) did not correspond to a grid point; it was frequently plotted as (2.5,1) which was a grid point. Some students were able to plot points only when the corresponding x- and y-axis values were clearly labelled and could not interpolate between them. Similarly, at the interviews, only 7 out of 12 students were able to read off from a graph a y-coordinate (3) which fell halfway between the scale labels 2 and 4; the others gave 2.5 as their answer.

On the pre-test, 42% of students were able to plot all four points correctly, whereas 53% were able to do so on the post-test. To test for significance, the number of changes from pre- to post-test were noted and a simple sign test for the significance of changes was used. This showed no significant change on this question. Further investigation showed that five classes showed improvement whereas the sixth class (a computer class) deteriorated markedly. The deterioration was in fact almost entirely due to a sharp increase in the number of students who interchanged x and y co-ordinates. The reason why this one computer class was so much worse is not clear, but the computer groups had much less practice at plotting points by hand than the calculator groups and at reading the scale (an automatic facility of the software).

The calculator groups also showed more improvement in reading points. Question 7, for example, showed the graphs of a linear and quadratic function $y = x^2 + 2x + 2$ and $y = 3x + 4$ on the one set of axes and their two points of intersection (both of which occurred at grid points and one of which was labelled). Students were asked to give the co-ordinates of the points of intersection. The errors once again involved interchanging the coordinates, scale and negative values. The computer group showed little improvement from pre-test to post-test (41 correct to 45 correct) but the calculator group improved significantly (21 correct to 47 correct), although from a lower base. The sign test for the significance of changes indicates significance at the 5% level ($p = 0.0226$). Results for the second part of question 7 which asked students to solve the equation $x^2 + 2x + 2 = 3x + 4$ were appalling: only one student on the pre-test (by solving the quadratic) and three on the post-test were able to give two correct answers for x! Further, only 24 students on the pre-test and 9 on the post-test were able to give even one correct value for x. Less than half the students attempted this part on either test. Of those that did, the most common source of errors was incorrect algebra.

We had assumed that Year 10 students would have mastered the skills of reading and plotting points. However in the light of the performance of the Year 10 students in this pilot study, future versions of the units need to give more explicit attention to reading and plotting awkward points.

Interpreting graphs: graphs to situations.

Question 5 involved a context leading to a quadratic expression for the area A m^2 of a rectangular enclosure in terms of the width x m; the context was similar but not identical on the pre- and post- tests. The quadratic graph showing how A varied with x was given and students were asked to read a value of A for a value of x, values of x for a value of A and the greatest value of A. All these questions were posed in terms of the context, e.g. "If the area enclosed is 3 m^2, what could the width of the garden be?". A significant proportion of students did not attempt this question (about 30% on the pre-test and 18% on the post-test). On the pre-test, 19% were correct on all three graph interpretation parts; this rose to 33% on the post-test. The sign test for the significance of changes shows there was an improvement significant at the 0.5% level ($p = 0.0046$) in performance overall on the interpretation of the graph.

Once again some students when reading A from x were unable to interpolate and went for the nearest grid point (about 10 instances on both tests). Nevertheless, on this specific item, there was a significant improvement in performance for the calculator group (sign test — $p = 0.0000$), but not for the computer group. In reading x from A, very few students gave two solutions (4 on the pre-test and 8 on the post-test); otherwise the one answer given was usually correct, probably because it corresponded to a grid point. Again the sign test applied to the number of changes from pre- to post-test for students in each group giving at least one correct answer on this item indicated a significant improvement in performance for the calculator group ($p = 0.0000$), but not for the computer group. Finding the maximum value seemed to be an easy aspect of graphical interpretation to teach. In this case the maximum value was a labelled grid point, and the percentage of students correct increased from 20% to 42%. Only the computer group showed statistically significant improvement (sign test — $p = 0.0024$). Many students who attempted to use algebra instead of the graph made errors because their own expression for the area was incorrect. In fact the results were extremely poor in every instance where students used algebra on the pre- and post-tests and at the interviews.

Graph to algebra.

Question 8 showed the graph of a straight line that passed through marked intercepts on the two axes and students were required to give the rule of the straight line (the equations on the two tests were respectively $y = 2x - 4$ and $y = 2x + 4$). On each test, only about two thirds of students gave a response. On the pre-test, 15% were correct and this rose to 17% on the post-test — there was no significant improvement. Further analysis showed that all the improvement was in any case due to one class in which the number of students correct was 3 on the pre-test and 17 on the post-test! (One assumes a highly effective teacher intervention at some stage of the project.) A large proportion of students (24% on the pre-test and 14% on the post-test) gave an equation of some linear appearance but with neither gradient nor y-intercept correct: $y = mx + c$ was one of the better answers. The lessons had not specifically taught students to find the equations corresponding to graphs, but we had expected that the link between algebraic form and the shape of a graph would have been strengthened. This is discussed below.

Algebra to graph.

The difficulties in making the translation from algebra to graphs were evident during the twelve interviews, where the only students to draw the straight line $y = 14 - 2x$ correctly were the three higher ability students from the calculator classes. Many students wanted to take the coefficients in the formula and directly give them a concrete meaning. For example, in attempting to graph $y = 14 - 2x$, four students put the graph through 14 (labelled on the y-axis) and +2 or −2 on the x-axis.

while another plotted the single point (2,14). When sketching the quadratic $h = -0.75x^2 + 27x + 2$ in another item about the height h above water of a distress flare x seconds after firing, one student explained the parabolic shape thus: "Because it starts on 0.75 then goes up by 27 then it moves across 2". Another commented: "It goes from −0.75 up to 27". Only four of the twelve indicated some reasonable sort of parabola.

As noted above, calculator students made better attempts at graphing $y = 14 - 2x$. However, the two groups performed similarly on the next question which required them to indicate on graphs they had drawn where $14 - 2x$ is greater than, equal to or less than $3x - 6$ (when students did not successfully draw the graphs, they were given a correct graph to use). The teaching program had used several "real-life" contexts which had required decisions of this nature. Only high ability students were able to go on to use algebra to check when $14 - 2x = 3x - 6$.

In addition to asking students to draw and sketch graphs, we tested to see if they could match algebraic and graphical form. Question 9 gave the graphs of one linear and two quadratic functions on separate sets of axes and listed six algebraic functions. Students were to match the graph with the corresponding function(s). No student got all matchings correct on either test. From pre-test to post-test, students became better at matching algebraic and graphical forms. On the pre-test, there were 47 occasions when a linear expression was matched to a quadratic graph; this occurred 18 times on the post-test. Similarly, 37 students on the pre-test and 8 on the post-test chose a quadratic expression for the linear graph. Once again, many errors arose when students looked at significant numerical values associated with the sketch (e.g. the intercepts with either axis, a maximum etc) and then chose any function that seemed to have these numerical values present in some obvious way.

IMPLICATIONS FOR FUTURE WORK

Significant improvements (although not large as one might hope) were observed for both the computer and calculator groups in most of the topics which the lessons targetted, i.e. interpreting graphs (with intersections, maximum values etc) and matching graph shape and algebraic form. There were few differences in the mathematical gains made by the calculator and computer groups, although the calculator groups made more improvements than the computer groups in reading and plotting points and in drawing graphs. On the other hand, the computer groups gained facility in using the technology. We conclude that for the Year 10 students at the school where the trial was conducted, practising these basic aspects by hand was still essential. It was not sufficient to use the ANUGraph feature to read off the co-ordinates of points or to work with the graphs drawn by the computer.

From this study we know that teachers with little previous experience of computers and with only a few hours of prior experience with ANUGraph can successfully manage the computer-based unit of work which we provided. Students can learn to use the computers and software whilst working on the unit, although they did not progress as far as students who used the parallel non-computer material. We have also established within the calculator lessons a variety of techniques (such as providing graphs and axes for students to add other components to by hand) which enabled the students to concentrate on interpretation of graphs and not become entirely bogged down in plotting.

Access to the laboratory of computers is still difficult in the school where we worked. The computer groups generally worked uninterrupted at the machines and seemed to miss opportunities for group reflection and discussion. This was probably because of the perception that every minute in the computer laboratory was highly valuable and should not be wasted. Our preferred mode would therefore now be a mix of time in and out of the laboratory with some of the hand plotting examples trialled by the calculator groups being used throughout the computer based unit. We believe that the ability to use a computer graphing program is a valuable skill for students to develop and recommend that all students have this opportunity at least by Year 10.

It seems worthwhile noting the extreme difficulty students experienced whenever they attempted to use algebra. Even basic calculator skills caused serious difficulties. For example, of the 9 students who did this question in the interviews, only one correctly evaluated $h = -0.75x^2 + 27x + 2$ when $x = 1$. Not having mastered basic skills makes further learning very problematic.

REFERENCES

Asp, G., Dowsey, J. & Stacey, K. (1992). Technology enriched instruction in year 9 algebra. In B. Southwell, B. Perry & K. Owens (Eds.), *Proceedings of the Fifteenth Annual Conference of the Mathematics Education Research Group of Australasia* (pp. 84 -93). University of Western Sydney: MERGA.

Asp, G., Dowsey, J., Hutton, B., McLennan, A. & Stacey, K. (1992a). Technology enriched algebra in year 9. In M. Horne & M. Supple (Eds.), *Mathematics: Meeting the Challenge* (pp. 425 - 432). Melbourne: Mathematical Association of Victoria.

Asp, G., Dowsey, J., Hutton, B., McLennan, A. & Stacey, K. (1992b). Technology enriched algebra in year 10. In M. Horne & M. Supple (Eds.), *Mathematics: Meeting the Challenge* (pp. 433-439). Melbourne: Mathematical Association of Victoria.

Hembree, R. & Dessart, D. J. (1986). Effect of hand-held calculators in precollege mathematics education: a meta-analysis. *Journal for Research in Mathematics Education*, 17(2), 83-99.

Kaput, J. (1992). Technology and Mathematics Education. In D. A. Grouws (Ed.), *Handbook of Research on Mathematics Teaching and Learning* (pp. 515 - 556). New York: MacMillian.

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[1] The research reported in this paper was supported by a grant to all three authors from the Australian Research Council for a project entitled "Development of algebraic concepts and logical reasoning by technology enriched instruction." The authors would like to thank Anne McLennan, Barry Hutton and the teachers and students who participated in the project.

A SOCIOLINGUISTIC PERSPECTIVE IN THE STUDY OF THE SOCIAL CONTEXT IN MATHEMATICS EDUCATION

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This paper illustrates the use of the Functional Theory of Language as presented by Halliday to compare the interaction between students and teachers in a mathematics classrooms as a function of gender and social class. The constructs of field and tenor presented by Halliday are used to compare the observation from two classrooms from a high social class boys' school and a low social class girls' school. Conclusions form the data as well as methodological implications are discussed.

During the past few years research in mathematics education witnessed significant shifts in the issues being investigated, the methods used and theoretical prespectives adopted (Atweh, Carrs, & Kanes, 1993). While process-product research, that aimed at developing generalised theories for the prediction and controlling of student learning, traditionally dominated research in mathematics education, more recently there was a significant increase in research that aimed to illuminate and explain the process of education in its diverse contexts. Researchers in mathematics education were quick to diversify their methodologies and perspectives in generating and analysing data to achieve these new concerns. One such new approach was sociolinguistics (Florio-Ruane, 1987, Evertson & Smylie, 1987). This paper illustrates the use of a particular sociolinguistic approach to understanding the mathematics classroom and to analysing student-teacher interactions.

SOCIOLINGUISTICS IN MATHEMATICS EDUCATION

Ellerton and Clarkson (1992) reviewed several studies conducted in Australasia, between 1988 and 1992, that were primarily concerned with language aspects of mathematics teaching and learning. The overriding concern of these studies was how the mode and form of language can be manipulated to increase students' learning of mathematics. The authors presented a model, suggested by Ellerton (1989) specifying sociolinguistics and psycholinguistics approaches as two aspects of the 'interface between language, mathematics and mathematics learning' (1992, p. 156). However, none of the studies reviewed followed either approach. Reviewing the research on the social context of mathematics education in the same period, Atweh, Cooper and Kanes (1992) identified four studies that used Wittensteinian approach that employed sociolinguistic concepts to explain aspects of the social context of mathematics education. These papers argued that mathematical concepts have no essential meaning outside the social context in which they operate. Hence, learning mathematics is mediated through language rather than through thinking.

To illustrate the use of sociolinguistics in mathematics education, this paper will adopt the Functional Theory of language as exhibited in the writings of Michael Halliday (1973,1974, 1978; Halliday & Hassan, 1989). According to this view "language is the main channel through which the patterns of living are transmitted to [the individual], through which he[/she] learns to act as a member of 'society'-in and through the various social groups, the family, the neighbourhood, and so on- and to adopt its 'culture', its methods of thought and action, its beliefs and its values" (1974, page 4). The child is "... socialised into the value systems and behaviour patterns of the culture through the use of language at the same time that he[/she] is learning it" (page 21).

To understand the language and its use one also needs to study its context. Halliday points out that the origin of the word context is CON-TEXT ie. 'with the text'. Con-text is an accompanying text that adds to the understanding of the intended text. Halliday presents a model with three components to analyse the context of a text. The three components of context are:

1 Field: refers to the institutional setting in which a piece of language occurs and embraces not only the subject matter but the whole activity of the speaker and participant in a setting[and we might add: 'and of other participants]...

2. Mode: refers to the channel of communication adopted, not only between spoken and written words but much more detailed choices [and we might add:' and other choices relating to the role of language in the situation] ...

3. Tenor: refers to the relationship between participants, ... not merely variations in formality ... but ... such questions as the permanence or otherwise of the relationship and the degree of emotional charge in it. (page 34) [brackets in original]

THE STUDY

This paper reports on analysis of a segment of the data collected as part of a long term project on the Social Context of mathematics education at the Centre of Mathematics and Science Education (Atweh & Cooper, 1989, 1991). The overall aim of the project was to study the nature and form of mathematics knowledge as presented in the school and to study the perceptions of students and teachers of the nature of mathematics and its relevance in the life of students. Of particular interest to the project was the investigation of the effect of gender and socio-economic background of students on the perceptions and how these perception affect the classroom interactions. The project consisted of a series of field studies (Popkewitz & Tabachnick, 1981). This paper is mainly concerned with the first study, carried out in 1989 in two private schools in Brisbane. The schools were selected to maximise the socio-economic and gender differences. The first school, Northside High, was an all girls' school from a low socio-economic part of the city, and the second school, Cityview, was an all boy's school that attracted most of its students from professional families. Both participating schools followed the same work program in mathematics, used the same textbook and were teaching the same chapter while participating in this study.

Grade nine (second year in the secondary school system in Queensland) was observed for the duration of one topic from the syllabus, about two weeks. Classroom interactions were recorded using a special classroom observation instrument developed by the authors (Atweh & Cooper, 1992). Class proceedings were tape recorded and later transcribed. Other sources of data collected were school publications and interviewees with principals, subject masters, cooperating teachers, and selected students. This paper is mainly concerned with classroom observations. In the following discussion the classroom observations will be analysed using the first and last components of Halliday's model.

RESULTS

The Field of Discourse

The field in both instances observed were mathematics classes where interaction oscillated between doing mathematics (ie mathematics as the primary field of discourse) and talking about doing mathematics (ie. mathematics as the subject matter or the second-order field of discourse) (Halliday, 1978). Both classes observed were teaching the same content in algebra: review of the Cartesian plane and plotting of linear funcfions. Both classes were very teacher dominated. However, a closer look at the classroom interactions reveals many differences in the hidden curriculum being constructed.

Ivor, the teacher at the boys school, presented rigorous definitions of the concept of a function in the following words:

> Domain: numbers where we get the x(independent variables)
> Range : numbers where we get the y (dependent variables)
> Set of ordered pairs is a function iff for a value of x there is one and only one value of y.

He then presented a more algorithmic version of the definition that may be used to determine if a relation is a function. Dealing with the same concept, Jeff, the teacher at the girls school, introduced the definition through the algorithm. In all his definitions, Jeff attempted to give many examples and counter examples of a concept and assisted students to generalise a pattern. He used these pattern as definitions. It was obvious that the two classes experienced mathematics as a formal system in different ways.

Further, Ivor stressed the convention nature of mathematics. In dealing with the co-ordinate system, after quickly reviewing what the students have done last year on plotting points, he asked the students why is it essential that the x- co-ordinate always comes first. Jeff's explanation of the same rule was " [since we say] x,y,z then x comes in first therefore the x-co-ordinate comes first". Once again the "justification" was presented an a rule of thumb to remember the order rather than a "formal" justification.

Another difference noticed in both classes related to application of the mathematics discussed to real world problems. At the end of his first class with the students, Ivor reviewed the work that the students have done on the pendulum in the science class. As homework he asked the students to perform an experiment on varying the length of the pendulum and measuring its period. Students were asked to tabulate the results and graph them. Similarly, to justify the need in mathematics to differentiate between functions and non functions he used an example from science.

This morning in science we were talking about classification of plants. We said that some [types] have some properties, and [other types] have other properties. Here we have a relations: we have functions and non functions. It is just a classification. You may wonder why do we need to classify anything. It helps us to analyse a particular area.

Jeff, on the other hand made no attempt to link the content studied to other subjects or to real world applications.

Hence, although the two classes are dealing with the same mathematics, and in rather similar way, the hidden curricula in both classes was quite different. In one class mathematics was presented as a formal system, while in the second class mathematics was based on generalisations and rule of thumb; one class stressed the reasons behind conventions while the second class presented conventions as "rules of the game"; finally, one class presented mathematics as meaningful activity related to the real world, while the second class presented mathematics as a set of rules to manipulate symbols. The question was not which mathematics is more valid or valuable, or which teaching style was more effective or relevant. Ivor presented mathematics that he perceived would be needed for higher studies in mathematics and Jeff is teaching mathematics that was "not very useful" for these students. Ivor was enculturating his students into the culture of "formal mathematics" and Jeff was enculturating his students into a culture of "following rules".

The Tenor of Discourse:

Ivor believed that most of his students are university bound. Being independent in assessing one learning and achievement was seen as a useful behaviour to nurture for higher education settings. This was reflected in the way that Ivor interacted with his students. While working on the exercises from the textbook, Ivor expected his students to check their own work. He was quite sarcastic when a student asked him to check their work for them. "You have the answers in the back of the book, check it your self." "Your are a big boy now, you do not need me to check your work." Jeff has a different attitude towards students work. He corrected the mistakes that students were making by discussing them on the blackboard. He provided further examples and repeated the rules "for those who are still experiencing problems". The two messages from the two classes were clearly different. While one class expected and challenged students to engage themselves in checking their progress, the other provided an atmosphere of support and readily available assistance to deal with their difficulties.

Another difference in the tone of both classes was the general atmosphere in the classroom. Ivor conducted a class that gave a feeling of a battle ground, a field of contestation between students and teachers. Several times in the lesson students complained "I do not understand this" and were not hesitant to say "I still do not understand". When the Ivor was discussing the classification of relations into functions and non functions, a student complained "What is the practical use of studying that." When the teacher said "It is a mathematical use and solely that" the student insisted "What use is that?" Also noticeable in the class was the use of sarcasm by the teacher. Sarcasm was used in several places in the classroom: when students asked an apparently silly question, or when a student was not paying attention or disrupting the class. The affect of sarcasm could only be understood in context of the situation in which it arose. The teacher's sarcastic comments were not taken by the students as a put down. The atmosphere of contestation, either in challenging the teacher for more information or in forms of disruptive behaviour, did not subside in the class as a result of these comments. Rather these comments were understood as demands for attention, but also as means of establishing an atmosphere of equality and reciprocal privilege with the teacher. Students responded to these comments by laughing or by sarcasm of their own:

The teacher: Have you finished your chewing gum yet? You can go and put it in the bin. Is it one of the school rules around here that your are not allowed to have chewing gum in the school?

The offending student: I have not read it anywhere

Another student: (Opens a book at random) Rule thirteen, Alcohol and chewing gums are allowed.... (Class bursts in laughing).

Jeff had a very different way of interacting with his student. He was very formal and serious with his students. He was firm in his expectation of and demand for attention and co-operation yet he was noticeably polite in his interactions with the girls. To get the attention of some students he would say "Excuse me there, are you all right". He consistently called girls by their first names, and never reprimanded anyone for a wrong answer. The messages from the two classes are quite different. They are consistent with stereotypes that boys are independent, tough and are rebellious and girls are dependent, fragile and obedient.

A further interesting difference between the two classes was seen in positioning of the human agency with respect to mathematics knowledge. The following discussion relates to the two segments of interaction from both classes listed in the Appendix.

One feature of the mathematical language in the girls' class is that it was not "exact" or "complete". Secondly, every sentence has one personal pronoun. This sample is quite representative of the style of talk that Jeff engaged in with his students. No sentence has a mathematical term as its subject. Mathematical knowledge was not presented as an abstract content separate from what people do. It should be pointed out, however, that peoples' action that constitute mathematics was not presented as the everyday life action; that is, not "meaningful" action. Lastly, the role of the first person pronoun was interesting. The "we" indicates a group ownership of the example and the procedures to be adopted. However, the two references to "I said to you" refer to a rule that students are to follow, but the rule is given without an explanation. This is not the only such construct that appears in Jeff's talk. When students were connecting plotted points, obtained from a quadratic equation, by straight lines, the teacher said.

Now some of you have joined the points with a ruler to get a V shape. Now there is nothing wrong with that but I want to tell you that from now on when you get something like that do not join them with a straight line. Join them with a smooth curve.

No additional information was given.

It is worthwhile noting the proper and rigorous mathematical language that Ivor was using. Also, in contrast to Jeff, Ivor used much less first and second person constructs, thus giving the impression that mathematics was an objective discipline that had "truths" of its own independently of us. It should also be recalled, that these truths were presented as meaningful since they were useful in everyday life and agreed upon to facilitate communication.

Conclusions

The comparison of the context of discourse of mathematics in the two classrooms studied showed that even though the teachers and students were engaged in working from the same textbook, the actualised curriculum was quite different in both classes. One classroom was developing mathematics as a highly formal field of study, stressing mathematical structures, concepts and language, the second class was developing mathematics as a set of skills or rules. The first class stressed meanings and reasons while the second class stressed generalisations of pattern from within mathematics itself. Further students in one class were encouraged to be self reliant in checking their progress and to be participants in their development of mathematics ideas, while the second class, indirectly encouraged the dependence on the teacher as a source of knowledge and assessment. Using the sociolinguistic terms adopted in this analysis the two classes differed in the field and tenor of discourse.

Secondly, these alternative constructions of the context do not have the same social value. One construction was perceived as appropriate for students intending to go into higher education and useful for making scientific and business decisions while the second was perceived as appropriate for the "less able" students and appropriate for consumer transactions. If discourse in the area of language and mathematics is to provide useful information about the development of mathematical understanding in children, then, inevitably, it has to address the question of value.

Thirdly, the Social Context Project was conceived from within the social critical sociology perspective. It adopted the ethnographic methods of data collection, including interviews and classroom observations. The social critical sociology provided the conceptual tools for analysing the data, but not the practical tools to deal with the huge amount of data gathered. The Functional Theory of language is a sociolinguistic theory that provides a view of language use that is sympathetic with the social context views adopted by this project, and has provided a framework for dealing with the classroom observations. This analysis illustrated the benefit obtained of using both perspective in an attempt to make sense of the data. Naturally not all the constructs of that theory were used. Our interest in looking at the role of language in this project is a social context one and not a pure linguistic one. Other investigation into the role of language of mathematics and in constructing understanding may make use of other aspects of this theory.

REFERENCES

Atweh, B. & Cooper, T. (1989). A tale of two schools. Paper presented at the 12th Annual conference of the Mathematics Education Research Group in Australasia. Bathurst: Sturt University.

Atweh, B. & Cooper T. J. (1991). Hegemony in the mathematics curricula: The effect of gender and social class on the organisation of mathematics teaching for Year 9 students. In F. Furinghetti (Ed.), Proceedings of the Fifteenth Psychology of Mathematics Education annual conference, Assisi, Italy.

Atweh, B. & Cooper, T.J. (1992, July). The structured Classroom Interaction Schedule: A validation study. Paper presented at the annual conference of the Mathematics Education Research Group of Australasia, University of Western Sydney - Nepean.

Atweh, B. Cooper, T. & Kanes, C. (1992). The social and cultural context of mathematics education. In W. Atweh & J. Watson, *Research in Mathematics Education in Australasia, 1988-1991*. Brisbane, Queensland: Mathematics Education Research Group of Australasia and the Centre for Mathematics and Science Education, Queensland University of Technology.

Atweh, B.; Carss, M. & Kanes, C. (1993). XXX, In YYY, etc.

Ellerton, N. (1989). The interface between mathematics and language. *Australian Journal of Reading*, 12(2), 92-102.

Ellerton, N. & Clarkson, P. (1992). Language factors in mathematics learning. In W. Atweh & J. Watson. Research in Mathematics Education in Australasia, 1988-1991. Brisbane, Queensland: Mathematics Education Research Group of Australasia and the Centre for Mathematics and Science Education, Queensland University of Technology.

Evertson, C. & Smylie, M. (1987). Research on teaching and classroom processes: Views from two perspectives. In J. Glover & R. Rouning (Eds.), *Historical foundations of educational psychology*, New York: Pleunn Press.

Florio-Ruane, S. (1987). Sociolinguistic for educational research. *American Educational Research Journal*, 24(2), 185-197.

Halliday, M.A.K. (1973). *Explorations in the functions of language*. London: Edward Arnold.

Halliday, M.A.K. (1974). *Language and the social man*. London: Longman for the School Council.

Halliday, M.A.K. (1978). *Language as social semiotic: The social interpretation of language and meaning*. Baltimore: University Park Press.

Halliday, M. A. K. and Hassan, R. (1989). Language, context, and text: Aspects of language in a social-semiotic perspective. Victoria: Deakin University Press.

Popkewitz, T. & Tabachnick, B. (Eds.). (1981). *The study of schooling: Field based methodologies in educational research and Evaluation*. New York: Praeger Publishers.

APPENDIX

Jeff's Class

[Jeff is teaching the plotting of inequalities. He has taken an example of y<x. Without any justification, he says "How about doing this one first y=x? He then proceeds to plot the linear function and chooses several points on the plane some on the line and some are not]

So we have a Cartesian plane with points everywhere. Now, [there are] some important things to remember. What I have done here is the equation y = x. And, we [have] seen the quadrant and [have] see[n] whether the points lie on the line. [This was] a quick revision of the things we have done these two weeks. So how is this going to help us in graphing inequation? Most of you are thinking this. Very simply if we have this (points to y=x) and this (points to y<x) they are identical. The only difference is this here: the sign (points to the = and the < signs). And if you remember the time when we were solving inequations, I said to you to solve them exactly the same way as you would with equations. So if we had, 5 = x+4 and worked that out, and then we had 5<x+4, I said to you to solve these exactly the same way as [you would] with the equal sign. So that is exactly the same procedure we are going to do in graphing inequations. We are going to graph them the same way as we graph a normal equation. There are a couple of exemptions. When we have a symbol < or > we are going to use a dotted line. (teacher writes):

Summary:
1) if symbol < or > use a dotted line
2) only time to use heavy line is if we
have <=
or >=

Ivor's class.

T: Take out your homework form last night. You were asked to do nos. 6,7,8,9,12,14,16. You are asked to, first of all to state the domain for each of those graphs that were drawn, then you were asked to state the range, and then you were asked to state if it was a function or not. First of all .... Andrew! I told you to sit down five minutes ago, (Andrew complains that he was picking up paper from the floor.) You do a good job. You should be a cleaning lady. (Class laughs).

Right, (Teacher writes on board and reads out loud):

Domain: numbers where we get the x(independent variables)
Range : numbers where we get the y (dependent variables)

(Teacher asks for help from students to name variables)

Then we end up with [a] set of order[ed] pairs. (writes (x,y)) ordered pairs x,y. (says and writes)

Set of ordered pairs is a function iff (what ...)for a value of x there is one (not one, but one) and only one value of y.

It is important to say one and only one. We can not have zero. There must be a value, for every x there must be a value of y, and there must be only one value of y.

Let us look at number 6. What is the domain, in other words from what numbers do we choose the x values? There is only one x value, what is that? The domain is -2. That's it. What is the range?

(Teacher writes) Domain (-2)

S: -2,-1,0,1,2,3

(Teacher writes) Range (-2,-1,0,1,2,3)

For that one value of x, how many values are there of y?

S: 6

T: Is there one and only one value of y for a given value of x? No. there are six in this case. Then it is not a function. Precisely.

MATHEMATICS AND CHILDREN WITH MILD INTELLECTUAL DISABILITIES: THE SUCCESS OF A PROCESS ORIENTED INSTRUCTIONAL MODEL

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Children with Mild Intellectual (IM) Disabilities are often taught addition and subtraction algorithms through the use of rote learning models which fail to enhance their problem solving skills. Recent developments in other curriculum areas suggest that there are alternative process oriented models which may be more successful in ameliorating successful problem solving strategies among such students. This research attempts to contrast one such model, Quality-Sameness Analysis (QSA), with more conventional rote learning systems such as the Computational Skills Program (CSP). In addition it is proposed that the use of calculators to solve algorithms will free cognitive functioning of IM students and enhance the acquisition of problem solving skills. Preliminary results confirm the QSA as a powerful process oriented model with considerable potential in the IM classroom and the major contribution that the use of calculators can make in the acquisition of problem solving skills among IM children.

Some of the recent developments in the field of mathematics education have evolved from the need for different approaches to teaching and learning as a consequence of the large numbers of children who either fail or become disinterested in developing mathematical skills. Research in this field must therefore logically relate very strongly to the needs of children with learning difficulties, especially in the areas of specific learning disabilities and mild intellectual disabilities. Of particular interest are the debates which relate to the development of problem solving skills and the use of the calculator as a major emphasis in mathematics education.

Bruner (1966) saw as central to the nature of learning and instruction the notion that material to be taught must necessarily be organised in such a way as to optimally facilitate learning, retention, and problem solving. Similarly, Carnine, some years later claimed that "the specifics of how to organise and present the content of a discipline must be carefully considered if students are to succeed at higher order thinking" (1991, p. 263).

A number of studies has illustrated the importance of employing strategies in attempts at problem solving. (Neimark, Slotnick & Ulrich, 1971; Chi, Feltovich & Glaser, 1981; Flavell, 1985). Recently however, children's learning and problem solving strategies have come under increasing scrutiny. For example, there is evidence that too much time is dedicated to teaching computational skills at the expense of concept understanding and problem solving strategies (Porter, 1989, cited in Engelmann, Carnine & Steely, 1991) and that the presentation of problem solving strategies lacks coherence.

Engelmann et al., (1991) proposed a strategy known as Quality-Sameness Analysis which they claim is concerned with "designing material in a way that will maximise the amount of teaching that is needed to demonstrate how related concepts are the same and to minimise the amount of teaching required to teach 'differences'" (p. 294). In their view, instructors should design problem solving methods which allude to critical details in the same manner.

Various investigators have suggested that quality-sameness analysis be utilised in a variety of curriculum areas for example, spelling (Dixon, 1991), and social studies (Kinder & Bursuck, 1991). Indeed such an approach might also serve to reduce memory load (Carnine, 1991) and may be viewed as the process underlying higher order thinking which incorporates the integration of concepts, rules, strategies, schemas etc. Kameenui (1991) further argued that through sameness analysis the learner has the capacity to link a range of complex concepts. He suggested that it allowed teachers to teach more in less time, and that the samenesses acquired function as "cognitive building blocks" which then led to more complex cognitive structures that could be used for identifying similarities.

A further application of quality-sameness analysis has been made in the area of students with special needs and research with these students emphasises the importance of making sameness explicit (Engelmann et al., 1991). Other investigators have observed enhanced performance in special needs students using sameness analysis in teaching fraction and other mathematical concepts (e.g., Kelly, Gersten & Carnine, 1990; Kinder & Bursuck, 1991). It seems therefore, that special needs students might benefit from the application of quality-sameness analysis primarily because it is time efficient, reduces memory load, seems to 'free-up' cognitive space and enhances performance.

Another variable of interest to the present investigators which promises similar benefits to students with special needs is the use of a calculator. For perhaps fifteen years an increasing number of commentators have called for a shift in emphasis away from laborious computational methods to the use of the calculator. Girling, as early as 1977 defined basic numeracy as the ability to use a calculator and others, as well (e.g., McIntosh, 1990), have called for the abandonment of pencil and paper methods in favour of mental computation and calculators. The Australian Association of Mathematics Teachers (1987) recommended that calculators be used by all students at all year levels and suggested that calculators may be of some benefit to special needs students and may provide students with learning difficulties an opportunity to think about mathematical relationships rather than the mechanics of computation.

In summary then, recent research would therefore suggest that two major areas of weakness for students with learning difficulties in mathematics include the inability to follow systematic strategies for solving problems and the fact that few students are fully competent at the basic facts. These weaknesses point to a real need to develop not only computation skills but also alternative methods for solving 'word' and 'real world' problems. This research reports on a project which offers such alternative methods of instruction in order to improve the problem solving abilities of students with Mild Intellectual Disabilities (IM) through teaching addition and subtraction relationships and interrelationships using the Quality-Sameness Analysis developed by Engelmann et al. (1991). An additional aim was to show that access to a calculator for computations has a positive effect on maths problem solving abilities. The way in which problem solving is taught in schools today (particularly word problems) can lead to a range of misconceptions among students usually formulated from the presentation of limited, restricted samples in step by step teaching, particularly with low achieving students. The Quality-Sameness Analysis which we use in this project is an attempt to design curriculum procedures that can virtually eliminate the development of such misconceptions.

In the Metropolitan West Region of Sydney children with Mild Intellectual (IM) disabilities are often taught addition and subtraction algorithms and their application to problem solving situations through rote learning and practice models such as the Computational Skills Program (CSP) developed by Macquarie University. This research aims to contrast a process oriented model, Quality -Sameness Analysis (QSA) with CSP to establish whether it is superior in raising achievement levels in problem solving for children with IM disabilities. In addition, it attempts to establish whether the use of a calculator in conjunction with either model has positive benefits for the development of problem solving skills in children with IM disabilities.

To test the hypotheses that one method of instruction was superior to another and that the use of calculators would ameliorate the capacity for learning for this group of students, four experimental groups and one control group were required. Two of the experimental groups, one under each instructional type, were allowed to use calculators while the other two and the control groups were not. This design allowed the investigators not only to compare across groups but also to compare the performance of those who used calculators against those who did not, and to compare performance based on the type of instruction received, CSP or QSA.

METHOD

Subjects

Five schools in the Metropolitan West region of the NSW Department of School Education in Sydney were approached to participate in this study. The five schools reflect similar characteristics in terms of student backgrounds and teachers in these schools were all aware of the Computational Skills Program (CSP) promoted by Macquarie University. In addition, each school had a class of children with Mild Intellectual (IM) disabilities. Five qualified teachers who are students in the postgraduate program in Special Education volunteered to act as instructors for each of the experimental and control groups. The instructors received a formal training session from one of the researchers on the appropriate methodology required to instruct children through the use of either CSP or Quality-Sameness Analysis (QSA).

A total of 73 children (50 male and 23 female) from five schools (one class within each school) and identified as having Mild Intellectual Disability, here defined as those with IQ's in the range of 55 - 75, participated in the study. The sample was purposive because intact classes were used. These children ranged in age from 10 yrs to 12 yrs 6 mths with an average of 11 yrs and a standard deviation of 0.78. Although the sample of participants was purposive, instructors were randomly allocated to the schools.

Design

In order to test our theory four experimental groups were defined. Each methodology was used with two groups, but with only one of each of the groups allowed to use calculators to perform mathematical operations. Therefore the four experimental groups included CSP with calculator; CSP without calculator; QSA with calculator; and, QSA without calculator. In addition, there was the Control group which was taught using the CSP method as it is normally applied in schools within the region. This design facilitated analysis through the use of Analysis of Covariance with the pretest as the covariate and age and sex as possible moderator variables. It also allowed for analysis based on instruction type (QSA vs. CSP vs. Control) and calculator use (With Calculator vs. No Calculator vs. Control). All participants were required to complete a word problem pretest. A parallel form of the pretest was also completed at the end of the experimental sequence, ten weeks of instruction.

Instrumentation

The word problem test included a combination of sixteen problems requiring the application of a variety of addition and subtraction algorithms. The experimenters included 4 simple, 4 classification, 4 sequence of events, 4 comparison word problems as outlined by Engelmann et al. (1991).

Two of the experimental groups were supplied with Casio MS-7 desktop calculators which are solar powered, have a large viewing screen and easy to operate numerical and function buttons. The ease of use and low maintenance were considered to be essential qualities in encouraging students to make proper use of this equipment.

All participants using the CSP model were supplied with the appropriate workbooks and instructions together with a variety of word problems for each of the skill levels of this program, and appropriate independent worksheets. Those who were being instructed through the QSA method had access to individual and wall tables of the number families, arrow cards and a variety of word problems for discussion and independent work.

PROCEDURE

Initially permission was sought from Metropolitan West Region of the DSE to conduct the study in selected schools in the region, those with IM classes. Once this permission was granted five schools which were known to the experimenters were invited to participate. Teachers and parents were informed on the nature of the experiment and it was emphasised that the researchers wished to determine which method of instruction was most successful in improving student performance. Each of the experimental schools was then guaranteed an additional experienced and qualified teacher to implement the program in their school. The teacher in the control school was aware that we were interested in problem solving and allowed to continue with his normal classroom practices. All five programs involved exactly the same amount of face to face contact with the students, one hour per session and three sessions per week. Over an eight week period this amounted to approximately twenty hours of instruction taking into account various school activities and absences which disrupted the planned programs.

The program of instruction in both modes CSP and QSA was devised to take place over one school term, the third term of the 1992 school year in NSW. The first and last weeks of the term were used to administer the pretests and posttests. The CSP program was conducted with strict adherence to the guidelines set out in the manuals provided. Word problem probes were added to each skill level using the same types of problems taught using QSA. Instruction in these followed traditional methods. QSA on the other hand required the development of an instructional strategy and appropriate teaching materials.

Once student volunteers were assigned to each of the five participating schools, each instructor received three hours of instruction prior to the presentation of instructional material together with ongoing instruction as the research progressed. The pretest was conducted in the five schools by a research assistant and the instructors who followed the same protocols in each school. The control group was found to perform significantly better on the pretest than the other four groups which were found to be of similar ability. All students were tested again at the completion of the program, in the last week of the term. Each instructor and the research assistant were responsible for administering and collecting this information.

RESULTS

Analysis of covariance (ANCOVA) was used to determine whether there were any significant differences based on performance on the word problem posttest. Although gender and age were used as moderator variables, there was no significant main effect found in the analysis due to either variable. The experimental variable, instructional type however, was found to be a significant ($p < .001$) contributor to the variance of posttest results in both the two group and four group instructional modes . Further analysis of these results through the use of Newman-Keuls tests are outlined below. Table 1 outlines means and standard deviations for all groups and combinations of groups.

Table 1: Raw data means and standard deviations on word problem pretest and posttest scores by

Group Type	QSA[1]		CSP[2]		Total		Control	
	Pre	Post	Pre	Post	Pre	Post	Pre	Post
Calculator	m=1.33	10.36	1.92	10.10	1.63	10.24		
	sd=2.50	2.91	1.73	2.03	2.12	2.47		
	n=12	11	12	10	24	21		
Non-calculator	3.36	9.83	1.88	7.13	2.55	8.29	4.35	5.47
	3.27	2.89	1.45	2.96	2.51	3.18	3.38	3.04
	14	12	17	16	31	28	20	17
Total	2.42	10.09	1.90	8.27				
	3.06	2.90	1.54	2.71				
	26	23	29	26				
Grand Mean							2.81	8.18
							2.83	3.41
							73	66

Note: (1) QSA–> Quality-Sameness Analysis
(2) CSP–> Computational Skills Program

In comparing results it is clear that all of the experimental groups performed significantly better on the posttest than the control group. The two QSA groups (with and without calculator) and the

CSP group (with calculator) significantly ($p<.05$) outperformed the CSP group and all four experimental groups significantly (CSP - with calculator, QSA - with and without calculator, $p<.01$; and, CSP - without calculator, $p<.05$), outperformed the control group on the posttest.

In order to determine whether the type of instruction given had any effect on performance, the data was combined into three groups, those who were given CSP instruction (with and without calculators), those with QSA instruction (with and without calculators) and the Control group. Further analysis of the ANCOVA revealed that both instructional types resulted in a significantly ($p<.01$) better posttest performance than the Control group but no difference between Instructional types.

The data was also combined in accordance to the use of calculators thus comparing, through ANCOVA, posttest results of the group who received instruction with the use of calculators, to those of the group who received instruction without the use of calculators and the Control group. Further analysis indicated a strong effect due to the use of calculators with both groups significantly ($p<.01$) outperforming the Control group on the word problem posttest but the Calculator group significantly ($p<.01$) outperforming the Non-calculator group.

SUMMARY AND CONCLUSIONS

It was clear from these results that both the use of calculators and the implementation of Quality-Sameness analysis are potent strategies in achieving success on problem solving for a group of IM children. While the use of calculators was demonstrated to be a successful strategy on its own, the fact that QSA was so potent either by itself or in tandem with the use of calculators is indicative of the power of the process oriented model. In today's technological age there seems to be an ever decreasing argument for the rote learning strategies that have been dominant for so long in our educational systems. The fact that students classified as low in ability were able to dramatically improve their capacity to solve problems adds further support to the proponents of QSA as a viable alternative methodology across curriculum areas.

In our efforts to implement the programs of study, especially those which required the use of calculators, it was evident that at the primary level very few of the IM students were even aware of how to use a calculator. Given the availability of the technology and the relative low cost, it is difficult to surmise why more educational systems do not take advantage of its availability. There are many aspects of normal daily tasks which have been simplified and made available to the masses through ever developing technology, for example, the availability and relative ease of driving a motor vehicle in contrast to the skills required when they first beca me available. No one expects those same skills to be mastered by today's drivers yet, in educational terms, we fail to accept that the development of concepts required to become good problem solvers may be more important than the mastery of computational skills easily implemented through the use of a calculator.

Our research and its success with a group of students who are often doomed to failure does suggest that creative process oriented approaches can work. The study was limited by the method of sample selection and the difficulty in rigidly controlling learning sequences within schools. Nevertheless, despite a variety of difficulties there were clear experimental effects. Similar research needs to be carried out with larger groups and at a variety of age and ability ranges before we can make definitive statements but the body of evidence continues to grow in support of Quality-Sameness Analysis as a viable alternative to more rote oriented models of learning for students with mild intellectual disabilities.

REFERENCES

Australian Association of Teachers of Mathematics and Curriculum Development Centre. (1987). *National statement on the use of calculators for mathematics in Australian schools*, Canberra, Australia: CDC

Bruner, J.S. (1966). *Toward a theory of instruction.* New York: W.W. Norton.

Carnine, D. (1991). Curricular interventions for teaching higher order thinking to all students: Introduction to the special series. *Journal of Learning Disabilities*, 24, 261-269.

Chi, M.T., Feltovich, P.J., & Glaser, R. (1981). Categorisation and representation of physics problems by experts and novices. *Cognitive Science,* 5, 121-152.

Dixon, R.C. (1991). The application of sameness analysis to spelling. *Journal of Learning Disabilities,* 24, 285-291.

Engelmann, S., Carnine, D., & Steely, D.G. (1991). Making connections in mathematics. *Journal of Learning Disabilities,* 24, 292-303.

Flavell, J.H. (1985). *Cognitive development* . New Jersey: Prentice-Hall.

Girling, M. (1977). Toward a definition of numeracy. *Mathematics Teaching* , 81, 4-5.

Kameenui, E.J. (1991). Toward a scientific pedagogy of learning disabilities: A sameness in the message. *Journal of Learning Disabilities,* 24, 364-372.

Kelly, B., Gersten, R., & Carnine, D. (1990). Student error patterns as a function of instructional design: Teaching fractions to remedial high school students and high school students with learning disabilities. *Journal of Learning Disabilities,* 23, 23-29.

Kinder, D., & Bursuck, W. (1991). The search for a unified social studies curriculum: Does history really repeat itself? *Journal of Learning Disabilities,* 24, 270-275/320.

McIntosh, A. (1990). Becoming numerate: Developing number sense. In S. Willis (Ed.), *Being numerate: What counts.* Victoria: ACER.

Neimark, E.D., Slotnick, N.S., & Ulrich, T. (1971). Development of memorization strategies. *Developmental Psychology,* 5, 427-432.

PROCESSES INVOLVED IN MATHEMATICAL PROBLEM SOLVING IN YEAR 12 CALCULUS

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Mathematical problem solving is a complex activity that necessitates very thoughtful consideration about which are the best ways to teach problemsolving. The purpose of this study is to uncover information about and gaina greater insight into the processes involved in solving problems in year 12 (V.C.E.) Calculus. A qualitative methodology (thinking aloud) has been employed in the study. The verbal protocols of six students are currently being analysed for regularities in overt behaviour and especially the utilisation of potential heuristics in their solutions. The paper will present the research questions, the research method and the plan for the categorisation of the verbal protocols.

Mathematical problem solving is an extremely complex activity that necessitates very thoughtful consideration about what are the best ways to teach problem solving. The process of solving mathematical problems is very important in the mental and conceptual development of the children. Lester (1980) claimed that the ability to solve problems is:

> "the ultimate aim of learning mathematics at every level" (p.287).

That problem solving is considered important, can be confirmed by the fact that it has been the subject of research since the turn of the century.

Before proceeding further, two key terms will be defined in this paper, problem and problem solving. A careful examination of many research papers reveals the fact that there is no agreement among researchers as to what is a "problem" and even more disagreement is evident when trying to determine what problem solving is. This is mainly due to the fact that mathematical problem solving appears to a certain extent to be so complex and subtle, as to defy an all-encompassing definition. Lester (1980) writes that:

> *"A problem is a situation in which an individual or group is called upon to perform a task for which there is no readily accessible algorithm which determines completely the method of solution". (p.287)*

Sweller and Low (1992), have proposed their own conceptualisation of what a problem is:

> *"No task can be classed as either an exercise or a problem simply by referring to its structureand components. Alone, the structure of a task cannot reveal its "problem" status. Its status is revealed fully by the novice-expert distinction." (p.84)*

In an attempt to define what "universal" problems are, Davis (1992), writes that they are:

> *"Those problems that are known not to have been solved at a particular time." (p.1-183).*

Problems of that kind are therefore problems to every person to whom they are posed. As a result of this short discussion I shall define a mathematical problem as a task posed to an individual or group, who will attempt to decipher the task and obtain a mathematically acceptable solution by not initially having access to a method which completely determines the solution. The task being a problem or not for a particular individual or group depending and being a function of the mathematical knowledge [general and task specific], the executive control mechanisms, the memory capacity, the automation of appropriate skills, the mathematical ability, the utilisation of potential heuristics and the mathematical maturity and creativity of the given individual or group. Problem solving can therefore be defined as the set of actions taken to perform a task, assuming that there exists a desire on the part of the individual or group to perform the task.

HEURISTICS AND OTHER PROBLEM SOLVING STATEGIES:

A useful categorisation of dimensions of mathematical behaviour [incorporating heuristics] has been proposed by Schoenfeld (1985b):

> "**COGNITIVE RESOURCES:** The mathematical knowledge possessed by the individual that can be brought to bear on the problem at hand.
>
> **HEURISTICS:** Strategies and techniques for making progress on unfamiliar or non-standard problems; "rules of thumb" for effective problem-solving.
>
> **CONTROL:** Global decisions regarding the selection and implementation of resources and strategies.
>
> **BELIEF SYSTEMS:** One's "mathematical world view," the set of (not necessarily conscious) determinants of an individual's behavior." (p.364)

The importance of teaching problem solving heuristics in school mathematics, as a means of improving problem solving performance, is one of the aims of the study to be described in this paper. Heuristical methods for solving problems include plausible but uncertain actions of a general yet natural character.

Some researchers (Lucas,1974; Kantowski,1977) of mathematical problem solving have endeavoured to teach explicitly task-specific heuristics or general heuristics, in attempting to improve the student's problem solving competence via instruction. The hypothesis they tested was that teaching students problem solving heuristics would improve their problem solving performance. The findings of these research studies suggest that systematic instruction in specific problem solving skills enhances the problem solving ability of the students. The investigators were able to observe positive effects on students being taught to use heuristics. It is disappointing however that few of these studies offer testable evidence that the students learned or were able to transfer the taught heuristics to isomorphic or non-isomorphic problems.

In search for a comprehensive framework of heuristic instruction, Schoenfeld (1985b) identified three major issues regarding heuristic strategies. The first issue is that one of the primary reasons for the lack of success in problem solving instruction through heuristics is attributable to the fact that the strategies were very much unspecified. A second issue is that the description of different ways of implementing the strategies must be done in sufficient detail so that our students can learn to implement them. Schoenfeld argues that the complexity of the implementation process has been greatly underestimated. The fact that in most research studies, investigators did not pay sufficient attention to "fleshing out" such strategies in detail, could partially explain their failure in heuristic instruction. The third issue is related to the detailed specification of strategies:

> *"If each heuristic label represents a class of a dozen or so more finely defined strategies, then any collection of ten powerful techniques becomes a collection of over 100 specific strategies."* (p. 370)

It is evident from the foregoing discussion that teaching heuristics is not as simple as some investigators might have thought a few years ago. Teaching explicitly the strategies along with the underlying substrategies and taking under consideration the student's belief systems and metacognitive behaviour does indeed seem to produce the desirable results.

A different approach has been advocated by Sweller and his colleagues. Owen and Sweller (1989), question the emphasis placed by curriculum developers and mathematics education researchers, on the use of heuristics in problem solving, by arguing that effective problem solving performance is not the result of the use of superior heuristics but the result of students having acquired the appropriate schemata and automation of domain specific skills. A critical comment on Sweller's argument that schema acquisition and automation of domain specific skills are the most effective tools of skilled problem solving performance, has recently been offered by Davis (1992). He claims that good problem solving and the automation of domain specific skills are by no means equivalent, one reason being that such automation is, in many cases, necessary but not sufficient to solve a given problem.

I would argue that a Grand Unified Theory Of Mathematical Problem Solving (GUTOMPS), incorporating the heuristic approach based on a refined Polya model and the approach advocated by Sweller and his colleagues, are two of its constituent components. This is what is actually needed by the mathematics teachers who as practitioners are utilising elements of both theories on a day to day basis.

THE STUDY

The purpose of this investigation is to uncover information about and gain a greater insight into the processes involved in solving problems in year 12 (V.C.E.) Calculus. One of the main hypotheses of this study is that heuristics are techniques which enhance problem solving performance. It is therefore a logical inference that incorporation of heuristics into problem solving strategies is a valuable asset for the problem solver. A number of challenging questions follow: Is it possible to teach heuristics to Year 12 Calculus students? Can a teacher device a set of heuristics and then produce conditions so that students will be enabled to embed these plausible techniques into their problem solving schemata? Can V.C.E. students be guided to adopt more effective strategies than those they are accustomed to apply to problems? How does instruction on heuristics effect performance in terms of time, accuracy of solutions and elegance of solutions?

RESEARCH QUESTIONS

The following hypotheses are proposed by the investigator in the light of the previous discussion:

(a) Implementation of an explicit, carefully planned instruction program on heuristics based on Polya's four stage model can provide a vehicle for shifts in strategy on mathematical problems [in Calculus] among a group of year 12 students.

(b) These shifts in problem solving strategy can be observed by the application of a diagnostic instrument deviced by Schoenfeld (1985) and modified slightly by the investigator.

(c) Heuristic-oriented instruction can be incorporated into teaching methods within the framework of the newly designed V.C.E. Mathematical Methods and Specialist Mathematics courses to be implemented in 1994 in Victoria.

(d) Heuristic-oriented instruction will enhance problem solving performance in terms of time, frequency of errors, score, and amount of difficulty.

RESEARCH METHOD

A qualitative methodology (thinking aloud) was employed in this study. The verbal and written protocols of six (three male and three female) Year 12 mathematics students are currently being analysed in detail for regularities in observable sequences of behaviours. Analysis will be made of the various stages of the development of problem solving in elementary Calculus. The instruction phase comprised a major part of the study. "Heuristic teaching" and how it effects problem solving is being considered. The main study was conducted in three phases during the third term of 1990. It included a pretest, the instruction in calculus via heuristics and a posttest. Protocols were recorded during the solution of pretest problems to determine processes used by the students and to enable the investigator analyse possible traces of any (pre-existing) heuristic techniques used by the students, before the instruction phase was initiated. During the instruction phase, the content of Calculus was introduced. During the posttest phase students solved task problems similar to those in the pretest, and protocols were recorded in order to trace the use of heuristics and investigate any improvement in problem solving during its evolution.

PLAN OF ANALYSIS

Categorisation of the interview data

A variation of Schoenfeld's framework for the macroscopic analysis of problem solving protocols (Schoenfeld, 1985) will be used to categorise the transcribed protocols of the students. The written protocols will be parsed into episodes of problem solving behaviour. The episodes and the associated questions [relevant to that type of episode] in each one of them will be matched to the heuristics used in the study. A full analysis of a protocol is obtained by anwering the questions associated with each episode. The episodes used in this study are listed below:

- PRE-READING ACTIVITY
- READING
- ANALYSIS
- EXPLORATION

- UTILIZATION OF NEW INFORMATION AND LOCAL ASSESSMENTS
- PLANNING-IMPLEMENTATION
- VERIFICATION (LOOKING BACK)
- TRANSITION BETWEEN EPISODES

A sample of the questions associated with one of the episodes follows:

NEW INFORMATION AND LOCAL ASSESSMENTS (NILA)

New information points include any previously unnoticed piece of information and they also include the mention of potentially valuable **heuristics** (new processes, new approaches). Local assessment is an evaluation of the current state of the solution at a **microscopic** level.

NILA (1) Does the problem solver assess the current state of his knowledge? (Is it appropriate to do so?)

NILA (2) Does the problem solver assess the relevancy or utility of the new information? (Is it appropriate at this stage?)

NILA (3) What are the consequences for the solution process of these assessments or the absence of them?

PRELIMINARY FINDINGS

The preliminary findings of the study reveal that the students devoted more time in reading the problem questions and planning during the posttest rather than in the pretest. The total time devoted to solving each problem was reduced for most students in the posttest. Finally the students appeared to be more consistent during the posttest in terms of monitoring their progress, by attempting to make local assessments during the solution process.

REFERENCES

Davis, G. (1992). Cutting through chaos: A case study in mathematical problem solving. ***In W. Geeslin and K. Graham, eds., Proceedings of the Sixteenth PME Conference (Durham), Vol. 1.*** *(pp.177-184) Durham: University of New Hampshire Department of Mathematics.*

Kantowski, M.G. (1977). Processes involved in mathematical problem solving. ***Journal for Research in Mathematics Education, 8,*** *163-180.*

Lester, F. K. (1980). Research on mathematical problem solving. In R. J.Shumway (Ed.), ***Research in mathematics education*** *(pp. 286-323). Reston, VA: National Council of Teachers of Mathematics.*

Lucas, J. F. (1974). The teaching of heuristic problem-solving strategies in elementary calculus. ***Journal for Research in Mathematics Education 5,*** *36-46.*

Owen, E., & Sweller, J. (1989). Should problem solving be used as a learning device in mathematics? ***Journal for Research in Mathematics Education, 20,*** *322-328.*

Schoenfeld, A. H. (1985ba). ***Mathematical problem solving.*** *Orlando: Academic Press.*

Schoenfeld, A. H. (1985b). Understanding and teaching the nature of mathematical thinking. In I.Wirszup & R. Streit (Eds.), ***Developments in mthematical thinking around the world*** *(pp.362-368). Reston, VA National Council of Teachers of Mathematics.*

Sweller, J., & Low, R. (1992). Some cognitive factors relevant to mathematics instruction. ***Mathematics Education Research Journal, 4*** *(1), 83-94.*

CASE METHODS AND TEACHER CHANGE: SHIFTING AUTHORITY TO BUILD AUTONOMY

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This study investigates how math case methods support teachers' professional development by shifting their perception of authority from external to internal and collective sources. The primary data include transcripts of case discussions and interviews, as well as math assessments of teachers. The findings demonstrate that case discussions provide opportunities for (1) realizing that capability and wisdom exist within the group, (2) developing a critical stance, and (3) developing stronger, more refined content and pedagogical content knowledge. Teachers that capitalize on these opportunities have a richer sense of their own autonomy.

Lisa freely admits that she was a "drill and kill" teacher, never straying far from the basics in the sixth grade text book. She blamed her lack of confidence on her weak mathematical background saying, "I never learned it when I was in school."

Lisa joined case discussions at the insistence of her fellow teachers, three of whom had participated in case discussions the year before. She rarely contributed to the early discussions. When she finally ventured into the conversations, she asked questions that others might have been reluctant to ask, such as, "I'm like the kid [in the case], I don't understand why 100% isn't 100", or "Why doesn't the teacher just tell them how to do it?".

Lisa's confidence and trust grew with each discussion. She began to tap her fellow teachers as resources for new ideas and materials. Instead of skipping percent this year, as she had always done previously, she did a six-week unit using the text as a guide supplemented by "real-world" problems and manipulatives. She prepared her unit collaboratively with Elena a sixth grade bilingual teacher. She even admitted to her students that she did not learn percent until she was an adult.

Lisa experienced dramatic changes in her beliefs and teaching practices, but there were no specific beliefs, approaches, or materials explicitly advocated by the case discussion program, nor was there an expectation of change stated as a program goal. As the designers of the case discussion program we ask the question: What invoked Lisa, and other teachers in her case discussion group, to initiate change?

The language used to discuss professional development, reform, or teacher empowerment often holds an implicit presumption that the developing, reforming, and empowering is done by someone other than teachers themselves. At one extreme, programs may actually predicate their goals on the assumption that teachers are resistant to change and must be prompted by accountability or top-down expectations. Other programs recognize teachers as professionals, but still require them to implement reforms that others think are best for them. Even change itself is viewed as something imposed on teachers. As Richardson points out, a critical feature in the literature on teacher change is that change, research based or otherwise, is defined as "teachers doing something that others are suggesting they do" (Richardson, 1990, p. 11). We propose that programs based on these assumptions may have unintended, and possibly undesirable, consequences. That is, they may be "disempowering for teachers, simply because [they] provide an external source of legitimation for change" (Kemmis, 1987, p. 82).

Lee Shulman (1986) first advocated adapting case methods from other professions to education. For the past six years, we have studied the use of cases as a professional development tool. We use cases written by classroom teachers, as stimuli for discussion. One of our primary goals has been to help teachers recognize themselves as their own change agents, both within the program and in their day to day experiences as teachers.

To begin, let us explain what we mean by a case. A case can be either framed as "an instance of exemplary practice" or it can carry "no presumption that the case itself illustrates either exemplary or ineffective practice" (Merseth, 1991 p. 2). Our cases follow the latter model (Barnett, Goldenstein, and Jackson, in press). They are narratives designed to promote discussion of significant issues, portray a variety of teaching strategies and philosophies, and highlight the complexity, rationality, and flaws in student thinking. They were written by fourth through eighth grade teachers about the surprises, perplexing situations and dilemmas encountered while teaching rational numbers.

The structure of case discussions is designed to maximize teacher direction and control. The discussions are lead by a facilitator whose role is to ask teachers to clarify and elaborate their ideas, justify their positions, and critically examine alternative perspectives. Teachers frame the issues to be discussed, decide where the discussion should start, and assess the process and content of the discussion. The aim is for discussion groups to experience a gradual shift from predominately facilitator-guided discussion to more teacher-guided discussion, and ultimately for the teachers themselves to become their own facilitators.

Since its inception, this program has also relied on teachers to be involved in decision making and control at a broader level. Teachers serve as advisory board members, help coordinate and plan the logistics of the program, write and revise cases based on field testing results, and contribute to the design of the research component. They are now taking leadership roles by learning to facilitate case discussions.

We believe that the structure of the case discussions and the design of the case discussion program itself helps shift teachers' perceptions of authority from external sources to internal and collective sources. However, our observations indicate that something more fundamental also contributes to the changes in ways that teachers are thinking about their roles, both within the discussion group and with their colleagues and students outside of the discussion group. This study identifies and describes other aspects of case discussions that influence the ways teachers perceive their own authority and establish autonomy as individuals and as a group.

THE STUDY

This study is part of an on-going research program at Far West Laboratory which examines the effects of case writing and case discussion participation on teacher knowledge, beliefs, and practices (Barnett and Tyson, 1993; Barnett and Sather, 1992; Barnett, 1991; Shulman, 1990, 1991, 1992). The purpose of this study was to learn more about the nature of case discussions as stimuli to professional growth. Specifically, this is an interpretative study (Erickson, 1986) addressing the following question:

Research Question

How do case discussions contribute to a shift in locus of authority and to the development of individual and group autonomy?

Data Collection

This study is based on many sources of data collected over a period of two years from 20 teachers who participated in case discussions for 1 or 2 years. The discussions were generally held once a month after school, for a total of 30 hours during each school year. Our data include the following:

1) videotapes and/or audio-tapes of all case discussions, as well as transcriptions of these tapes
2) videotapes of three teachers' reflections about their own teaching shown on videotape
3) pre-assessments and post-assessments of teachers' understanding of rational numbers (for 20 teachers)
4) structured oral interviews, each lasting 45 minutes to 1 hour (for 20 teachers)
5) researchers' notes from all case discussions.

This report will focus primarily on a subset of the data collected from a case discussion group formed by seven teachers. We targeted this group because they were all from the same school site and could experience collegial opportunities outside of the case discussion group. This school serves a high concentration of underprivileged students and ethnic minorities, and includes a large Latino population and many Asian immigrants who have limited English proficiency. The teachers are also a diverse group, representing a range of teaching experience (from 2 years to more than twenty), professional development exposure (from none to extensive), and grade levels (from fourth to sixth).

Findings and Discussion

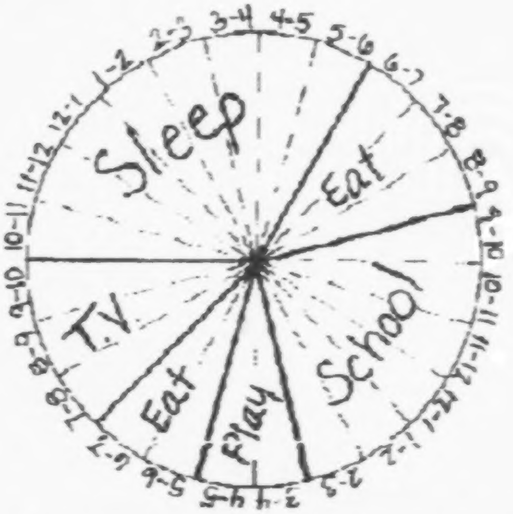
To investigate the ways that case discussions contribute to a shift in locus of authority and to the development of individual and group autonomy, we analyzed transcripts from case discussions and interviews with teachers. We looked for ways that autonomy and authority changed over the course of the year and considered what aspects of the case discussions might be responsible for these changes. From this analysis, we determined three ways that case discussions contribute to teachers' perceptions of their own authority and thus build autonomy. Case discussions provide opportunities to:

- realize that capability and wisdom exists within the group
- develop a critical stance
- develop stronger and more refined content and pedagogical content knowledge bases.

To illustrate how teachers' individual and collective autonomy is influenced by these different aspects of case discussions, we have constructed a narrative that includes excerpts from one case discussion. This discussion was selected because it contains several vivid examples that illustrate our findings. It took place February, 12, 1992 and was the fifth case discussion for this group. We realize that by summarizing data and condensing events we may misrepresent the conversation among teachers, but we have tried to preserve the essence of the discussion.

SIX HOURS ISN'T ONE SIXTH OF A DAY: EXCERPTS AND COMMENTARY FROM A CASE DISCUSSION

In this case the teacher-author writes about a lesson in which she asked her students to make circle graphs representing the amount of time they spend on activities during a day. The circles given to the students were divided into 24 parts, one part for each hour of the day. The case included a circle graph that one of her students drew. The teacher asked the students to name the fractional part of the graph that was spent conducting each activity.



Teachers in the discussion group raised several issues with regard to this case. One of the issues discussed was why students in the case had difficulty seeing that the section of the graph representing time spent in "school" was one fourth of a day. Sue argued that it is "much more visible" (p.2) to start with a circle divided into fourths and figure how out many twenty-fourths it represents than to see one-fourth imbedded in a circle "presubdivided" into twenty-fourths. ("Presubdivided" was a term coined by this group and used frequently in their case discussions.) After some discussion, most teachers agreed with Sue, even though the idea seemed counterintuitive at first.

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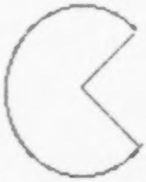
The case discussion participants also speculated that maybe students couldn't name the one-fourth because they were accustomed to the text book rule of counting the number of equal parts to determine the denominator, and the number of shaded parts to determine the numerator of a fraction. If students focused on the unequal sized parts representing time spent on the different activities, they would be unable to name the fraction based on the textbook rule that they had learned.

This prompted Sue to question whether or not unequal parts could even represent fractions. According to a definition she had been using, the parts had to be equal or it wasn't a fraction. This began a serious debate. Sue went to her classroom to get some examples from a resource book she had been using with her students.

Returning to the group, Sue described how she had shown her students different drawings and asked them to determine if the drawing represented a fraction and what fraction it represented, based on the definition from the book. The drawings were not the usual textbook drawings of circles and rectangles, and one drawing in particular (shown below) stimulated a spirited debate among her students (transcript 2/12/92, pp. 7-8).



Some students claimed that the drawing represented three-fourths, some that it represented three-thirds, and others that it wasn't a fraction at all, but a whole. Sue reported that she had accepted all of these answers because the students were able to defend their answers based on a three-part definition that she had presented earlier to the class.

In response to Sue's "classroom case", Nancy and Jim discussed whether or not Sue should "allow" her students to consider this drawing as a whole, even though her students had justified their assertion saying that it was a "whole PacMan."

Nancy: I'd let them go ahead and see it as a whole with a part missing, but I would be really happy when they saw the PacMan shape as the whole. Therefore, they're getting themselves out of that rut saying the whole has to be the circle.

Jim: I have a problem with the directions of this thing and maybe it's because I'm simpleminded, but I want my students, to get something firmly fixed in their mind which is this visual sense of what fractions might be. What I'm afraid of is if I've led them through this, their original visual sense that this represents three-quarters of something is going to be confused by a lot of these things that will leave them less intuitively able to solve problems than they were beforehand. I don't think very many of them are going to be the brilliant intellects that will be able to apply a definition and a procedure and so on to a new problem and come out with a creative answer.

Facilitator: So what would you recommend?

Jim: Well I would recommend staying away from irregular shapes, giving them a variety of regular shapes divided into fractions or relatively regular numbers, not giving them these ambiguous things.

Sue: But somehow we want them to get the concept of the one, and that's all my discussion was.

Jim: But I would not show students [irregular shapes] at this point, except for a handful, like the PacMan one.

Sue: Maybe it does depend upon the level of the student, but I feel that at some point it's good to get them out of that thinking that the one square is the whole, the one circle is the whole, or the one long strip.

Lisa: Maybe they're not ready for [irregular shapes]. Maybe if we worked to get 4th graders to understand the parts, then by the time we get to 6th grade we could take them farther. I see what Jim is saying. . . But they need to have some kind of understanding, and if they don't have that basic understanding, then how can they know what one-fourth is? Why is that a quarter? Because I'm the teacher?

The facilitator sensed that Elena was troubled and asked her to comment. Elena referred back to Jim's earlier statement that not very many of these kids are going to be "brilliant intellectuals." She was concerned about saying that "this is as far as these kids are going to go." As a Latina, teaching in a school with a high concentration of Latino students, and as one who is dedicated to addressing equity issues in the Latino community, this was salient for Elena. So she raised the issue for the group to consider. There was discomfort among the group members in talking about this issue. Jim reassured Elena that it was not his intention to underestimate the students' abilities. Nevertheless, the conversation prompted them to realize that this issue might need further attention. As a result they discussed the issue informally after the case discussion and decided to initiate case discussions about equity issues in the future.

The conversation turned back to a safer topic as everyone joined the deliberation about what represents a whole and what represents a fraction. They debated examples from Sue's resource book and referred back to examples from earlier cases. Although there was no clear consensus about the teaching implications, there was an awareness that something that looks like a part, might be considered a whole depending on the situation. For some teachers, using ambiguous examples with students was desirable, for others it was questionable.

Sue was clearly unsettled by this discussion. She was questioning the role of a definition, her uncritical use of the resource book, her colleagues' opinions, and her own understanding of fractions. Near the close, she reflected back on the case discussion.

Sue: I don't know how many times I've heard people say things, and I go wait a minute! I looked at this and I said oh boy, and I didn't look at it with a discerning eye. I questioned the circle [graph] myself, and it doesn't fit the definition. And I was telling Elena I'm the kind of person who has to plow through it [the resource book]. I accept it when it's written, and that's the way I'm going to go with it. p 21But right now I am getting a little anxiety. I'm pushed and pulled about it.

This discussion, like many others, closed with unsettled issues that continued to arise in subsequent discussions. It is clear from the complete transcript of this case discussion that the teachers had begun to rely on each other, rather than the facilitator or an external resource to frame, examine, and resolve issues. In the sections below, we will discuss how the substance of case discussions like this one contribute to an internalized sense of authority for individuals and the group. Although each aspect is discussed separately, they are likely interrelated.

CAPABILITY AND WISDOM WITHIN THE GROUP

Our analysis leads us to believe that case discussions can lead to a professional community among teachers by providing them with opportunities to realize and rely on the capability and wisdom of the group. As the discussions progress, they began to develop similar, though not identical, philosophies and values, a common language, a set of shared experiences, and mutual goals.

The professional community evidenced by the above case discussion and subsequent discussions was demonstrated in many ways. Teachers began to co-plan, test, and revise instructional units together. They talked to each other about content or methods that they didn't understand. They developed what Ingvarson and Laughran (1992) call a "mutual accountability", where they questioned and supported each other about issues such as expectations of students, curriculum articulation among different grade levels, and concerns about achievement tests.

DEVELOPING A CRITICAL STANCE

The model for discussing cases is designed to elicit thoughtful criticism. The facilitator challenges teachers --- and the teachers challenge each other — to justify their responses, to examine both the desirable and undesirable consequences of recommendations, and to identify the constraints and limitations of different ideas. By becoming better informed of both sides of the issues, teachers decide for themselves based on their own experiences and situations.

In the beginning, the facilitator had to probe for critical analysis and ask for different perspectives. Through the course of the discussions, we found evidence that teachers developed an increasing predilection to offer critical analysis of the cases, of each other's comments, and of their own teaching. This process was not spontaneous for teachers, possibly because they needed to first develop an environment of trust, were anxious to portray a "politically correct" position, or because they were uncomfortable with the authority role that criticism requires. As illustrated in the case discussion above, teachers had developed the trust within the group to expose their own teaching struggles for criticism and to confront delicate ethical issues.

CONTENT AND PEDAGOGICAL CONTENT KNOWLEDGE

In earlier studies we found that discussing a collection of cases in a relatively narrow domain of mathematics learning, such as rational numbers, enhances both teachers' content and associated pedagogical content knowledge bases (Barnett, 1991; Barnett & Sather, 1992). We do not think that the same results would have occurred in this study if teachers had not learned more about rational number teaching and learning from the cases. In the case discussion above, teachers struggle to understand the meaning of a fraction, the arbitrary nature of the whole, and how the mathematics might look from a student's point of view. With each discussion their understanding of rational numbers becomes deeper and more refined. This study suggests that strengthening content and content pedagogy may be essential in developing a sense of internal authority and autonomy. Stronger content and pedagogical content knowledge also appears to contribute to the development of a critical stance. If one's own knowledge is "shaky", how can one confidently analyze someone else's work or comments? It may also appears to increase risk-taking which is closely tied to autonomy.

CONCLUSIONS AND IMPLICATIONS

In this paper we presented an overview how the structure of the case discussion program allows teachers to take more ownership in their own professional growth. The subject of our investigation was how other, subtler aspects of the case discussions might have contributed to teachers' perceptions of authority and the development of autonomy. We identified three ways that this occurs: the realization that capability and wisdom exists within the group, the development of a critical stance, and the development of a stronger and more refined content and pedagogical content knowledge base.

We argue that in addition to changing the structure of a professional development program to offer more opportunities for teachers to be in charge, teachers need opportunities to enhance their pedagogical knowledge. We agree with the conclusions of Wilcox, Lanier, Schram, and Lappan (1992) that, "One of our biggest challenges may lie in how to develop in preservice [and inservice] teachers a disposition to ask critical questions—about curriculum, instructional practices, educational policies, testing, their own learning and that of others" In our view the development of professional judgment and teacher empowerment may depend on creating environments where teachers learn to become their own authorities, holding themselves mutually accountable and responsible for what occurs in their classrooms.

REFERENCES

Barnett, C. (1991). Building a case-based curriculum to enhance the pedagogical content knowledge of mathematics teachers. *Journal of Teacher Education, 42*(4), 263-272.

Barnett, C., Goldenstein, D., & Jackson, B. (1993) *Dilemmas of teaching: Math cases to promote inquiry, discussion, and reflection.* Unpublished manuscript.

Barnett, C. & Tyson, P. (1993, April). *Mathematics teaching cases as a catalyst for informed strategic inquiry.* Paper presented at the annual meeting of the American Education Research Association, Atlanta.

Barnett, C. & Sather, S. (1992, April). *Using case discussions to promote changes in beliefs among mathematics teachers.* Paper presented at the annual meeting of the American Education Research Association, San Francisco.

Erickson, F. (1986). Qualitative methods in research on teaching. In M. C. Wittrock (Ed.), *Handbook of research on teaching* (3rd ed.) (pp. 119-161). New York: Macmillan.

Ingvarson, L. & Loughran, J. (1992). Loose connections: The content of science teachers' work. The Science Education Professional Dvelopment Project.

Kemmis, S. (1987). Critical reflection. In M. F. Wideen & I. Andrews (Eds.), *Staff development for school improvement a focus on the teacher* (pp. 73 - 90). New York: The Falmer Press.

Merseth, K. (1991). *The case for cases in teacher education*. Washington, DC: American Association of Higher Education and the American Association of Colleges for Teacher Education.

Richardson, V. (1990). Significant and worthwhile change in teaching practice. *Educational Researcher, 19,* 10 - 18.

Shulman, J. (1992, April). *Tender feelings, hidden thoughts: Confronting bias, innocence, and racism through case discussions.* Paper presented at the annual meeting of the American Education Research Association, San Francisco.

Shulman, J. (1991). Revealing the mysteries of teacher-written cases: Opening the black box. *Journal of Teacher Education, 42*(4), 250-262.

Shulman, J. (1990). Now you see them, now you don't: Anonymity versus visibility in case studies of teachers. *Educational Researcher, 19*(6), 11-15.

Shulman, L. (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher,15*(2), 4-14.

Wilcox S., Lanier, P., Schram, P., & Lappan, G. (1992). *Influencing Beginning Teachers' Practice in Mathematics Education: Confronting Constraints of Knowledge, Beliefs, and Context.* East Lansing, MI: The National Center for Research on Teacher Education.

METACOGNITIVE CONTEXT CUING IN MATHEMATICAL THINKING AND MATHEMATICS EDUCATION.

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Metacognition is here characterised by three themes: i) 'thinking as the object of thought', ii) 'thinking regulating thought' and iii) 'the feeling of knowing'. Traditional formulations of metacognition are critisiced on their theoretical, operational and applied bases via these three themes, mainly because they denigrate the crucial role of affect in metacognition. Metacognition is more usefully defined as double context cueing: firstly to an appropriate ego-state of awareness, and secondly to a target/content emotional state with appropriate knowledge and ability response tendencies. Examples are discussed of metacognitive context cuing by internal/external co-constructed contexts, internal contexts, finely differentiated contexts and restrictive contexts that inhibit mathematics education. The necessary role of metacognitive context cuing in mathematical thought is discussed. The results of a research exploration into metacognitive context cuing are presented, including a simple credibility index for Likert responses. Directions are suggested for further research applications in mathematics education.

Critique of the Current Formulations of Metacognition

Flavell, in 1971, was one of the first to use the term 'metacognition' and later defined it as "knowledge or cognition that takes as its object or regulates any aspect of cognitive endeavor" (Flavell, 1978, p. 4). Later researchers have elaborated on Flavell's two themes - thinking as the object of thought, and thinking regulating thought. So Moshman (1979) distinguishes metacognition as thinking about one's theories as opposed to thinking with one's theories. Tyler (1983) further elaborates "This is to do with the thinking one does about thinking, the evaluation of one's own abilities and mental states, the selection of strategies to meet the task goals, and the monitoring and regulation of progress . ... Metacognition is becoming an increasingly popular topic for research." (p. 123).

Operational Restrictions - Down to the Choice of Strategies

Operationally Flavell's second theme has now degenerated. The 'regulation of thought' has operationally become 'the choice of strategies'. Kuhn, Amsel and O'Loughlin (1988) remark on this operational emphasis: "Most studies of metacognition are studies of cognition about how to learn and remember or studies of cognition involved in choosing and monitoring strategies to solve problems. Yet there remains a whole, relatively unstudied and very significant realm of metacognition consisting of cognition about what one knows one knows and how one knows it (though not in any formal, epistemological sense). " (p .7) The "how one knows it" introduces a third theme to metacognition, 'the feeling of knowing or one's confidence in one's answers'.

Non-Affective Strategies for Facile Tasks

Since then we have met-a lot of metacognitive sub-categories such as metacognitive capacity, metacognitive knowledge, metacognitive activity, metacognitive strategies, metacognitive frameworks, metacognitive skill, metacognitive judgments, metaconsciousness, metamemory, etc. (Pinard, 1992; Borkowski, 1992; Slife & Weaver 1992; Weisberg, 1992). However, my criticism still applies that the regulation of thought entailed in these sub-categories has been operationally reduced to the mere selection of strategies. Further, the strategies operationalised frequently refer to facile forms of thinking more fitting to machines than to human cognition e.g. closed problem solving which minimises the role of affect, such as one to five step arithmetic word problems solvable by selection from the strategies taught (Melot & Corroyer, 1992; Montague, 1992; Okamoto, 1992).

Predominantely, studies in metacognitive regulation have avoided the generation and cuing of affect, and ignored other preconscious involvement in what they have studied - even though the regulation of cognition is so dependent on this preconscious involvement. For example Fischbein writes of the role of metacognition in mathematics education: "One of the fundamental tasks of mathematical education - as has been frequently emphasized in the present work - is to develop in students the capacity to distinguish between intuitive feelings, intuitive beliefs and formally supported convictions. ... Through metacognitive techniques the student has then to learn to see clearly the formal properties of the mathematical concepts used and to understand the intuitive sources of his misconceptions." (Fischbein, 1987, pp. 207,209) Unfortunately the same cognitive-affective structures may be responsible for both understanding the formal concepts and for being the source of the misconceptions. I believe researchers in metacognition need to turn their attention to the role of these cognitive-affective structures.

Who's Hiding Humunculus? - ad hoc 'Theory' and the Infinite Regress of Reductionism

Currently, metacognition is one of Psychology's hardest hit victims of reductionism. The mind is a bunch of functions - one for this and one for that - and one for executive control, regulating thought, telling all the others what to do. When we analyze the executive control to find which part controls the control we generate ad hoc new parts whose names are merely the description of their functions, and these functions generally ignore the interactions between the parts. Remember, we met-a lot of these parts above - the invented sub-categories of metacognition like the metamemory that reminds you to remember, but what reminds the metamemory to remind you? This is the infinite regress of reductionism, the ghost in the machine, the humunculus man inside the man, Plato's problem of who guards the guards, what monitors the monitor. Its metacognition pretending to explain cognition regulating thought and failing. Perhaps we could exchange one humunculus for a parallel processing model with cognitive-affective input (incorporating perceptions of small internal and external changes), where control was a partial feedback distributed function of the accommodating structure.

Tools of the Trade - Task-Talking Confounds the Process it Probes

Many of the research conclusions about metacognition are derived using the prevalent technique of task-related talk (Meloth, M.S. and Deering, P.D. 1992). The subject talks about the task while preforming it. This task-talking is then used as a window onto how the subject's thoughts directed the performance. What this ignores is the considerable confounding interaction between task-talk and tasks of increasing cognitive-affective complexity. Try dancing while explaining each step and movement of the body. Try giving a running commentary during sexual intercourse. In her review article on the mechanisms of metacognition Ann Brown (1987) notes that: "Asking subjects to report on internal events that are not readily available to such inspection may significantly impair the processes on which they must report. ... Requiring an expert golfer to describe his or her swing may, in fact, ruin the expert's swing, rather than help the novice." (pp. 76-77).

Misguided Application - Conscious Monitoring Paralyzes Expert Performance

An aim of this applied research is for a person who has been taught metacognitive strategies to be able to improve their task performance by consciously choosing the most appropriate strategy of which they are aware. However, rather than improving performance, this intrusive conscious monitoring to initiate conscious intervention paralyzes that powerful efficient intuitive spontaneity of reflexive thought and action we observe in expert performance. Consider: this concrete block is falling on my head. Thinks "shall I step back, or jump forward, or shall I quickly move to the side. Perhaps its time to just stand here and take this opportunity to end it all."

Thinking about Thinking - Next 'Thinking About Metacognition' the Sudy of Hyper-Metacognition?

Metacognition is cognition at a higher (meta) level of cognition. So there is no need to assume or investigate a new kind of cognition. As Flavell (1978) originally said it is still cognition but 'thought' is now the object of that cognition. This non-affective aspect of metacognition hardly warrants a new name. It warrants even less a field of study to itself.

Feeling of Knowing and Confidence in Being Right - Extensively Studied in Research on Intuition

Some researchers emphasise that third theme 'the feeling of knowing' and 'confidence in the correctness of an answer' as metacognitive abilities (Costermans, Lories and Ansay, 1992; Reder & Ritter 1992; Okamoto, 1992). My criticism of this research is that it has been done to much greater effect in research on intuition and insight (Bastick, 1982). Extensive measures of feeling of knowing and confidence in correctness of answer were used by Bastick (1979) in his studies of intuition and insight - e.g. Likert confidence ratings, narrowness of double estimates, variations in subject's physiological responses (gsr, heart rate, respiration etc.) on threat of electric shock for incorrect answers, etc. However, insight studies are not consistent with current formulations of metacognition and there have been logical and empirical criticisms of recent attempts to combine the feeling of knowing aspect of metacognition with experiments on insight problems (Weisberg, 1992). Researchers need some new theoretical formulation of metacognition that encompasses these important cognitive-affective aspects.

REDEFINITION OF METACOGNITION

I would like to more usefully redefine metacognition as context cuing of internal state learning via ego-state control. That is, my two themes are i) the cuing of ego-state control to elicit an appropriate state of awareness on the continuum from focussed to regressed and ii) the cuing, at that level of awareness, of a target state with appropriate learned responses.

This definition has two important commonalities with traditional metacognitive research. Firstly, in reviews of the literature on context cuing of cognition it is usual to distinguish between cuing by the physical environment in which learning tasks place - the external context - and the cuing due to the subject's own emotional state - the internal context (e.g. Davies, 1986). We will find it useful to take into account that, from the prospective of the subject, internal and external contexts co-construct, but for some purposes of analysis we can maintain the traditional separate identifications of internal and external cuing contexts. Secondly, a useful intention of metacognition, as opposed to its restricted practice, is to study conscious, reflective awareness and deliberate control of the "higher intellectual functions." (Kuhn, et al. 1988, p. 32). I would retain and widen this intention and use the term 'metacognition' to mean the initiation and direction of higher cognitive-affective states and responses (like curiosity, insight, intentions, expectations, facts and strategies etc.) which, through practice, may become the reflexively initiated and directed components of expert performance.

Internal/External co-Constructed Ccontext C+uing

We are very familiar with the fact that "thinking occurs in a context that significantly shapes its form and expression" (Kuhn, et al. 1988, p. ix) However, it is not only extreme contexts that evoke extreme emotional states like love, hate, fear or jealousy, for, as we shall see, contexts precisely differentiated - e.g. by the subtle nuances of synonyms - can evoke different emotional states in those sensitive to these differences. Dolan (1983) gives examples which illustrate what I have called the co-construction of internal and external contexts, their cuing of different states of awareness and the cuing of target emotional states with their associated learned responses:

> For example, handing a hungry person his favorite kind of cookie or sandwich may elicit an externally observable response of reaching for the food, putting it up to his mouth, and beginning to chew. Internally, the cookie or sandwich may be eliciting certain associated feelings, remembered conversations, visual images, etc., connected to previous experiences with that particular object. ...When people are feeling ill, they tend to respond favorably to childhood foods associated with feelings of comfort and security. ... listening to music that was popular during the time of one's youth or a couple's courtship often functions to elicit happy feelings. Conversely, Vietnam veterans suffering from Post-Traumatic Stress Disorder sometimes evidence an unpleasant associational response to stimuli that remind them of events they experienced in combat. (pp. 75-76)

This context cuing is also so familiar to us from TV advertisements that associate benefits and preferred life-styles with brand names so that either context cues the other.

Internal Context Cues

Anxiety - test anxiety, deadline anxiety - is a common internal cuing context. Rodin's 'The Thinker' illustrates another common example: using internal contexts of muscle sets to cue ego-control to appropriate states of awareness. We scratch our heads to puzzle. We pace the room to ponder. These learned muscle sets cue and maintain a state of awareness appropriate to resolving puzzlement.

Maladaptive Choice of Restrictive Context Cues

From research on creativity, what is particularly noticeable about some famous examples of contexts chosen to cue ego-control to a level of awareness suitable for creative work, is that they are unfortunately maladaptive in that they restrict the amount of work that can be done - and hence, presumably, are difficult for the creator to change. A well known anecdotal example is of the fiction writer who needs to get blind drunk to tempt the muse. Then to stare at the blank sheet of paper in the typewriter to evoke content responses. Howard Gardner tells us about: "Richard Wagner, who had to work himself up into nearly psychotic frenzy before finally finding himself able to put pen to score" (Gardner, 1982, p. 123). Similarly Rosen, (1953) reports on the psychoanalysis of a mathematician with: "the marked inhibition of any desire to work at mathematics during the daylight hours " (p. 123). Famous mathematicians also used such contexts to cue states of awareness suitable for their work. Newman, (1956) reports that the mathematicians Helmholtz and Poincaré worked sitting upright at a table. Hadamard paced the room. J. Teeple (chemist) took two baths. Shaw ate vegetables (p. 2039).

Nearer our own experience are the context cuings of ego-control we find in mathematics classrooms. "Sit up straight at your desks and pay attention" focuses awareness. "Children, come and sit with me on the story mat" cues a regressed awareness. The non-verbal cues teachers individually use for their class management are examples of context cuing for their children. What is important for mathematics education is that, unfortunately, these context cues are implicit and not directly transferable out of that teacher's classroom. So we severely restrict our children's mathematical thinking because they become dependent on these context cues in the mathematics classroom but the children can not independently initiate them for themselves or use them out of the classroom.

Rituals and habits developed over a lifetime - habits like Kant's obsessively punctual routine - are commonly used for context cuing. Some affect-laden context cues for target states are so powerful that they directly cue both the ego-control to the appropriate level of awareness and also cue the target task/content relevant state. Marcel Proust's, now famous description of how a 'madeleine' biscuit and cup of tea evoked multimodal memories from his childhood with his aunt, is an example of how a powerful context can double cue (Proust, 1928, p. 54,58). In comparison to Proust's real life affect-laden multimodal example, many experiments on state dependent learning use tasks involving a minor modality and little affect - such as arithmetic or simple recognition - and those concerning internal states commonly use drug induced internal states (Slangen, 1991). These designs do not fully utilise affect and learned cuing. We would expect greater affect-dependent cognitive tasks to show greater experimental effects as Davies' (1986) review of the literature indicates.

Cuing by Finely Differentiated Contexts

Examples of finely differentiated context cuing are the target emotional sets evoked for the interpretation of language nuances. There are cultural cues that indicate regression is the appropriate mode of perception. Forms of religious observance and the use of unusual grammatical structures or archaic (poetic) words can cue a regressed, noncritical mode.

Any small difference in a context that can be sensed may act at a cue. "differences in the structure of stimuli do lead to different forms of processing." (Shepp, 1978, p. 123). Hermeneutic and discourse analysis shows us the possible cuing effects of even finer differentiations within the contexts of scientific literature. "The various scientific disciplines are differentiated by their respective discursive practices, or the modes of discourse propagated by scientific readers and writers." (Frawley, 1986, pp. 75-76). I can give as examples: i) The use of 'organism' when referring to 'people' which cues the reader into a behaviorist interpretation set; ii)

The choice of the word "input" in the following quote, instead of 'a stimulus field' or 'an apperceptive field' cues the 'information processing' interpretation set in the sensitised reader rather than a psychoanalytic or behavioural one; "the older child perceives input in an articulate, specific, or differentiated fashion." (Shepp, 1978, p. 123).

METACOGNITIVE CONTEXT CUING IN MATHEMATICAL THOUGHT

Ego-states required for mathematical thought range from the extremely regressed to the highly focused. Regressed states allow for heavily affect-weighted cognition (primary process thinking). It is in regressed states that we are open to holistic ideas, solutions guides, the beauty and aesthetic of mathematics. It is in such an ego-state that mathematical discoveries are made. It is in such an ego-state that we can appreciate Einstein's praise of Bohr's theory of the atom as "the highest form of musicality in the sphere of thought" (Einstein, 1949, pp. 46-47). At the other ego-state extreme we have minimum affect-weighted cognition (secondary process thinking) suitable for precision in detail - e.g. exact step-by-step deductive logic. During any particular problem solving task we need to change our ego-state appropriately to source ideas, select and apply techniques. Busse and Mansfield (1980) review the work of Kris (1952) which describes this traditional view of ego-state change during problem solving :

> He (Kris, 1952) hypothesized that during this (inspirational) phase the ego temporarily loosens its control of thinking processes to permit a regression to a preconscious level of thinking. ... Kris believed that it is essential in creative thinking to temporarily abandon logical, rational thought that constricts thinking and prevents the formulation of new solutions. The person must deliberately allow day-dreams, fantasies, and the like to intrude upon his thinking. Later in an elaborational phase, the ideas are subjected to rigorous logical evaluation." (p. 32).

It is metacognitive context cuing of our ego-state control that gets us into the appropriate state of regressed or focussed awareness suitable for successive phases of mathematical thinking - then problem related contexts cue the relevant knowledge and ability responses (we hope). We have seen how muscle sets are a commonly used context cue. "I doubt if Einstein actually used his muscle fibers in his logic. He was rather referring to the feeling that he used to navigate his way through work. ... The organic senses are intimately involved in problem solving and send signals of impending success, unacceptable risk, and even unacceptable mediocrity." (Adams, 1986, pp. 75-76)

RESEARCH EXPLORATIONS - 'GETTING IN THE MOOD'

I have conducted some empirical research explorations to advance some tentative suggestions for future research designs and applications of metacognitive context cuing. 60 Australian, Education students aged 18-22 years old were asked how they got in the mood to do i) a task they liked, ii) a task they disliked and iii) a mathematics task. They were also trained in 'range answering' and asked to rate how much they liked or disliked the tasks and how easily they could do the exercises. Range Answering is a technique I have developed to assess the credibility of a Likert response - to assess, for example, a self-report of one's confidence in one's answer. Simply, if you asked two decorators for upper and lower estimates of how much it would be to decorate your house and the first said between $1,000 and $10,000 whereas the second said between $2,000 and $2,200 we can reasonable conclude from the smaller range that the second is more confident of what is involved. Range answering asks for an upper and lower response ("between what two numbers ..") rather than only asking for one number, which would have been the mean. Using the two numbers, this mean is readily calculated and in addition it can be weighted by the range to give a more credible index for subsequent statistical analyses. I used a -100 to +100 scale and an index of the mean weighted by scale/range i.e. $0.5|x_{u}+x_{l}| \times 200/|xu-xl|$ which suitable weights more confident responses for least squares type analyses - xu being the upper response, of course, and xl being the lower response.

It was expected that subjects would find the unpleasant exercise more difficult than the pleasant one. This was not the case. The stronger the affect involved in the task, whether positive or negative, then the easier it was to re-experience this for the exercises. It was common for subjects to use personal ritual for context cuing the state of awareness required for the task - e.g. ritualistic dressing preparations for partying, ritually displaying all books pens and papers in preparation for homework. The most common preparation for liked tasks was to imagine what was going to happen. This cued an enjoyable anticipation. The most common cuings to motivate preparation for unpleasant tasks were 'resignation', and 'fear of punishment'. It was culturally interesting that descriptions of states for liked tasks commonly used metaphors for 'freedom' and 'power' while those for disliked tasks were of 'wasted effort'. In agreement with Yapko (1984) it was also noticeable that subjects changed the major modality used for describing liked and disliked tasks: "the dominant modality changes from context to context" (p. 123).

Task 'objects' were also used for cuing a state of readiness e.g. sports equipment, a particular piece of music. Results duplicated Strauch's (1983) observations on context cuing that: "A song may stir pleasant memories, the sight of a particular individual might cause anger or fear, or the sight of blood may cause nausea." (Strauch, 1983 p. 123). For example, subjects reported using particular songs to cue the required moods, a girl intimated that no matter how she felt - even if she was crying - the sight of her lover's face always made her happy. An important finding for future research designs is that - in agreement with attribution theory - the same external context can be co-constructed with the current internal context to cue different states in different subjects. So we should not expect to utilise one context for cuing similar states in all subjects. The results also showed care must be taken when interpreting Likert scales. Named bipolar dimensions for rating stated feelings were not as would be culturally expected and their meanings were differently co-constructed - for example a subject might state a feeling as 'boring', then give positive and negative extremes as 'powerful' and 'deadly', whereas another subject who coincidentally stated the same feeling might describe it using different extremes.

SUGGESTIONS FOR FURTHER RESEARCH

The typical person senses no control over their seemingly random metacognitive context cuing. Aims of research on metacognitive context cuing should be to understand more about these potentially useful experiences for the purpose of applying this knowledge to enhance expert performance. Teachers and children need to be aware that these processes have been working below their conscious level. Even brief classroom observations will reveal many convincing personalised examples that can be shown to teachers and children. Facilitating ego-states need to be identified as do ways of cuing their initiation and maintenance.

Teachers could explicitly choose context cues that their children could use to initiate their own thinking in or out of the classroom. Maladaptive context cuing could be interrupted and replaced by some facilitating context utilization e.g. using cognitive redefinition (Riskind, 1989). Current techniques from hypnoses e.g.

facilitating regression from secondary to more primary process thinking (Levin, & Harrison, 1976, p. 400), attribute theory, state dependent learning e.g. training associational cues (Dolan, 1983, pp. 176-177), Neuro-Linguistic Programming, etc. can be extended to facilitate reflexive application of metacognitive context cuing for enhancing individual's chosen expert performances.

REFERENCES

Adams, J.L. (1986). *The Care and Feeding of Ideas: A Guide to Encouraging Creativity.* Reading Mass: Addison-Wesley.

Bastick, T. (1979). *Psychophysiological investigation of intuitive thought processes and creative mathematical problem solving.* Ph.D. dissertation, Department of Education, Brunel University, England.

Bastick, T. (1982). *Intuition: How we think and act.* Chichester: John Wiley.

Borkowski, J.G. (1992). A framework for teaching literacy, writing, and math skills. *Journal of Learning Disabilities. 25(4),* 253-257.

Brown, A. (1987). Metacognition, executive control, self-regulation and other more mysterious mechanisms. In F.E. Weinert & R.H. Kluwe (Eds.), *Metacognition, Motivation and Understanding.* (chap. 3, pp. 65-116). Hillsdale, New Jersey: Lawrence Erlbaum Associates.

Busse, T.V. & Mansfield, R.S. (1980). Theories of the Creative Process. *Journal of Creative Behavior, 14(2),* 91-103,132.

Costermans, J., Lories, G. & Ansay, C. (1992). Confidence level and feeling of knowing in question answering: The weight of inferential processes. *Journal of Experimental Psychology Learning, Memory, and Cognition, 18(1),* 142-150.

Davies, G. (1986). Context effects in episodic memory: A review. Special Issue: Context and cognition. *Cahiers de*

Psychologie, 6(2), 157-174.

Dolan, Y.M. (1983). *A Path With A Heart: Ericksonian Utilization with Resistant and Chronic Clients.* New York: Brunner/Mazel.

Einstein, A. (1949). Autobiographical notes. In P.A. Schilpp (Ed.), *Albert Einstein: Philosopher scientist* (pp. 1-93). Evanston, IL: Open Court.

Fischbein, E. (1987). *Intuition in Science and Mathematics: An educational approach.* Dordrecht, Holland: D. Reidel Publishing Company.

Flavell, J.H. (1978). Metacognitive development. In J.M. Scandura & C.J. Brainerd (Eds.), *Structural process theories of complex human behavior.* Alphen a.d. Rijn, The Netherlands: Sijthoff and Noordhoff.

Frawley, W. (1986). Science, Discourse, and Knowledge Representation: Toward a Computational Model of Science and Scientific Innovation. In M. Amsler (Ed.), *The Languages of Creativity: Models, Problem-Solving, Discourse. Studies in Science and Culture,* (vol. 2, pp. 68-91). Newark: University of Delaware Press.

Gardner, H. (1982). *Art, Mind, and Brain: A cognitive approach to creativity.* New York: Basic Books Inc.

Kris, E. (1950). On preconscious mental processes. *Psa. Quart., 19,* 540-550.

Kuhn, D., Amsel, E. and O'Loughlin, M. (1988). *The Development of Scientific Thinking Skills.* San Diego: Academic Press, Inc.

Levin, L.A. & Harrison, R.H. (1976). Hypnosis and Regression in the service of the ego. *The International Journal of Clinical and Experimental Hypnosis, 24(4),* 400-418.

Melot, A.M. & Corroyer, D. (1992). Organization of metacognitive knowledge: A condition for strategy use in memorization. *European Journal of Psychology of Education, 7(1),* 23-38.

Meloth, M.S. & Deering, P.D. (1992). Effects of two cooperative conditions on peer-group discussions, reading comprehension, and metacognition. *Contemporary Educational Psychology, 17(2),* 175-193.

Montague, M. (1992). The effects of cognitive and metacognitive strategy instruction on the mathematical problem solving of middle school students with learning disabilities. *Journal of Learning Disabilities, 25(4),* 230-248.

Moshman, D. (1979). To really get ahead, get a metatheory. In D. Kuhn (Ed.), *Intellectual development beyond childhood: New directions for child development,* (vol 5, pp. 59-68). San Francisco: Jossey-Bass

Newman, J.R. (1956). The Elusiveness of Invention. In J.R Newman (Ed.), *The World of Mathematics.* New York: Simon & Schuster.

Okamoto, M. (1992). The study of metacognition in solving arithmetic word problems. *Japanese Journal of Educational Psychology, 40(1),* 81-88.

Pinard, A. (1992). Metaconscience et metacognition. *Canadian Psychology, 33(1),* 27-41.

Proust, M. (1928). *Swann's Way.* New York: The Modern Library.

Reder, L.M. & Ritter, F.E. (1992). What determines initial feeling of knowing? Familiarity with question terms, not with the answer. *Journal of Experimental Psychology Learning, Memory, and Cognition, 18(3),* 435-451.

Riskind, J.H. (1989). The mediating mechanisms in mood and memory: A cognitive-priming formulation. Special Issue: Mood and memory: Theory, research and applications. *Journal of Social Behavior and Personality, 4(2),* 173-184.

Rosen, V.H. (1953). On mathematical illumination and the mathematical thought process. *Psychoanalytic Study of the Child, 8,* 127-154.

Shepp, B.E. (1978). From Perceived Similarity to Dimensional Structure: A New Hypothesis about Perceptual Development. In E. Rosch, & B.B. Lloyd (Eds.), *Cognition and Categorization.* Hillsdale, New Jersey: Lawrence Erlbaum Associates.

Slife, B.D. & Weaver, C.A. (1992). Depression, cognitive skill, and metacognitive skill in problem solving. *Cognition and Emotion, 6(1),* 1-22.

Strauch, R. (1983). *The Reality Illusion.* Wheaton, Ill: The Theosophical Publishing House.

Tyler, L.E. (1983). *Thinking creatively.* San Francisco: Jossey-Bass.

Weisberg, R.W. (1992). Metacognition and insight during problem solving: Comment on Metcalfe. *Journal of Experimental Psychology Learning, Memory, and Cognition, 18(2),* 426-431.

Yapko, M.D. (1984). *Trancework: An introduction to clinical hypnosis.* New York: Irvington Publishers, Inc.

TEACHING THE UNDERSTANDING OF MATHEMATICS: USING AFFECTIVE CONTEXTS THAT REPRESENT ABSTRACT MATHEMATICAL CONCEPTS

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This paper argues that we do not teach students to understand mathematics. We only teach them mathematics, and leave the understanding - or lack of it - up to them. The reason is that the affective aspects of students' mathematical experiences - feelings that are essential for understanding increasingly abstract mathematical concepts - are, perversely, continually reduced as the mathematics becomes increasingly more abstract. The two major causes of this anomaly are i) the mismatch between the structure of understanding mathematics and its 'logical' structure mirrored in mathematics curricula together with ii) the influence this has, in conjunction with Piagetian theory, on mathematics teaching. The role of inner affective contexts in organising students' mathematical knowledge, developing their mathematical intuition, in their learning and applications of mathematics is discussed. It is also considered how these organisational contexts are conceived and developed, and how to explicitly encouraged this during teaching children how to understand mathematical abstraction.

CONFUSING USES OF 'ABSTRACTION' IN MATHEMATICS AND IN MATHEMATICS EDUCATION

'Abstraction' in mathematics education has been discussed by Dienes (1989), Ginsburg and Asmussen (1988). This is not the same as 'abstraction' as used in mathematics (Nolt 1983). I argue that this currently preferred Piagetian practice in mathematics education is misguided by the confusions of 'abstraction', as used in mathematics and 'abstraction', as used in mathematics education. In particular, mathematics may be described by 'logical' hierarchies e.g. counting comes before addition, which is the foundation for multiplication, which leads on to algebra, which is the basis for ... and so on; or categorisation comes before sets, which are the foundation for groups, which leads on to number systems, which are the bases for ... and so on. These hierarchies are used to describe both mathematics curricula and the steps in understanding - and so teaching - these curricula. Sometimes these steps are described as minute sequential behavioural details, as in the literature on diagnoses and remediation in mathematics education. The variations in possible hierachical descriptions simply illustrates that their 'logic' merely reflects preferences of the practitioners.

Teaching Mathematics From the Cconcrete to the Abstract

'Good' mathematics teaching currently follows the Piagetian stages of development from concrete sensori-motor 'hands-on', through pre-operational and concrete operational stages, to abstract formal operations. Mathematics educationalists traditionally agree that: "In terms of learning mathematics, the ability to cope with abstractions would depend on the emergence or development of formal operational thinking" (Orton, 1992). So 'good' maths teachers introduce concrete objects and use them for the purpose of abstracting mathematics from them. The highest objective of the 'good' maths teacher is to abstract mathematics from real world contexts so that their children understand mathematics as an abstraction in the sense of being context independent. Oh that we could go directly to the abstract rather than being directed by pedagogy through this meandering subterfuge of introducing concrete examples merely so we may later reject them in favour of their abstract mathematical properties. Fortunately the social credibility of teaching through applicable maths makes this 'good' teaching practice more acceptable. For even if our children fail to reach the intended level of abstraction, we have at least served the utilitarianism of human capital education in skilling them for employment.

level, Cartesian coordinates, which depend on an understanding of algebra, are even less understood, and considered even more complex, rarefied and abstract. So abstract describes the difficulty and complexity non-mathematicians associated with increasing hierachical levels of mathematics. However, mathematics educators recognise that mathematics understanding need not follow any such hierachical description of mathematics - for example a child who can find his row and seat number in a theatre, understands much about Cartesian coordinates without necessarily understanding the assumed prerequisites of algebra or even arithmetic. Structural assertions that 'A' must be taught as prerequisite foundation for 'B' are tautological statements generated by ascription to a given 'logical' hierachical description of mathematics. A child centered mathematical educator would rather ask "Does the child now need to know 'B'?" If the answer is yes, then it will be found that the child already has an intuitive understanding of 'A' and the maths teacher may encourage the child to make this explicit. If the child has no intuitive knowledge of 'A' then, from the perspective of a child centered pedagogy, 'A' cannot be taught - only imposed.

Mathematical Abstraction as Context Independent Generalisation

Mathematicians on the other hand, in contrast to non-mathematicians, consider abstraction to be generalisation from particular contexts to context independence. So mathematical abstraction may occur at any level in a descriptive hierarchy. For example, 7 year old's base 10 arithmetic my be generalised to other bases and the properties abstracted for any base 'b'. This meaning of 'abstraction' as generalisation of content and strategies from concrete exemplars to context independence is consistent with Piagetian theory (Dubinsky, 1986) used by our 'good' mathematics teacher and operationalised in mathematics education research (Bettge, 1992; Gonzalez, 1990; Reed, 1989; Iben, 1989; Kouba, 1989 and Cobb, 1987).

Understanding Mathematical Abstraction

I believe that the *understanding* of mathematics does not parallel this 'concrete to abstract' description that is used to describe generalisation in mathematics and used to apply Piagetian theory to the teaching of mathematics - no more than it parallels any particular 'logical' hierachical description of mathematics - as shown above. In particular, as pointed out by Ginsburg and Asmussen (1988), purely cognitive explanations of mathematical thinking ignore the involvement of feelings, personal meanings and motivations in mathematical experiences. I believe that a person's individual understanding of mathematics - no matter how generalised the mathematics - is always organised by an increasingly personalised internal contexts incorporating their mathematical knowledge and beliefs. I suggest that mathematics education should be concerned with how we construct and change these internal contexts.

> In a number of areas there has been a persistent search for units of analysis appropriate for the description of the organization of knowledge and beliefs. In developmental psychology, in problem solving research, in work in artificial intelligence, and in research in expert systems, various approaches have been explored. These can be divided into two main categories: the specific and the general. Among the specific kind are schemata, scripts, and plans. .... Among the general kind are to be found principles, structures, themes, and Keegan's candidate, thought-forms. (Gruber & Davis, 1988, p. 251).

Other approaches are Thematic Appreciation Units - TAUs, Explanation Patterns - XPs (Schank, 1988), Generative Metaphors (Schon, 1979) and Networks of Metaphors (Gruber & Davis, 1988). Another way these approaches may be ordered is on a dimension from non-affective to affect-laden. The traditional view of concrete to abstract parallels affect-laden cognition to affect-free cognition. However, in contradiction to this traditional Piagetian practice in maths education, "highly intellectual individuals engaged in abstract mental activities resort to 'primitive' imagery to solve their problems." (Mavromatis, A. 1987 p. 197). Because these internal organising contexts are so affect-laden I call them 'feeling symbols'. The literature is replete with examples of highly affect-laden organisations for the most abstract concepts in art, literature, science and mathematics.

FEELING SYMBOLS THAT ORGANISE ABSTRACT CONCEPTS IN ART, SCIENCE, LITERATURE AND MATHEMATICS

Feeling symbols in art and literature

Edvard Munch painted The Scream in 1893. The art critic Robert Hughes tells us that Edvard Munch's childhood:

> was ghastly. His father was a ranting religious bigot, his mother a submissive wreck; his beloved sister Sophie died of tuberculosis, and, as he put it later, 'Disease and insanity were the black angels on guard at my cradle. In my childhood I felt always that I was treated in an unjust way, without a mother, sick, and with threatened punishment in Hell hanging over my head.' Thus Munch's main image of family life was the sickroom. (Hughes, 1980, p .277).

Similarly, describing his own feeling symbols, Picasso said: "The painter passes through states of fullness and of emptying. That is the whole secret of art. I take a walk in the forest of Fontainebleau. There I get an indigestion of greenness, I must empty this sensation into a picture. Green dominates in it. The painter paints as if in urgent need to discharge himself of his sensations and his visions" (Ghiselin, 1952, p. 59). In her analysis of Richardson's novel 'Pilgrimage', Wallace (1982) highlights the 'garden' metaphor for women and the 'brow/mouth' metaphor for men that Richardson uses as feeling symbols to illustrate gender differences.

Feeling Symbols in Psychology

"Jung ... often used the images of water to represent the depths of consciousness and of diving and returning (with 'the treasure, the priceless heritage') to represent the growth of self-knowledge and enlightenment." (Shear, 1982, p. 157). Osowski (1986) studied William James' use of metaphor in producing his Principles of Psychology (over the 12 years from 1878 - 1890). Osowski identifies four main metaphors James used as his feeling symbols: stream of thought, flight and perching of a bird, herdsman and fringe of felt relations. Similarly, Gruber and Davis tell us that: "Locke relied on a small set of mutually complementary metaphors for knowledge. Some of the more salient ones were as follows: material object, closed space, acquisition, possession, tool or instrument, and wax tablet." (Gruber & Davis, 1988, p. 259).

Feeling Symbols in Science and Mathematics

Keegan (1985) traces Darwin's thought-form of 'gradualism' in a variety of contexts - change by the accumulation of many small or infinitesimal steps. Keegan suggests that the thought-form of gradualism permeated the whole of Darwin's thinking. Another science example given by Mavromatis (1987) is "Kepler's comparison of the sun, the stars or planets, and the space between them to God the Father, the Son, and the Holy Ghost. .. Kepler's comparison of the planets to the Son is not merely unusual but entirely irrelevant: there is neither external similarity (the Son is one , the planets are many) nor an internal one (the Son does not 'revolve' round the Father)... (yet) ... On one occasion he (Kepler) specifically stressed that 'it is by no means permissible to treat this analogy as an empty comparison; it should be considered by its Platonic form and archetypal quality as one of the primary causes,' " (Mavromatis, 1987, pp .212-213). Gerald Holton (1973) also chronicles many such affect-laden organising themes in science and mathematics around which he has written his book 'Thematic Origins of Scientific Thought: Kepler to Einstein'.

FEELING SYMBOLS - THE INTERNAL CONTEXTS ORGANISING MATHEMATICAL THOUGHT

How feeling symbols organise mathematical thought

It has long been generally recognised in psychiatry, though not in traditional maths education, that affect is central to the organisation of cognition (Ciompi, 1991). We represent mathematical abstractions by these internal affective-cognitive contexts which I call feeling symbols. "Even when dealing with highly abstract concepts, one tends to represent them almost automatically in a way which would render them intuitively accessible." (Fischbein, 1987, p. 212). These feeling symbols influence how we learn mathematics and how we use our mathematical knowledge. "What has been shown in this work is that, beyond the dynamics of the conceptual network, there is a world of stabilized expectations and beliefs which deeply influence the reception and use of mathematical and scientific knowledge." (Fischbein, 1987, p. 206). These feeling symbols direct our mathematical reasoning. "The dynamics of mathematical reasoning - and, generally, of every kind of scientific reasoning - include various psychological components like beliefs and expectations, pictorial prompts, analogies and paradigms. These are not mere residuals of more primitive forms of reasoning. They are genuinely productive, active ingredients of every type of reasoning." (Fischbein, 1987 p. 212). Also: "what is critical to inferential behaviour is the context and goals involved in reasoning, not an abstract logical form it may resemble." (Kuhn, Amsel and O'Loughlin, 1988 p. 19). "Very often, in a reasoning process, the search and solution strategies are influenced by such models functioning tacitly, which are then beyond direct conscious control." (Fischbein, 1987 p. 203).

Why feeling symbols organise mathematical thought

Mathematical feeling symbols condense huge amounts of information and have a substantial identification component. By using empathy and special-kinesthetic representation to understand mathematics, we partially identify with mathematical objects. I think the instinctive aspects of intuitive mathematical thought - speed, lack of conscious effort, personal involvement etc. - result from empathic projection hooking into the drives. (In the case of great intuitions, this is via the cognitive-affective bridge of deep feeling symbols) "condensation mechanisms facilitate drive discharge" (Rothenberg, 1988, p. 70). Our instincts and drives direct our reflexive thoughts and actions which are our mathematical intuitions. Simply, at some level we are the objects of our mathematical understanding. The properties of these objects are our personal characteristics. During this identification, when we take part in a problem we behave according to 'our' characteristics.

How intuitive mathematical understanding develops

Firstly, some people have not learnt to identify with mathematical objects and use spacial-kinesthetic representations of them, so have not developed mathematical intuition, whereas, they may have developed such intuition is other areas. Later I recommend that explicitly teaching such intuitive understanding - via empathic identification and spacial-kinesthetic representations - should be part of mathematics education.

Initial representation of mathematical understanding in the mathematically naive is not the same as the understanding of the trained mathematician (Dubinsky & Lewin, 1986). To understand the initial state and how it changes I will liken the initial naive representation to a small black and white photograph of the exemplars that epitomise the understanding. For example, a child may initially represent the idea of 'two' by remembering the actual two objects that were used to illustrate two. The memory is a close likeness of the objects and there is little involvement in the memory. When many examples of 'two' are known, the life-like photographic representation will change into a more impressionistic picture of, perhaps, the preferred exemplars with the others as shadow images. As more experiences of 'two' cause the memory to be re-constructed with empathic assimilation of the current experiences, the picture further transmogrifies, gradually on each iteration, into a large involving, more encompassing, surrealist colour abstract painting of many exemplars and their many inter-relations - which may not necessarily be mathematical. Using this 'photographic to surrealist" analogy: a deeper understanding of 'two' is like living in an idiosyncratic surrealist world of exemplars, pairs, bonding, mirror symmetries, evenness, parents, Noah's ark, opposites, complementariness, double hops, yin and yang - all one's being is emotionally centered, during the moments of ideation, in the living surrealist world of personal affect-laden connotations of 'two'. If the memory could be re-constructed without assimilation of new experience, then the representation would ossifies rather than transmogrify. Schank's Explanation Patterns - XPs - are similar to such ossified representations: "understanding requires an active memory, full of knowledge based on repeated ossified experiences and also full of novel experiences that are unmerged with other events." (Schank, 1988, p. 221). However, memory never exactly re-constructs itself because there is always a different internal context that co-constructs it. Its more like the game of whispers, where each re-construction slightly changes the previous representation. So repeated memories transform the original experience into a surrealist copy which is the abstract feeling symbol. Albert Rothenberg gives the following example from the long term psychoanalysis of a poet. It shows how, over a long period of creative work - re-constructing the memory and assimilating new relevant experience - the initial feeling symbol can transmogrify.

> Similarly, prior to the creation of the metaphor 'the branches were handles of stars', that author had thought only of the sound and shape connections between branches and handles. Afterward, he became dimly aware of images of branchlike maternal arms encompassing a child. During further creative work related to this metaphor, the fire like intensity of the star led to conscious thoughts of warm, erotic sensations and to unearthed unconscious fantasies of erotic sensations in the held child. The unravelling stopped short, however of connecting himself to the held child. (Rothenberg, A. 1988 p. 70)

I believe mature mathematical understandings are, like those above, similarly repeated transmogrifications built on mathematical contexts - e.g. feelingly of motion and perspectives of space - that may have been experienced simultaneously with infantile instincts. Cooney (1991) considers that fundamental intuition of mathematics, the abstraction of the relation of n to n+1, can be understood in terms of such infant experiences. These older mature feeling symbols may contain irrational intentionality contributed by the child's early experiences. Rothenberg (1988) says:

> On the basis of recent research on memory and development, there is reason to believe that all childhood events are construed in adulthood in accordance with the child's level of cognitive and affective development at the time they occurred (one citation given). At certain levels of development, for instance, only sensory and motor aspects of an event will be apprehended and experienced. This plays a role in the substance and structure of memories (Rothenberg, 1988, p. 178).

Such mature and pervasive feeling symbols are most widely accessible in regressed primary process thinking. Then they may direct cognitive-affective associations that guide mathematical solutions and strategy selections. Suler (1980) relates that: "Rapaport (1950) in fact described primary process as a drive organization of memory, since all objects, images, and experiences are organized according to their relationship to some instinctual tension." (Suler, 1980, p. 144). "Psychodynamic theory predicts that children who can permit drive-laden material to surface in fantasy and play, and who can cognitively integrate and master that material, should be open to ideas and flexible in their problem-solving approach. Thus, they should be better learners than children who do not actively integrate emotional material." (Russ, 1982, p. 570). This indicates that applications to mathematical education should utilise ego-regressing environments - like storytelling. Only context relevant reduced perspectives of a deep feeling symbol are available in more focused thought.

RESEARCH EXPLORATIONS

To explore the internal contexts subject's use to organise their mathematical knowledge I first asked 80 tertiary maths education students to imagine that they were born again and grew up in 'Flat Land' as either an acute angle or an obtuse angle: Which one they would prefer to be and why? I also asked them to role play their chosen identity with appropriate sounds (not words), body shapes and movements so that the other angles could easily tell which they were. The second part of the investigation was similar except that they had to choose to be one of the following: pi, fractions, division, functions or graphs.

Many subjects represented the open or closed shapes of their chosen angle. Many used cross-model transfer from sharp angle to sharp sounds - and similarly for the sounds made by obtuse angles. Some subjects incorporated the aerodynamic properties of the acute angle in their point first rapid movements, whereas other acute angles moved in rapid, tiny steps. The most interesting representations were complete character identifications - e.g. an acute angle is slim and sophisticated, an obtuse angle is open and friendly.

The second part of the investigation involved the understanding of more abstract mathematical concepts. The results showed, as above, that subjects organise their understanding of the properties of mathematical objects through highly personalised identifications based on empathy and spacial-kinaesthetic representations, sometimes derived from cross-model transfer. In addition, because these mathematical objects were more abstract this investigation allowed for various levels of understanding and different emphases in understanding. The results clearly showed subjects had these differential levels and emphasis of understanding. The lowest levels indicated only symbol recognition - subjects would take the shape of the symbol e.g. pi would stand with legs apart and arms horizontally to the side. The next level of understanding was represented by being an exemplar e.g. a subject folded double would be the fraction 'one-half'. The next higher level of understanding would be one property of the object e.g. the subject who was pi pulling his infinite tail, behind him, or an unfinished movement representing a fraction. The highest level of understanding was shown by a holistic representation of a personality whose characteristics were defined the subject's identification with properties of the object e.g. snobbish and haughty, or evil and dangerous. It was interesting that low level symbol representation accompanied movement to represent lack of understanding e.g. a pi that jumps erratically, disappears from here and turns up there. Whereas 'stationary' represented certainty e.g. 'bar-graph' subject was stationary because he was, "more definite, easier to read, easier to get information from me, than if I was moving".

The emphases subjects gave to the representation of their understanding indicated strengths, weaknesses, misconceptions, personal priorities the object held for them and the way their understanding was likely to correctly or incorrectly evolve e.g. a 'fraction' subject shattering does not emphasise the equality of the fractional parts, a 'fraction' subject with independently moving torso and legs emphasised rules to change the top and bottom numbers, a 'fraction' subject with an arm hidden was incomplete.

Another important result was the wealth of metaphors subjects used to organise their understanding at all levels of mathematical abstraction. These could be suggested to other students, when teaching the relevant mathematics, as options for organising new understanding e.g. bar graphs are going up and down stairs.

CONCLUSIONS FOR MATHEMATICS EDUCATION

The construction of mathematics understanding need not follow the 'logical' hierarchy of a curriculum, nor the 'particular to context independent' generalisations of mathematics, nor the 'concrete to formal operational abstraction' of Piagetian pedagogy. In contrast to traditional Piagetian practice it should be recognised that internal affect-laden contexts are essential organisers of intuitive understanding of mathematics at all levels of abstraction, that these feeling symbols drive new learning and mathematical applications. Children should be taught how to develop these personalised contexts through such techniques as encouraging identification with mathematical objects using empathy and spacial-kinaesthetic representations of their properties. At the moment we do not teach children to understand mathematics, we only teach them mathematics and leave the understanding - or lack of it - up to them.

REFERENCES

Bettge, S.H. (1992). The wording of mathematical problems: Consequences for girls' and boys' expectancies of success. *Zeitschrift fur Sozialpsychologie, 23(1),* 46-53.

Ciompi, L. (1991). Affects as central organising and integrating factors: A new psychosocial/biological model of the psyche. *British Journal of Psychiatry, 159,* 97-105.

Cobb, P. (1987). Information-processing psychology and mathematics education: A constructive perspective. *Journal of Mathematical Behavior, 6(1),* 3-40.

Cooney, J.B. (1991). Reflections on the origin of mathematical intuition and some implications for instruction. Special Issue: Mathematical cognition: I. Emerging theoretical perspectives. *Learning and Individual Differences, 3(1),* 83-107.

Dienes, Z. (1989). "Something unusual": Teaching mathematics at the elementary level. *Journal of Structural Learning, 10(1),* 83-94.

Dubinsky, E. (1986). Teaching mathematical induction: I. *Journal of Mathematical Behavior, 5(3),* 305-317.

Dubinsky, E. & Lewin, P. (1986). Reflective abstraction and mathematics education: The genetic decomposition of induction and compactness. *Journal of Mathematical Behavior, 5(1),* 55-92.

Fischbein, E. (1987). *Intuition in Science and Mathematics: An* *educational approach.* Dordrecht, Holland: D. Reidel Publishing Company.

Ghiselin, B. (1952). Conversation with Picasso. In B. Ghiselin (Ed.), *The Creative Process* (pp. 55-60). Berkeley: University of California Press.

Ginsburg, H.P. & Asmussen, K.A. (1988). Hot mathematics. *New Directions for Child Development, 41,* 89-111.

Gonzalez, L.M.J. (1990). Level of abstraction in geometric analogies. *Revista de Psicologia General y Aplicada, 43(1),* 23-32.

Gruber, H.E. & Davis, S.A. (1988). Inching our way up Mount Olympus: the evolving-systems approach to creative thinking. In R.J. Sternberg (Ed.), *The Nature of Creativity: Contemporary psychological perspectives* (chap 10 pp. 243-270). Cambridge England: Cambridge University Press.

Holton, G. (1973). *Thematic Origin of Scientific Thought: Kepler to Einstein.* Cambridge Mass: Harvard University Press.

Hughes, R. (1980). *The Shock of the New: Art and the Century of Change.* London: British Broadcasting Company.

Iben, M.F. (1989). Mathematics classroom effects on student development of spatial relations and abstract mathematical thought: The U.S. and Japanese experience. *Journal of Mathematical Behavior, 8(1),* 123-136.

Keegan, R.T. (1985). *The development of Charles Darwin's thinking on psychology.* Unpublished dissertation, Rutgers University, Newark, NJ.

Kouba, V.L. (1989). Children's solution strategies for equivalent set multiplication and division word problems. *Journal of Research in Mathematics Education, 20(2),* 147-158.

Kuhn, D., Amsel, E. and O'Loughlin, M. (1988). *The Development of Scientific Thinking Skills.* San Diego: Academic Press, Inc.

Mavromatis, A. (1987). *Hypnagogia : The unique state of consciousness between wakefulness and sleep.* London: Routledge & Kegan Paul.

Nolt, J.E. (1983). Mathematical intuition. *Philosophy and Phenomenological Research, 44(2),* 189-211.

Orton, A. (1992). *Learning mathematics: issues, theory, and classroom practice.* London: Cassell.

Osowski, J.V. (1986). *Metaphor and creativity: A case study of William James.* Unpublished dissertation. Rutgers University, Newark, NJ.

Reed, S.K. (1989). Constraints on the abstraction of solutions. *Journal of Educational Psychology, 81(4),* 532-540.

Rothenberg, A. (1988). *The Creative Process of Psychotherapy.* New York: W.W. Norton & Co.

Russ, S.W. (1982). Sex differences in primary process thinking and flexibility in problem-solving in children. *Journal of Personality Assessment, 46(6),* 569-577.

Schank, R.C. (1988). Creativity as a mechanical process. In R.J. Sternberg (Ed.), *The Nature of Creativity: Contemporary psychological perspectives* (chap 9 pp. 220-238). Cambridge England: Cambridge University Press.

Schon, D.A. (1979). Generative metaphor: A perspective on problem-setting in social policy. In A. Ortony (Ed.), *Metaphor and Thought.* London: Cambridge University Press.

Shear, J. (1982). The universal structures and dynamics of creativity. *Journal of Creative Behaviour, 16(3),* 155-175.

Suler, J.R. (1980). Primary process thinking and creativity. *Psychological Bulletin, 88(1),* 144-165.

Wallace, D.B. (1982). *The fabric of experience: A psychological study of Dorothy Richardson's 'Pilgrimage'.* Unpublished dissertation, Rutgers University, Newark, NJ.

ALGEBRAIC MANIPULATION: ACTIONS, RULES AND RATIONALES.

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Twenty year-10 students in a boys' private school in Melbourne were interviewed individually while they worked on algebraic tasks. The tasks included solving very simple linear equations and simplifying expressions. The taped interviews provide examples of informal, and often misleading, ways in which many students describe algebraic manipulation procedures. Informal terminology (e.g., "You move the 2 over there and put it on top") was used successfully by some students, but for many others it was associated with making mistakes. Students often appeared to be guided by memory of actions they might carry out rather than by general principles. They showed difficulties understanding the ways in which numbers can be used for checking algebraic work and the purposes of basic algebraic tasks such as simplifying and solving.

In no part of mathematics is the tension between rule and understanding stronger than in algebra. Long-standing research shows that error-free manipulation is rare, errors such as $3^2 = 6$, $4(n+5) = 4n+5$, $(x+8)/(x+2) = 8/2$ being common (Carry, Lewis and Bernard, 1980; Kuchemann 1981, Foxman et al, undated). Other work has shown that even 16 year-old students whose manipulative skills are good have very little understanding of how to use algebra in the processes of generalising, reasoning or justifying statements (Lee & Wheeler,1987). Lee and Wheeler (1987) have also pointed out the fragility of the connection that many students make between arithmetic and algebra. There are also indications of deep-seated misinterpretations of the meanings of algebraic symbols. For example, many students regard an equation such as $6s=p$ as a description of a loose association between quantities (e.g. a comparison of relative sizes) rather than as a strict equality between numbers. (Clement, 1982; MacGregor, 1991). The tensions between rule and understanding are evident in all these instances.

The focus of the present work is on performance in algebraic manipulation involving several steps. The context in which the work is carried out is that of comparison between expert and novice performance in algebraic manipulation, in particular in solving linear equations and simplifying expressions. Our research aims to understand how the additional conceptual awareness of experts develops and how it relates to performance skills based on symbol movements. We seek to understand how students' understanding commonly breaks down in the course of this development and how such breakdowns might be minimised. In the present paper, we present extracts from interviews with Year 10 students which illustrate the following three aspects of this problem.

(i) Students seem to be guided by images of actions which are not related to underlying general principles. This is reflected in the language of actions and goals used by both students and teachers to discuss algebraic transformations. Some students denied that they work with general principles, but rather claimed that they could just see what to do.

(ii) Students showed a lack of understanding of the rationale of using numbers to check algebraic results. The extracts from the interviews illustrate three different modes of checking which use numbers in different ways and there are probably others.

(iii) The purpose of algebraic tasks is unclear and students lose sight of the aims as they work. Even within the purely symbolic tasks that are reported in this paper, one student wavered between simplifying and solving, using actions appropriate for one in the other.

The data used here was collected as part of an experiment focused on algebraic manipulation procedures used for solving one- and many-variable equations, and, to a smaller extent, on the forming of equations. Two classes of

Year 10 boys ('middle' sets in a relatively 'good' school) were taught these topics for three weeks by the first author, using a method in which the checking of procedures was an integral part of the work. Errors were immediately rectified or discussed and students generated and checked their own examples. Concurrently, some twenty of the students were being interviewed on video by the first author, one to one, on the material being taught, to expose more fully their methods and the justifications which they might have for them.

In this paper we report extracts from interviews with two of the students, which illustrate behaviour that was commonly observed. About one half of the interview with Andrew is in Appendix 1, a further extract is in Appendix 2 and an extract from Steven's interview is Appendix 3. In the extract in Appendix 1, Andrew makes a manipulation error which was observed in several interviews. He knows that to "undo" the division by 2 on the left hand side, he must multiply. However, he *replaces* the division by multiplication, thus in effect multiplying the first term by 4. An attempt to clarify the error by looking at numerical fractions is made difficult by the fragility of Andrew's knowledge of fractions. Andrew checks a solution to an equation by substituting the number found throughout the whole equation and then repeating the steps of the algebraic solution with the numbers. In Appendix 2, we see Andrew using false cancelling to simplify an expression and, as he tries to check it numerically, possibly confusing simplifying with solving. In Appendix 3, Steven shows a similar confusion and also illustrates strongly how his work is guided by images of moving numbers and symbols rather than by principles.

ACTIONS, GOALS AND PRINCIPLES.

It is clear that perceptions of symbol movements play a part in algebraic manipulation carried out by both experts and novices. For example, in a recent study (Bell & Malone, 1992) of approaches to the reading and writing of algebraic formulae, students were asked to construct and select valid rearrangements of the formulae $C = V \div R$ (for current, voltage and resistance), $S = D \div T$ (speed, distance and time) and $A = B \div C$ (three numbers). Later in the test, they were asked to say which most closely described their way of thinking about these questions:

> Look back at the question on C, V and R. Tick the one or two of the following statements which is closest to the way you were thinking.
> a. I thought which would be the biggest number, so the others would be divided into it.
> b. I tried some actual numbers in my head.
> c. I just remembered the formulae.
> d. I thought the one on top of the right hand side would go underneath on the other side.

Of the school pupils, aged 13-15, about 25% reported thinking which was the bigger number, 40% reported trying actual numbers, 20% just remembered the formulae, and only 10% reported thinking of physical movement of the symbols. But of the two samples of experts (teachers or prospective teachers in training) the great majority reported thinking of physical movements, and none considered numbers or sizes. (Bell & Malone, 1992).

In contrast to the Bell and Malone sample of students, the students we interviewed in this study seemed to think predominantly in terms of symbol movements. In Appendix 1 (interchange 2), Andrew makes a manipulation error which was observed in several interviews. To "undo" the division by 2 on the left hand side, he *replaces* the division by multiplication, thus in effect multiplying the (f - 3) by 4. When the interviewer tries to clarify what has gone wrong (interchanges 10 - 31), note the confusion that is shown by Andrew between multiplying and dividing, especially in the denominator (interchanges 11 - 14) and the interviewer's "action-oriented" language (multiply, divide, throw away: interchange 15). In discussing algebraic manipulation, most of our language is like this, stressing goals and actions, rather than the principles involved. We do not seem to have alternatives readily available. We had hypothesised that differences would be observable between the students' and the interviewer/teacher's language, in that students' language would be purely descriptive of the surface aspects of the procedures (e.g. "Move the 3 over here") while the teacher's language would refer to underlying principles and justification (e.g. ""We must do the same thing to both sides, to keep the balance"). Study of the complete interview records shows that both teacher and students spoke primarily in terms of surface aspects of symbolic procedures, although the teacher's talk contained the additional dimension of validity or justification, drawing a distinction between the goal (e.g., "I want to get rid of the 3 in the denominator") and the method of achieving that goal (e.g., "I will multiply both sides by 3"). The students' talk often referred only to the goal, and when a general rule was quoted, which was not often, it would be expressed in terms of "what you have to do" rather than what is a valid operation on the equation or fraction. Both teacher and students speak about symbol movements such as "getting rid of the denominator", but for the teacher there is a distinction between goals and valid methods for moving towards these goals whereas for the students there appears to be no such distinction.

All of the errors seen in here in Andrew's work were common to many of the subjects. We sought for clues as to their origin in the language used. Note, for example, that when Andrew was asked if he was using a rule, he does not speak of doing something to both sides but says he was "times-ing everything along there by 2" and says he is not sure whether he should multiply the *f* term (interchanges 7 - 10). He appears to have a visual image of what he should do rather than an explicit rule to apply. Another boy said of the 2 in the denominator, "It's divided, so you have to multiply". Other boys' words and handwaves seemed to suggest that it was that particular 2 on the bottom which was being used to multiply the top (or to multiply the number on the right hand side), so it was thus felt to have been appropriately used up and did not still remain as a denominator.

The interviewer frequently asked the boys about the rules they used. Most of them denied having any rules governing their manipulations, as for example does Steven in Appendix 3, interchanges 2 - 4. Another boy, in a similar situation, said " No, I just do it. Rules just go right over the top [of my head]." One of the few boys who did talk about doing things to both sides of an equation actually failed (or forgot) to do so. He professed to subtracting 4 on each side, then "times-ing" both sides by 3, and in both cases actually did it only on the left hand side until reminded by the interviewer. He also collected terms onto one side by simply moving them, that is, by bringing a term to the other side of the equation without changing its sign. It is clear that the ability to state a correct rule does not guarantee its correct application. Rules need to be taught along with discussion of the ways in which they may be applied, with exposure to possible errors and, of course, with practice.

USING NUMBERS TO CHECK ALGEBRAIC PROCEDURES.

During the twenty interviews, many students showed they were confused about the different ways in which numbers are used to check the results and validity of algebraic procedures. In fact, there are several conceptually different ways in which this must be done and three can be seen in the extracts from interviews in the Appendices. In the first segment of the interview with Andrew in Appendix 1 (interchange 4) we see an instance where a solution of an equation is being checked. The checking procedure that Andrew uses, which is to repeat the complete solving process with numbers, fails because Andrew repeats his algebraic error with the numbers. A similar checking procedure was also observed in several other interviews. Later on (Appendix 1, interchange 15 *et seq*), we see a quite different use of numbers for checking algebra, when the interviewer tries to illustrate, almost by analogy, that when (f - 3)/2 is multiplied by 2, the answer is not 2(f - 3). Unlike the equation solving where only the solution is to be used in the check, the interviewer chooses an arbitrary fraction, in this case 3/4. Andrew's vagueness may indicate that he does not comprehend the relevance of the interviewer's questions about 3/4 to his problem about (f - 3)/2. The reference to a numerical fraction was an attempt by the interviewer to find some basis for demonstrating to Andrew that this was an incorrect operation and to enable him to replace it by the correct one. It ran into trouble because his conception of fraction multiplication by an integer was faulty - he thought you multiply both the top and the bottom. This error is also shared by many of the Year 10 students interviewed. However it is worth noting that this does not correspond to Andrew's algebraic misconception so, on this evidence, we see that he does not perceive a link between the arithmetic and the algebra. The attempt to link with the real-life situation of four times 3/4 of a cake does not go easily either; it does not seem a natural recourse for him.

Andrew's interview continued (Appendix 2) with questions about simplifying expressions. He appeared at first (interchange 2) not to be entirely clear about what this meant, and how it was different from the previous kinds of manipulation (i.e., solving equations). Note how the checking that he is doing here uses numbers in yet another way. Andrew does not appreciate that any number will do for checking whether something is an identity (in this case whether (2x+6)/3 is equal to 2x+2). In fact, to be precise in this case, any number *other than* the solution of the equation (2x+6)/3 = 2x+2 will do, but by the end of the episode, Andrew seems to be searching for this solution.

After the twenty interviews with boys from the middle Year 10 sets, six further interviews were conducted with boys from the top set of the same year. Some of them used the same language as the boys in the first sample, and made the same mistakes though perhaps not so frequently. The most noticeable difference was the much more ready way in which they recognised and were able to pick up and use the checking and verifying procedures suggested by the interviewer.

SOLVING OR SIMPLIFYING.

Andrew made a false cancellation of the 6 and the 3 (Appendix 2, interchange 4) He was then guided by the interviewer through a checking procedure, substituting a number for *x*. He appeared not to understand the rationale for this initially, but did agree that the numerical result from the two expressions should be the same. But when they were not, his suggestion for remedying the situation was to try a different value for *x*, only to find that it still doesn't work out. (interchanges 12 - 13) Note the language here; he appears now to think that the task is to solve an equation, i.e. find a value for *x* which will equate the two expressions. Simplifying has mutated to solving, during the checking episode. Steven also shows (Appendix 3, interchanges 6 - 10) how symbol moves which are appropriate to simplifying tend to be employed in a solving situation, although there are other explanations for these moves.

CONCLUSIONS

To summarise, we suggest that major causes of errors in manipulation in algebra for these students include the rules about how to "move" numerals and letters, that they have developed by generalising from simple cases in early algebra learning and now remember as actions to be carried out, rather than as principles. The memory of spatial patterns associated with the fraction bar and the equals sign, which guide experts, can easily be misused by novices. Even when verbal rules are remembered by rote they may be given no meaning, and therefore be unusable.

The implications for teaching are, we suggest, that there needs to be much more exploratory work, in which possible rules and interpretations are experimented with and their validity checked by such methods as substituting numbers. Explicit attention needs to be directed to the different underlying logic that is involved when numbers are used to check algebra. There needs also to be, of course, practice to gain fluency - but checking in its many forms should be built in as a natural part of the task.

REFERENCES

Bell, A. & Malone, J. (1992). Learning the Language of Algebra, *Proceedings of the Mathematics Education Research Group of Australia,* Richmond, NSW: University of Western Sydney.

Carry, L.R., Lewis, C., & Bernard, J.E.(1980), *Psychology of Equation Solving*. Austin, Texas: University of Texas.

Clement, J. (1982). Algebra word problem solutions: thought processes underlying a common misconception. *Journal for Research in Mathematics Education, 12*, 16-30.

Foxman, D., Ruddock G., Joffe, L., Mason, K., Mitchell, P., Sexton, B. (undated).. *A Review of Monitoring in Mathematics 1978 - 1982.* London: Assessment of Performance Unit, Department of Education and Science.

Kuchemann, D.E. (1981). Algebra. In K.M. Hart (Ed.) *Children's Understanding of Mathematics, 11-16.* London: Murray.

Lee, L. & Wheeler, D. (1987). Algebraic Thinking in High School Students: Their Conceptions of Generalisation and Justification. Concordia University, Montreal.

MacGregor, M.E. (1991). *Making sense of Algebra: cognitive factors influencing comprehension.* Geelong: Deakin University

Appendix 1: Interview with Andrew

I.(1) Can you solve this equation?

A.(2) *Andrew, working silently, multiplies all the terms of the equation, but when multiplying the fraction, multiplies the top as well as removing the 2 in the denominator.*

$$\frac{f-3}{2}+4f=2$$
$$2f-6+8f=4$$
$$10f-6=4$$
$$10f=10$$
$$f=1$$

I.(3) Right, will you now check your answer?

A.(4) *When he comes to check the solution, A. writes the whole equation, including the right hand side, and repeats his solution procedure. Thus his check repeats his error.*

$$\text{LHS}=\frac{1-3}{2}+4=2$$
$$=2-6+8=2$$
$$4=2$$

I.(5) Oh, so you've got 4=2. If you are checking, do the left hand side on its own like this (Interviewer indicates one side of the equation and writes *LHS =* on the left.

A.(6) *A crosses out =2 and starts a fresh check, working out the LHS only, but still performs the same operations.*

$$\frac{1-3}{2}+4$$
$$2-6+8$$
$$=4$$

This doesn't equal the right hand side so I. asks

I.(7) Have you any idea what's wrong? *(Little response)* Look here, (*pointing to the top line*) you had f - 3 over 2 and in the next line you had 2f - 6. What equation rule were you using at this point?

A.(8) I was times-ing everything along there by 2 (*waving his hand over the whole equation).*

I.(9) You multiplied everything there by 2....?

A.(10) I'm not sure whether I should have multiplied the f or not by 2.

I.(11) Well you've multiplied the 4f by 2 and the right hand side by 2 and that's all correct. But multiplying this fraction by 2 you've done wrongly. You multiplied the top by 2; and what have you done with the bottom?

A.(12) Multiplied that by 2

I.(13) But that would be 4, wouldn't it?

A.(14) But isn't that divided?

I.(15) *(Pointing to the very first line)*What this says is, f - 3 divided by 2. You've multiplied it by 2 *and* thrown away the bottom. That's equivalent to multiplying the top by 2 and dividing the bottom by 2.
If that was a fraction like 3/4 ... . If I multiply 3/4 by 4, what do I get?

A.(16) 12/16

$$\left(\frac{3}{4}\times 4\right)\quad \frac{12}{16}$$

I.(17) What's the value of 12/16?

A.(18) 3/4

I.(19) Well if that's *equal* to 3/4, it can't be *multiplied by* 4, can it?
Think of it in real life, ...in ordinary language, what is 3/4 times 4?

A.(20) *No response.*

I.(21) 3/4 of a cake, multiply it by 4, how much cake would you have?

A.(22) You'd get four 3/4s.

I.(23) How much would that be altogether?

A.(24) It would be 3. 3 cakes.

I.(25) How many quarters altogether?

A.(26) 16.

I.(27) How many quarters in 3?

A.(28) 12.

I.(29) So when you do this(*3/4* x *4)* this had better be 12 quarters, or 3. *(I writes this)*
So what does this have to come to? What rule should you use when you are working this out?

A.(30) Multiply the top. *(Looking at the writing from interchange 16 as above and comparing with the 12 just written in interchange 29.)*

I.(31) Yes, so when you multiply this *((f-3)/2)* by 2 you multiply the top by 2 and leave the bottom. That's what went wrong here (*pointing to the top line again*). So that sorts out an important problem for you.

Appendix 2: Andrew's interview continues.

I.(1) Now can you simplify that? i.e. (2x+6)/3

A.(2) What? Do you mean like I did the last one?

I.(3) No, replace it by a simpler expression.,..like if I said 4/2, and asked you to simplify that.

A.(4) *A. cancels 3s from the 3 and the 6 and gets 2x + 2 over 1, then 2x +2*

$$\frac{2x+\cancel{6}\,^{2}}{\cancel{3}\,1} \qquad \frac{2x+2}{1}$$

I.(5) Could you put a value in for x in this, and in this, to check?

A.(6) *A puts in 3 and does as before, getting 8.*

$$\overset{2x+2}{6+2=8}$$

I.(7) Is there any other way you could work this out?

A:(8) Do you mean with an x or something?

He works out (2x+6)/3 getting 12 / 3 and then 4.

$$\frac{12}{3} \; 4 \; \frac{6x+6}{3}$$

I.(9) Should this be the same?

A.(10) Yes.

I.(11) What's wrong?

A.(12) You could try putting a smaller number, to get it smaller - put 1 instead of x.

He does this and of course the expressions still do not tally.

$$\frac{2+6}{3} \frac{8}{3} \; 2+2=4$$

I.(13) So it still doesn't work out.

Appendix 3: Extract from Steven's Interview.

I.(1) *Presents Steven with this equations and asks* Any ideas?

$$\frac{a-3}{4} = \frac{a+2}{3}$$

Steven. (2) Yes. Put that in brackets and multiply that by 3, and multiply that by 4.
He points to the left hand side and the right hand side in turn.

I.(3) Do you have any rule that you are using for that? Like cross multiplication or something?

S.(4) No, I just know that - you know..*(Pointing with his finger to the two sides.)*

I.(5) *Interviewer I gives him a new equation.* Would you do the same thing here?

$$\frac{a-3}{4} = \frac{a+2}{3} + 1$$

S.(6) I think I would.

$$3(a-3) = 4(a+2)+1$$
$$3a-9 = 4a+8+1$$

S.(7) Would that equal zero, -9 +9?

I.(8) How do you mean?

S.(9) I think that -9 +9, oh! I think that's where I made my mistake.

I.(10) Yes, you've added the things on the two sides.
You have to think of this as two equal things on the two sides.
Pointing to the two sides of the equation.

S.(11) I was thinking you add them together.

I.(12) Yes, but if you add 9 to this side, you've got to add 9 to that side as well.

S.(13) Yes. I think I know where I've gone wrong.

I.(14) Yes, we can sort that out. There's something gone wrong up here too.
I. points to line X.
When you do this multiplication across like this it's only going to be valid if you can justify it by saying you have multiplied both sides by the same thing, and I don't think you have.

S.(15) Oh - I understand. So I have to multiply this by 3 as well. *Pointing to 4(a + 2).*

I.(16) No, what we have to do is multiply both sides by 12 and we've got to multiply this *(pointing to the 1 on the right hand side)* by 12 as well.

S.(17) So this has got to be 12.
He turns the 1 into a 12 and continues

$$3(a-3) = 4(a+2) + 12$$

$$3a - 9 + 9 = 4a + 8 + 12 + 9$$

$$7a\ 3a = 4a + 29$$

S. (18) So I've just got to add 9 to this and to that...now I minus this one *(pointing to the 4a)*

I.(19) Yes.

With some help, Steven works this out and reaches -a = 29:, answer a = -29: then, with more help, makes a correct check.

CHILDREN'S QUESTIONS ABOUT GEOMETRY

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Question-asking is a key process in mathematical thinking, and is recognised as such in the 1992 New Zealand National Mathematics Curriculum which emphasizes the need to encourage children to ask questions in various mathematical contexts. However, observations suggest that this is something which few New Zealand primary school teachers do and, further, there seems to be almost no research into children's question-asking in mathematics which might provide some guidance. The present paper reports the results of a small investigation into middle primary school children's ability to ask questions about geometry. It also provides a first glimpse into preservice primary school teachers' perceptions about both the value of children's questions for teaching and learning in mathematics, and difficulties involved in eliciting them.

Until the beginning of this year, one of the objectives of the New Zealand Mathematics Syllabus was to help children develop their urge to enquire (Dept of Education, 1985; 1986), but there was no indication that this meant children being encouraged to ask questions as part of their mathematics learning (Dept. of Education, 1986). However, in the mathematics curriculum which came into effect at the beginning of this year (Ministry of Education, 1992, 24) this key enquiry process has been made explicit, "Within a range of meaningful contexts, students should be able to pose questions for mathematical exploration." Further, there is mention that students should be exploring their own interests and brainstorming possible questions for investigation. But how does this new emphasis on children's questions fit within the perspective that most teachers, and indeed the mathematics education community generally, hold about learning in mathematics? And what research is there which might provide some guidance to practice?

In contrast to science education (see, for example, Biddulph, 1989; Biddulph and Osborne, 1984), little comparable recognition of the value of children's questions seems evident in the field of mathematics education. Perhaps not surprisingly, the focus seems to have been largely on teacher questions (see, for instance, Cemen, 1989; Sullivan and Clark, 1991; Wigley, 1992). Relatively few references to children's questioning appear in the literature on mathematics education. Biggs and MacLean (1969) propose that mathematics education should build on young children's natural curiosity by encouraging them to ask questions. Griffiths (1988) takes a similar view and comments that children who have the confidence to ask questions are less likely to become confused about mathematical ideas. Bobis (1991) and Whitin (1989) have suggested that children be encouraged to ask questions to enhance their number development. "When children ask the 'why' question in their mathematics classroom, they are on their way to strengthening their number sense" (Whitin, 1989, 26). Baker and Baker (1990) and Chambers, Cunningham-Craig and Hammond (1987) believe that children's genuine involvement in investigations relies upon them asking the questions to be investigated. And finally, Tobias (1987) makes the point that learners in mathematics should take their own questions seriously and persuade their teachers to take them seriously too; she accepts mathematician Peter Hilton's view that true mathematics takes one from answers to questions.

Research into children's questions in mathematics appears to be almost non-existent. Can primary school aged children actually generate investigable questions in mathematics? What questions might primary school children ask about geometry, for instance, if given the opportunity and encouragement? What difficulties do teachers experience in eliciting children's questions? Could children's questions about geometry lead to worthwhile investigations? What might be the effect upon children's learning in mathematics if they investigated questions which they themselves had raised? And if the effect were positive, how might teachers be helped to adopt children's questions as an integral part of their teaching approach? The second and third of these questions were the focus of a small-scale study reported in this paper.

METHOD AND SAMPLE

Questions about geometry were elicited from a total of 100 children in four classes in two Hamilton primary schools by 32 second-year primary student teachers under the guidance of the author. One school was semi-rural and served children from a predominantly middle income group. The 48 children in the two classes at this school were aged about eight to eleven years. The second school was an urban school situated in a mainly lower socio-economic area. The 52 children in the two classes in this school were aged approximately nine to eleven years. One of the classes contained 22 Maori children being educated in a bilingual programme. These children were more fluent in English than Maori so the 11 student teachers, who were themselves Maori, interacted with them in English to elicit their questions.

Since the experience in primary science had shown that children frequently need some exploratory-type experiences as a basis for asking questions, the 100 children in the present study were first provided with a set of geometrical experiences chosen from activities to do with (i) distinguishing 2D and 3D objects, (ii) identifying angles on drawings of various objects, (iii) bilateral and rotational symmetry, (iv) tessellation - modelling a cobblestone driveway, (v) enlargement - of an umbrella drawn on a 3x3 grid, and (iv) nets - drawing a flat plan of a pyramid on paper, and then folding it to make a pyramid. In the course of these experiences, and at the end, the children were invited to say what questions they had about various aspects of geometry. The student teachers were advised that they may have to frame their invitations in several different ways, e.g. "What questions do you have about what we have been doing?" What things about these activities have puzzled you, or have you found a bit tricky?" "What have you wondered about as we have done these activities?"

The sessions with the groups of children lasted approximately 45 to 50 minutes. During this time the author observed different groups at work, and modelled for about 12 student teachers a way of helping children turn some of their comments into questions.

The questions were recorded by the student teachers as they occurred. The student teachers were also asked to reflect on the difficulties and usefulness of eliciting children's questions, as they perceived them, and to write these in their reports about their teaching experiences with these children.

CHILDREN'S QUESTIONS

The children's questions reveal a number of interesting features.

1. Number of questions asked

One feature that stood out immediately was the difference in the number of questions elicited in the two schools. Whereas the 48 children in the semi-rural school asked a total of 50 questions, the 52 children in the urban classes asked only 23 questions altogether. Whether this stemmed from the children in the urban school being more socialised into non-questioning, or from the group of student teachers who worked with these children being less able to elicit questions, is not known. Experience in science education suggests that the larger groups of urban children, with whom the 11 student teachers worked, would have produced more questions, not less, so the difference probably had nothing to do with group size.

2. Focus of questions

As would have been predicted from the research into children's questions in primary science, the questions asked about geometry were largely to do with the aspects raised in the exploratory activities. Apart from some general questions asked about geometry itself, there were questions raised in both schools about angles, 2D and 3D shapes, symmetry, and tessellation. The children in the semi-rural school also asked about enlargement. This suggests that teachers can have considerable, if subtle, control over the directions that investigations might take.

3. Similarity of questions

Some of the questions from the two schools were remarkably similar. For example, "What is geometry?" was asked by children in both schools, with someone in the first school also wanting to know, "Is geometry basically shapes?" and a child in the urban school inquiring, "Is geometry like shapes, or what?" In both schools children wanted to know whether circles have angles and, more generally, what are angles or how do you know something is an angle. Further, some children in both schools wondered what 2 dimensional and 3 dimensional things are, and what shapes will tessellate. An inquiry in each school about symmetry was not too dissimilar either. In the semi-rural school a child asked, "What is the line that separates the two sides [of a symmetrical tile] called?" and in the urban school a child wanted to know, "How can you find the line of symmetry?"

These responses suggest that research may be able to alert teachers to the kind of questions they can reasonably expect children in their primary classrooms to ask about geometry. If teachers can anticipate some of the questions that their children are likely to ask, then they can be better prepared for resourcing investigations that may flow from them.

4. Questions can indicate children's thinking

Many of the children's questions revealed their current understanding about aspects of geometry. The children's questions above about symmetry suggest that the children already have some understanding of this concept. So do their questions about enlargement: "What would happen if there were heaps of small squares?" "Do you always need to draw squares or a grid to enlarge something?" "Would it be harder if the square sizes were different?" Children can be helped to build upon such understandings.

In other cases their questions revealed considerable lack of understanding. Take angles for example. Children asked, "Do angles cross?" "Are angles something that lean to one side?" "Is an angle something that isn't straight?" "Is something an angle as long as it bends?" "Are angles different lines joining together?" There is no hint in these questions of an angle being a measure of rotation, and perhaps it is not surprising when in everyday life children usually hear the term 'angle' applied to something that is sloping, and in school mathematics to the corners of a shape. The point about questions such as these is that they can alert a teacher to the need to assist children towards the basic mathematical idea of angle. Here it is a matter of helping children to restructure their ideas rather than extend them.

A few of the children's questions suggest that they are interested in the origin and status of some terms, for example, "Who made up the word 'geometry'?" "Who invented it?" A few children were also interested in how it connects with their world: "Are there lots of things that you use geometry for?" "Who uses geometry?" "Is a car part of geometry?"

Several of their questions were totally unpredictable and it is difficult to know, without further probing, what the children were thinking; for instance, "Is geometry safe?" "How come when you push a 3D wheel forward it returns?" "Why don't circles go crinkly round the edge instead of smooth?" "Is 3D something coming true?"

5. Questions and investigations

Inspection of the children's 73 questions indicates that perhaps 80% could provide worthwhile investigations in geometry, and that many could be grouped for investigative purposes. The children's questions above about enlargement are in this category. So are many of their questions about tessellation: "What shapes can be in a tessellation?" "Can you use two shapes to pave a driveway?" "Can hexagons do tessellation?" "Can it work with a shape with wavy lines?" "Does it change if the piece of paper [tile or paving stone] is different sizes?" Questions such as these, and others such as, "What is a pyramid?" "If you make a pyramid from a plan, how many sides has it got?" "How many different 3D shapes can you get?" "How many angles have different shapes got?" could lead to the development of ideas which the curriculum writers suggest are appropriate at this level of schooling.

STUDENT TEACHERS' EXPERIENCES OF ELICITING AND USING CHILDREN'S QUESTIONS

This seemed to be partly because the children themselves were not used to asking substantive questions during mathematics learning. As one student teacher reported, "I don't think that they [the children] were used to being asked to ask questions" and another noted, "I found it hard, initially, getting questions from the children. They were more used to me asking them questions and often when I asked them if they had any questions they would say to me, 'Yes, what are we doing this for?' "

The difficulty in eliciting children's questions was also partly due to the inexperience of the student teachers. As one of those at the urban school commented, despite having obtained eight questions from her group of five children, "I need more practice with that skill." She was also aware of a strategy that she might use, "If I could have, like, turned what some of the things they were saying around so that they could see that they could make a question out of it, I would have got more questions."

Some others had managed to use this strategy in the course of their initial work with their children: From discussions, the children's thoughts and queries can be transformed into questions which can be investigated in future lessons." Another also found an indirect approach more effective: "I found it more effective to discuss as a group each activity, rather than asking them directly for questions. I found that through discussions the children asked me questions."

And did the children's questions provide a basis for meaningful investigations? The results seem to have been mixed. In many cases they did, for example, "Focussing on children's questions and allowing the children to investigate them proved stimulating, meaningful learning for the children." In at least one case this was not the student teacher's perception; her group of children "seemed to do the activities because they had to, not because they wanted to find out the answers."

CONCLUDING COMMENTS

This study represents a small first step towards understanding factors relating to children's questioning in mathematics. It suggests that although it is a new experience for many children at the primary school level, they are capable of beginning to ask questions in mathematics. These questions are likely to focus on aspects of mathematics incorporated in exploratory activities, often provide valuable insights into the children's thinking, and can frequently lead to useful investigations.

The results also indicate that special skill is needed to elicit children's mathematics questions at first. The primary science experience suggests that most primary teachers will need considerable support to develop this because it is also likely to require a considerable shift in a teacher's mental picture of what one does as a mathematics educator.

This investigation leaves many questions unanswered. Apart from those raised in the introduction to this paper which were not addressed by this study, it would be interesting to know whether a different set of exploratory activities would have produced more and/or different questions, whether children with some experience of asking questions in mathematics would ask more, and what is the prevalence of various questions across a range of schools. It also raises the issue of whether printed curriculum expectations about children engaging in question-asking in mathematics are likely to be realised without a relevant research foundation. Obviously there is scope for considerable research in the area of learners' questions in mathematics.

REFERENCES

Baker, A. and Baker, J. *Mathematics in process.* Perth: Eleanor Curtain, 1990.

Biddulph, F. Children's questions: Their place in primary science education. Unpublished D.Phil. Thesis, University of Waikato, 1989.

Biddulph, F. and Osborne, R. *Making sense of our world: An interactive teaching approach.* Hamilton: SERU, University of Waikato, 1984.

Biggs, E. E. and MacLean, J. R. *Freedom to learn: An active learning approach to mathematics.* Ontario: Addison-Wesley, 1969.

Bobis, J. Using a calculator to develop number sense. *Arithmetic Teacher,* 1991, 38, 5, 42-45.

Cemen, P.B. Developing a problem-solving lesson. *Arithmetic Teacher,* 1989, 37, 2, 14-19.

Chambers, P., Cunningham-Craig, J. and Hammond, J. *Bright ideas teacher handbooks: Maths.* Warwickshire: Scholastic, 1987.

Department of Education *Mathematics: Junior Classes to Standard Four.* Wellington: Government Printer, 1985.

Department of Education *Mathematics Forms 1 to 4 teachers guide: Introduction.* Wellington: Department of Education, 1986.

Griffiths, R. *Maths through play.* London: Macdonald, 1988.

Ministry of Education *Mathematics in the New Zealand curriculum.* Wellington: Learning Media, 1992.

Sullivan, P. and Clarke, D. Catering to all abilities through "Good" questions. *Arithmetic Teacher,* 1991, 39, 2, 14-21.

Tobias, S. *Succeed with math.* New York: College Entrance Examination Board, 1987.

Whitin, D. J. Number sense and the importance of asking "Why?" *Arithmetic Teacher*, 1989, 36, 6, 26-29.

Wigley, A. Models for teaching mathematics. *Mathematics Teaching*, 1992, 141, 4-7.

VISUALISATION AND THE DEVELOPMENT OF MENTAL COMPUTATION

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The purpose of this paper is to report findings of a study intended to investigate the effect of instructional activities designed to facilitate the development of visualisation strategies in young children. Utilising this intuitive capacity can provide a basis for developing number relations and devising strategies to learn basic facts. Given practice children can develop a reasonably large collection of set dot patterns that they recognise without counting. These patterns begin to be related mentally to one another as they enhance part-part-whole relations. Organising the dots into recognisable subgroups and patterns facilitates the ability to recognise and give number names to groupings, especially when the groupings are larger than five.

Instructional activities and materials that were used to facilitate the development of visualisation strategies in two kindergarten classes over a one year period are described. Students' verbal and pictorial responses are analysed and the implications for fostering the development of basic facts in this manner are discussed.

The results indicated that, with practice, children can become capable of mentally combining and separating patterns, thus able to instantly recognise the whole and its related parts - a very efficient strategy for aiding the recall of basic addition and subtraction facts.

VISUALISATION AND THE DEVELOPMENT OF MENTAL COMPUTATION

Should teachers consciously try to develop children's abilities to compute mentally before the time of independent development? From personal experience and that of teachers working with me in recent years, the advice is that developmentally appropriate experiences can and should be offered in the kindergarten and Year 1 mathematics programs. This view is one that other researchers have expressed (Fuson & Secada, 1986; Thornton, 1989) and one that expounds Vygotsky's zone of proximal development (1978). In this zone children can be helped by more capable peers and adults to learn new concepts and skills related to those already possessed. Work with children in their first year of formal schooling has provided evidence of the value of emphasising visualising skills. The aim is to increase the likelihood that prerequisite skills, such as subitisation, are well established before children are asked to apply them for obtaining quick responses to addition and subtraction facts.

The purpose of this paper is to report findings of a study intended to investigate the effect of instructional activities designed to facilitate the development of visualisation strategies in young children. Students' responses revealed that the visualisation activities used fostered the development of skills that enhanced learning of their basic facts.

Visualising number patterns

The process of instantaneous recognition of number patterns without counting is known as subitising (from the Italian word subito - immediately) (Folk et al., 1988). Piaget found that children, by the age of four, are capable of instantaneously recognising groups of one, two, three and four objects. Having been surrounded by groups of objects from a young age, children develop stable mental images of these patterns and they are soon able to reconstruct them for themselves. Constant exposure to the number names used to describe the arrangements help children form pattern-name associations (though they may not fully understand the quantitative aspects of the patterns till later). Utilising this intuitive capacity can provide a basis for developing number relations and devising strategies to learn basic facts. For example, given practice with dot patterns children can develop a reasonably large collection of set patterns that they recognise without counting. These patterns begin to be related mentally to one another as they enhance part-part-whole relations. Organising the dots into recognisable subgroups and patterns facilitates the ability to recognise and give number names to groupings, especially when the groupings are larger than five. Joining and separating visual patterns provides a basis for developing future number relations. For example, a child might recognise a pattern for "two" and a pattern for "three" in the "five" pattern (see Figure 1).

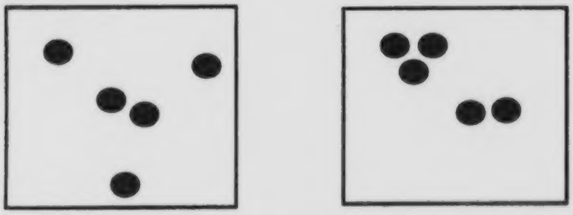


Figure 1 Two dot patterns for the number five. The second pattern is easier to recognise because the dots are arranged into smaller subgroups that facilitate subitisation.

THE STUDY

The following section describes instructional activities and materials that were used to facilitate the development of visualisation strategies in two kindergarten classes over a one year period. Students' verbal and pictorial responses will be analysed and the implications for fostering the development of basic addition and subtraction facts in this manner will be presented.

Participants

Two mixed ability kindergarten classes from a metropolitan Sydney school participated in the study. The classes contained 32 and 35 children that ranged between the ages of 4 years 5 months and 5 years 7 months at the start of first term. Boys and girls were equally represented in the former class but the girls slightly outnumbered the boys in the latter. Two female classroom teachers (one of whom was the chief investigator), participated in the study. Both teachers had more than 5 years teaching experience on a variety of grades. The teachers held meetings almost daily during the initial stages of the study and fortnightly during second, third and fourth terms.

Instructional Materials and Activities

Dot Patterns

A set of dot cards was made that represented the numbers from one to ten as an arrangement of dots. Except for the numbers one, two and three, each number was represented by at least three different dot arrangements. Figure 2 shows the cards that were presented for the number six.

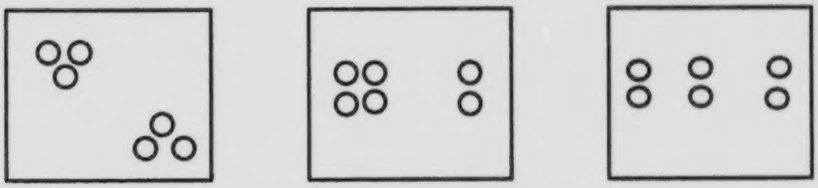


Figure 2 Dot cards for six

The exact size and construction of the cards are not significant. The important aspect of the cards is that they utilise the ability of children to subitise and they encourage mental shifts from thinking about a number as one type of dot arrangement (or part-part-whole combination) to another. For example, the arrangement for the number five could be seen as a group of three dots and a group of two dots or as a group of four dots with one left over.

Activities with the cards were done with the whole class and with small groups of children working independently. Some of the activities included 'flash-and-tell' and 'find a pair' (see Van de Walle, 1988, for a description of these and many others).

The Ten Frame

Ten frames are two-by-five rectangles into which counters or dots are placed to illustrate numbers less than or equal to ten. This model for numbers was originally designed by Wirtz and is explained in detail by Labinowicz (1985) and Thompson and Van de Walle (1984). Like the dot patterns, the ten frame utilises the developing capacity of young children to subitise. Children learn to recognise patterns and their number names according to the number of dots and empty frames occurring on a flash card. For example, if the ten frame for eight was flashed, the child could be asked:

- How many dots were shown on the card?
- How many empty spaces?
- How many if we add one more (or less) to the number shown?
- How many more to make ten?
- How many less to make five (or any other number)?

Other activities encouraged children to manipulate their mental images by adding or taking away dots to make a new number and then explain how they got their new number to a classmate using drawings, concrete materials and verbal descriptions. The overhead projector was used extensively by the classroom teachers to model these procedures to the class. Once one arrangement for a number was given, children were immediately challenged to think of another way to represent the same number. For a more detailed explanation of the activities see Bobis (1992). (For further activities see: Hoelter, 1987; Labinowicz, 1985; Thornton, 1989; Van de Walle, 1988.)

Activities emphasising visualising skills were integrated into various aspects of the ongoing mathematics programs and were incorporated into short sponge activities throughout the day both during and distinct from the regular mathematics lessons. Starting from week one of the first term in the school year we carried out activities like those described in the previous section and were supplemented by auditory patterning, counting back and counting on activities. Children were encouraged to verbalise the visual patterns they saw for different numbers and discussions concerning 'nice' (or easy to visualise) patterns were regular occurrences. The emphasis was initially on subitising skills using the dot cards for numbers less than five and was then extended to include larger numbers and activities with the ten frames. All the activities focused on identifying groups of numbers rather than counting one-by-one. They also required the children to focus on part-part-whole relationships of various number combinations - especially the decomposition of ten.

Students' Responses

The primary mode of obtaining information about pupils mental images was through their verbal comments in class discussion and individual interviews held in conjunction with their drawings and the instructional materials described above.

Whole class sessions (lasting for about 10 minutes) and individual interviews (lasting for 3 to 5 minutes) were regularly taped and transcribed so as to keep a record of the children's responses. In class discussions children's descriptions of the visual images they developed for the specific groupings varied widely. They formed mental images based on their own particular conceptualisation of part-part-whole relationships. Further, it was observed that as the pupils listened to each others' explanations, they began to think about the visually presented arrangements in more than one way and began to independently elaborate on and extend their own ideas. The following excerpts illustrate the mental strategies children were using to determine the number of dots on ten frames as they were flashed before them.

Excerpt 1:

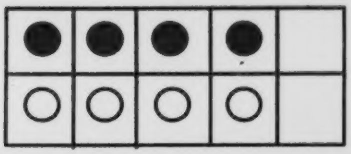
Teacher: (Flashing a ten frame) How many dots did you see?
Lara.

Lara: Nine.
Teacher: How do you know there are nine? You didn't have time to count all those dots.
Lara: There is one empty space and there are ten spaces altogether.
Teacher: How many more dots do we need to make ten dots?
Lara: One more.
Scott: How does she know that? How does she know that there are ten spaces altogether?
Teacher: Lara?
Lara: Because I counted them this morning when I was playing 'Flash' with Tess.
Teacher: Scott, you come and count them.......

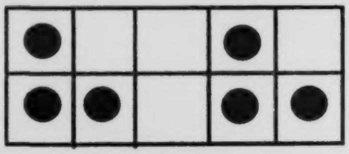
Excerpt 2:
John-Paul: I know why that's a pattern for eight.
Teacher: Come and show everyone then.
(John-Paul moves to front and points to the ten frame.)
John-Paul: See, there's four there and four there and that makes eight.
Teacher: Very good John-Paul.
John-Paul: I know another one.
Teacher: We'd love to hear another.
John-Paul: There's two spaces (pointing to the ten frame).
Teacher: How does two spaces tell you that there are eight dots?
John-Paul: (Pause)...Because..(Pause)..if there were two more there'd be ten.

Although neither John-Paul nor Lara had received any formal instruction in basic addition or subtraction facts, they were already thinking of ten in terms of its parts.

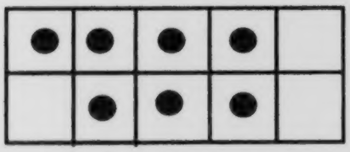
Although we were able to gain insights into students' mental activity through their verbal responses, their drawings also gave indications of how they thought of the visual patterns. The following figures are examples of drawings made by children to record the picture in their mind of their special number. Comments accompanying the drawings were made after reading interview transcripts. Light and dark shades indicate that children used different coloured dots.



Tara

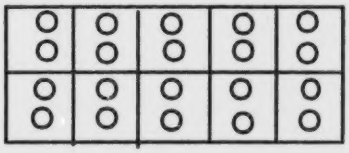


Timothy



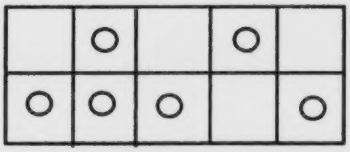
Dominica

Tara, Dominica and Timothy used the visual arrangement of dots to help them represent the mental image of their number. Timothy decided to use a doubling pattern (3 and 3) while Tara used colour to distinguish her pattern of 4 and 4. She decided to draw this number on her ten frame because "I see two spaces and I know there must be 8 dots".

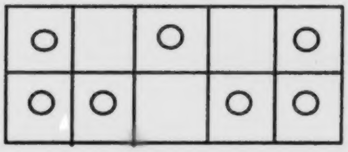


Natash

Natasha's representation for 20 is an example of how the brighter children were able to independently extend their own ideas to include number relationships beyond that traditionally taught at the kindergarten level. She shut her eyes and "saw two frames but I only had one frame to colour-in so I put two counters in each space and made twenty".



Patricia



Emily

Patricia's illustration did not immediately reveal any specific number relations and when questioned about her arrangement of dots it became evident that she did not understand the task and simply placed counters with no consideration to the arrangement of dots on the ten frame. An interview with Emily, on the other hand, revealed that she "saw" numerous part-part-whole relationships. Emily liked this arrangement because "there are 3 spaces and 7 dots and there are 3 here, 3 here and 1 in the middle. And there are 2 here and 2 here and 1, 2, 3 on top".

CONCLUSIONS AND IMPLICATIONS

The present study indicated that, with practice, children can become capable of mentally combining and separating patterns, thus able to instantly recognise the whole and its related parts - a very efficient strategy for aiding the recall of basic addition and subtraction facts. The ten frame imagery can be used as a visual reference in later work instead of relying on counting to derive sums of ten or answers to subtraction facts with minuends of ten. Although counting is certainly important, the objective of this paper is to stress the importance of developing visualising strategies as an efficient method for reinforcing number relations prior to the introduction of addition and subtraction in the symbolic form. For the majority of children in this study, it was found that once abstract mathematical symbols for the basic facts were finally introduced they were able to represent what they had previously visualised, talked about and illustrated in a very concise way.

More time needs to be devoted in the kindergarten and Year 1 mathematics programs to providing developmentally appropriate experiences that foster intuitive skills like visualisation. A program such as that outlined in this paper has implications for exact and approximate mental arithmetic skills in later years.

REFERENCES

Bobis, J. (1992). Visualising number patterns. *Square One*, 2, 6, 4-5.

Folk, C. Egeth, H. and Kwak, H. (1988). Subitising: Direct apprehension or serial processing? *Perception and Psychophysics*, *44*, 4, 313-320.

Fuson, K. and Secada, W. (1986). Teaching children to add by counting-on with one-handed finger patterns, *Cognition and Instruction*, 3, 229-60.

Hoelter, Margaret. (1987). *The number picture*. Melbourne, Oxford University Press.

Labinowicz, Ed. (1985). *Learning from children*. London, Addison-Wesley.

Thomas, C. and Van de Walle, J. (1984). Modeling subtraction situations. *Arithmentic Teacher*, 32, 8-12.

Thornton, C. (1989). "Look ahead" activities spark success in addition and subtraction number-fact learning. *Arithmetic Teacher*,*36*, 8, 8-11.

Van de Walle, J. (1988). The early development of number relations, *Arithmetic Teacher*, *35*, 6, 15-21 & 32.

Vygotsky, L. (1978). *Mind in society: The development of higher psychological processes*. Cambridge, Mass: Harvard University Press.

EXAM PERFORMANCE AND THE GRAPHICS CALCULATOR IN CALCULUS

MONIQUE A. M. BOERS AND PETER L. JONES

At Swinburne University of Technology we have found that, after the introduction of a graphics calculator in introductory calculus subjects, female students for the first time significantly outperformed males on calculus tests. Students solution strategies were examined as a possible reason for the superior performance of females, but seemed to point to the opposite result. Comparison of scores of exam questions with graphical and purely algebraic content revealed that females received their better marks due to their better performance on purely algebraic questions.

There have been a number of recent studies in which a graphical calculator or graphical package was used to support the teaching of calculus (Heid, 1988; Ruthven, 1990; Ryan, 1992; Tall, 1989; Teles, 1990 (cited by Dunham, 1992); Vazquez, 1991 (cited by Dunham, 1992)). Some of these studies show that there are several ways that students appear to benefit from this more visual approach to calculus. For example, Heid (1988) found that by focussing on concepts instead of procedures for the main part of the course and by studying concepts with the help of computer generated graphs, students' conceptual understanding was improved while their performance on a regular technique based calculus test was not hampered. Ruthven (1990) found that students who have access to graphics calculators in a traditional calculus course showed a significantly better capacity to translate graphs into an algebraic form than students who did not have access to such calculators in their course. He explained his results by stating:

> Regular use of a graphic calculator is likely to rehearse specific relationships between particular symbolic and graphic forms, as it is through such relationships that the calculator itself is operated, albeit in the reverse direction to that tested. Moreover, reliable access to graphics calculators is more likely to encourage both students and teachers to make more use of graphic approaches in solving problems and developing new mathematical ideas, not only strengthening these specific relationships, but rehearsing more general relationships between graphic and symbolic form. (p. 447)

Ruthven also found that girls and boys appeared to benefit differentially from the graphics calculator: the girls in his study profited more than the boys. He explained the positive influence on women by observing that the calculator provided feedback to women which might have the effect of reducing their generally higher level of anxiety and that the greater exposure to graphical images might have increased competence and confidence of female students. Similarly in a technology enhanced pre-calculus course Dunham (1991) found that, although males showed a superior performance than females on visual items of the pretest, both groups of students gained significantly over a ten week period in their competence on visual items. Furthermore, there were no longer competence differences between males and females on visual items of the posttest. Dunham (1991) also studied the impact of the graphics calculator on calculator neutral test items, that is, those that could be solved either algebraically or graphically. She did this by interviewing eight students after each test in the subject and by asking them what method they used to solve the problems. The majority of solution methods she found were either purely graphical or purely algebraic. Only 23% of the students' solutions contained a mixture of graphical and algebraic methods; students did say that in the exam they preferred the graphical approach because of the speed with which you could find an answer, but also expressed 'algebraic-guilt' about using the calculator.

With regard to visualisation skills, Shoaf-Grubbs (1992) found that female students' visualisation skills improved with a graphics calculator. However, these tasks were not specifically mathematically related. Recently, Boers & Jones (1992) found that the introduction of the graphics calculator into the first year of a Mathematics/Computer Science degree was associated with differential performance levels of males and females. With the introduction of the graphics calculator in 1991, females in this group scored significantly higher than males ($F(1,68)=2.14$, $p=0.02$), whereas in the two consecutive years before its introduction the males of this major scored slightly higher than females, although not significantly (see figure 1). In 1991 the females scored on average 10% higher than the males. While in 1992, the difference was of a similar proportion, 9%, but due to the smaller sample size did not reach statistical significance ($F(1,34)=0.70$, $p=0.41$).

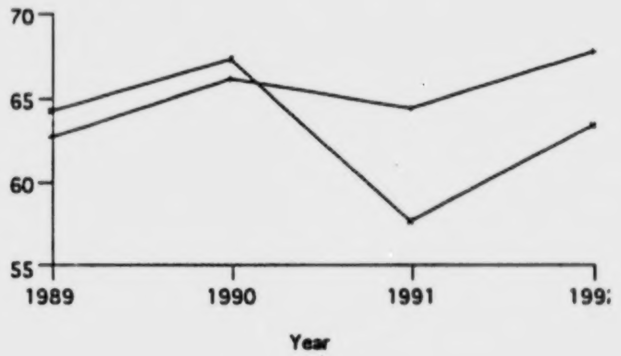


Figure 1: Comparison of mean Male and Female Math/Comp Science Scores. (+ Females, x Males)

This paper reports some recent investigations into the source of this apparent differential influence of the graphics calculator on the performance of males and females in which answers to the following research questions were sought:

1. Is there a difference in the kind of solution strategies, as defined by Boers & Jones (1993), used by males and females to answer questions with explicit graphical content?
2. Is the performance difference between males and females related to the question content?

METHOD AND RESULTS

Introduction.

In 1991 the TI-81 graphics calculator was prescribed for all first-year Applied Science students taking introductory calculus. The course content remained essentially unchanged from previous years, except that graphical solutions were given greater emphasis than in the past. In addition, the calculator was used as an integral part of the teaching process to provide alternative graphical representations of processes that were previously presented in symbolic form only, for example, limits. Students were also urged to use their calculator as checking devices. The assessment was not changed from previous years, therefore comparison with previous years was possible.

Comparison of Solution Strategies

In a previous paper Boers & Jones (1993) found that students used a variety of strategies when solving graphically oriented calculus problems. Some of these strategies were mathematically more sophisticated than others. It was thought that perhaps the superior performance of Mathematics/Computer Science females was because they used different strategies than males on graphically oriented calculus problems. To test this, the solution strategies of the Mathematics/Computer Science majors, a subset of 67 students, in which the females significantly outperformed the males after the introduction of the graphics calculator, were classified according to the scheme developed by Boers and Jones (1993) . Two questions on the paper were susceptible to this analysis.

The first question. The first question on which this analysis was done, was a routine graphing question involving a rational function with two discontinuities: one involving a vertical asymptote and the other being a removable discontinuity. The question was:

(a) For what values of x is the function $y = \dfrac{x^2+2x-3}{2x^2+3x-5}$ not defined?

(b) Find an exact value for the limit $\lim\limits_{x \to 1} \dfrac{x^2+2x-3}{2x^2+3x-5}$

(c) Sketch a graph of the function y.

Due to the limitations of the graphics calculator students needed to integrate algebraic information gathered from parts (a) and (b) of the question with the graphical image produced by the calculator to form a complete picture of the function. The graphics calculator will not show removable discontinuities unless they fall exactly on the centre of a pixel of the screen as was the case in this example. In the initial study of student solutions to this problem, three main solution strategies were identified:

Strategy I (successful integration of information):

the student successfully integrated both algebraic and graphical information in reaching their solution. An indication of successful integration was the presence of a gap in the graph or a circle around the point $x=1$ of the graph of $f(x)$.

Strategy II (failed integration of information):

the students attempted to integrate both algebraic and graphical information in reaching their solution but failed in their attempt. When conflict arose some tended to disregard the graphical information in favour of their algebraic work (Strategy IIA, failed integration: algebraic preference), that is, adjusting the graph by showing two vertical asymptotes, while others did the opposite (Strategy IIB, failed integration: graphical preference). The latter group adjusted the answer to part (a) to only one point where the function was not defined, that is, $x=-2.5$ only.

Strategy III (no integration of information):

students made no real attempt to integrate algebraic and graphical information in reaching a solution. There was no apparent realisation that the solution to an earlier part of the question bore any relationship to the graph they had to produce in the end. Three subgroups could be identified: one group of students who used an algebraic method to solve the first part of the problem (Strategy IIIA, No integration: algebraic), one who used a non-algebraic method (Strategy IIIB, No integration: non-algebraic), and one for students whose solution to the first and the third part of the question seemed purely calculator generated; no evidence of algebraic work was shown (Strategy IIIC, No integration: calculator only).

A detailed description of each of these strategies with sample student responses can be found in Boers & Jones (1993). In classifying the responses of the students in this study new strategies were not found. This confirmed the validity of the initial classification scheme. The range of strategies used by males and females along with percentage usage, are shown in Table 1.

***Table 1*: The number and percentages of students adopting each strategy for question 1.**

	Strategy			
	I Integration	II Failed Integration	III No Integration	Not Classified
Males N=40	6(15%)	6(15%)	23(58%)	5(13%)
Females N=27	1(4%)	6(22%)	20(74%)	0(0%)

From Table 1 we can see that in general the females used lower order strategies than males. That is, there were more males than females who were able to integrate the algebraic information with the graphical information and there were more females than males who appeared not to integrate the two pieces of information (74% versus 58%). Due to the small cell sizes appropriate statistical tests could not be performed.

The second question. The second question on which comparisons between strategies were made, was:
(a) Sketch a graph of $y = x^{1.4}e^{-x}$ $(x \_ 0)$, labelling the coordinates of the two stationary points.
This question was more straight forward, requiring less integration of information than the previous question. Three strategies were identified:

Algebraic
  the student primarily relied on their algebraic skills to arrive at their solution, only using the calculator to help determine the general shape of the function and possibly check the reasonableness of their solution.

Failed algebraic to calculator
  the student attempted an algebraic solution but failed, then attempted a primarily graphically based solution.

Calculator only
  solution appeared to be purely graphically and calculator generated; no evidence of any algebraic work shown.

The strategies found for the two questions are different because the second question demanded fewer integration of information skills from the students than the first question. The second question could be solved by calculator alone, whereas the first question needed either algebraic supplementation or other reasoning to solve entirely by calculator.

In classifying the responses of the students no new strategies were found, again confirming the validity of the initial classification scheme. The range of strategies used by males and females along with percentage usage, are shown in Table 2.

***Table 2*: Percentage and number of students adopting the three strategies for question 2.**

	Strategy			
	Algebraic	Algebraic/ Calculator	Calculator only	No graph
Males N=40	13(33%)	5(13%)	18(45%)	4(10%)
Females N=27	10(37%)	3(11%)	11(41%)	3(11%)

From the table it can be seen that women had a slightly higher preference for the algebraic strategy, whereas men were slightly more inclined towards the calculator strategy. This finding is similar to that of Dunham (1991). The mixed solution method (algebraic/calculator), which was very similar to the calculator only strategy in that the final solution was based on the calculator alone, was similarly preferred by males. For this question statistically significant differences between the choices of strategies of males and females were not found.

Conclusion. Comparison of the strategies used by males and females to answer questions with a graphical content does not seem to support the idea that the strategies chosen by females would make their scores on the exam higher than those of males. Based on the first question one might actually expect females' scores to be lower due to their less sophisticated responses to the question. Therefore a comparison of scores that males and females received on graphically and algebraically oriented problems might shed more light on the reasons for the difference in overall performance. This is done in the next section.

Comparison of performance on graphical and non-graphical questions.

In this section we investigate the relationship between the question content and the performance of males and females on the exam. The sums of scores of males and females on questions with a graphical content and those questions which required only algebraic work were compared. These are shown in Table 3. Problems that could be solved either graphically or algebraically, such as limits, were excluded from this analysis. In 1991, the females outscored the males on both sorts of questions, marginally on the graphical and significantly on the non-graphical questions. This does not support the contention that females outscored the males on the whole test because the graphics calculator helped their performance on questions with a graphical content. The difference was primarily due to their superior performance on non-graphical questions. When we look at a similar analysis for 1989, we see the reverse. In 1989, males and females scored equally well on algebraic questions, whereas on questions with graphical content females outscored males (see Table 3). It might therefore be the case that the performance of females did not necessarily improve, but that the males performed more poorly with the introduction of the calculator. Possible explanations could be that the males were more interested in the tool and therefore no longer as keen to study and exercise the problem solving skills necessary for a calculus exam that had as its major focus the assessment of manipulative skills.

***Table 3*: Comparison of scores of males and females on graphical and non-graphical questions in 1989 and 1991.**

	1989		1991	
	Algebraic	Graphical	Algebraic	Graphical
Males	24.2 (0.8)	10.0(0.4)	17(1)	5.3(0.3)
	N=71	N=71	N=27	N=40
Females	24 (1)	11.1(0.6)	20(1)	5.6(0.3)
	N=30	N=30	N=40	N=27
	p=.9	*p*=0.13	*p*=0.10	*p*=.42

SUMMARY AND CONCLUSION

In this article we have tried to explain a difference in performance of males and females that occurred when a graphics calculator was introduced into a first year mathematics program. We have looked at differences in strategies males and females use on questions requiring a graphical response and at differences in performances on graphical and algebraic questions. With respect to the strategies, when integration of graphical and algebraic information was required, more males than females were successful. When students had an option to solve a problem either algebraically or graphically, slightly more females preferred the algebraic strategy and, vice versa, slightly more males preferred graphical strategies. Based on the analysis of performances on graphical and purely algebraic questions we must draw the conclusion that the better performance of females was related to a better performance on algebraic questions. Considering the trend of the performance scores over the last four years, one might come to the conclusion that with traditional technique based calculus testing, males are disadvantaged by the introduction of the graphics calculator.

REFERENCES

Boers, M. A. M., & Jones, P. L. (1992). The graphics calculator in tertiary mathematics: An evaluation of the effect of the graphics calculator on attitudes and performance at the secondary/tertiary interface. Swinburne University of Technology, Melbourne.

Boers, M. A. M., & Jones, P. L. (1993). Students' use of graphics calculators under examination conditions. Submitted for publication.

Dunham, P. H. (1991). Mathematical confidence and performance in technology-enhanced precalculus: Gender-related differences. (Doctoral dissertation, The Ohio State University). Dissertation Abstracts International, 51, 3353A.

Dunham, P. H. (1992). Teaching with graphing calculators: A survey of research on graphing technology. In L. Lum (Ed.), Proceedings of the Fourth International Conference on Technology in Collegiate Mathematics (pp. 89-101). Reading, MA: Addison-Wesley.

Heid, M. K. (1988). Resequencing skills and concepts in Applied Calculus using the computer as a tool. Journal for Research in Mathematics Education, 19, 3-25.

Ruthven, K. (1990). The influence of graphic calculator use on translation from graphic to symbolic. Educational Studies in Mathematics, 21, 431-450.

Ryan, J. (1992, July). Integrating computers into the teaching of calculus: Differentiating student needs. Paper presented at the Fifteenth Annual Conference of the Mathematics Education Research Group of Australasia, University of New South Wales, Richmond.

Shoaf-Grubbs, M. M. (1992). The effect of the graphics calculator on female students' cognitive levels and visual thinking. In L. Lum (Ed.), Proceedings of the Fourth International Conference on Technology in Collegiate Mathematics (pp. 393-397). Reading, MA: Addison-Wesley.

Tall, D. (1989). Concept images, generic organisers, computers, and curriculum change. For the Learning of Mathematics, 9(3), 37-42.

Teles, E. J. (1990). Numerical and graphical presentation of functions in precalculus. (Doctoral dissertation, University of Maryland) Dissertation Abstracts International, 51, 777A.

Vazquez, J. L. (1991). The effect of the calculator on student achievement in graphing linear functions. (Doctoral dissertation, University of Florida) Dissertation Abstracts International, 50, 3508A.

YOUNG CHILDREN'S REPRESENTATIONS AND STRATEGIES FOR ADDITION.

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This is a report of a study of the representations and strategies for addition, used by a sample of 55 children in years 1,2, and 3 in three schools in Brisbane. Children were presented with operations represented in symbols and asked to explain their procedures as they worked. The teachers were also interviewed to determine the representations that they were introducing. The general developmental sequence was from use of objects, to use of counting to mental calculations using knowledge of number facts and place value. The results are discussed from the perspective of the demand that the procedures make on children's information processing capacity. We suggest that some of the difficulties occur because teachers introduce procedures that are recommended in curriculum documents without being aware of the cognitive load that they impose.

The intention of the research was to find out how children interpreted the symbols and the addition operation, and how they represented them with analogs and used strategies to solve the problems. The results were analysed from an information processing perspective and compared with the strategies and representations that the teachers introduced. The hypothesis was that some of the strategies and representations that teachers currently accept as good practice do not seem to be as successful as they should be and therefore need closer examination.

THEORIES OF COGNITIVE DEVELOPMENT AND REPRESENTATIONS

Addition is a binary operation which requires three elements and a relation between them to be mapped from one structure to another. It is therefore a concept at the system mapping level and should be possible for children to cognize in its most basic form from about 5 years onwards (Halford, 1988; in press; forthcoming). However not all children have developed the requisite capacity by that age to cognize such a concept and those who have may not perform to the limit of their capacity because of lack of requisite knowledge based on experience. Case and Sowder (1990) predicted, on the basis of the number of mental number lines that the child would need to take into account, that at about 6 years children should become capable of computing single digit sums, at about 8 years they should become capable of mentally computing double digit sums, and at about 10 years they should become capable of computing double digit sums with some form of carrying or regrouping. Their research showed this to be the case. We would explain the complexity of their tasks differently, and not tie the capacities so closely to ages and stages, but rather predict that after about 5 years of age the child should become capable of cognizing binary operations and that subsequent performance would depend on the complexity of the operation in terms of the number of relations that need to be considered and the child's prior knowledge.

Teachers of young children use concrete materials as representations in teaching mathematics in the belief that they will facilitate the learning process. The concrete representation should mirror the structure of the concept and the child should be able to use the structure of the representation to construct a mental model of the concept. In theory concrete materials as analogs should reduce learning effort and serve as memory aids; provide a means of verifying the truth; indirectly facilitate transition to higher levels of abstraction and be used generatively to predict unknown facts. There are however some possible disadvantages. Mapping from a concrete representation to a concept imposes a processing load and this can interfere with the understanding of a concept;

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[1] This research was supported by a grant from the Australian Research Council. We acknowledge the contributions made by Shauna Schulz in data collection and categorising, Paul Burnett for advice concerning statistical analyses, and Graeme Halford for the application his theory.

if an analog is poor or not properly understood it can generate incorrect information; and if it is not well mapped into the material to be remembered it can actually increase the processing or memory load of a task (Halford and Boulton-Lewis, 1992).

There is a growing body of evidence which shows that concrete materials often fail to produce the expected positive outcomes (Boulton-Lewis, in press; Bobis, 1993; Hart, 1989; Sowell, 1989; Lesh, Behr & Post, 1987; Dufour-Janvier, Bednarz & Belanger, 1987; Lesh, Landau & Hamilton, 1983).

Most of the concepts taught in the early years of school, such as addition and subtraction operations require two way mappings if analogs are used. The child is often expected to start with symbols or numbers expressed in word problems, to map the symbols into concrete representations, to perform the operation with the concrete materials and to map the resulting quantity back into symbols (verbal or written). We have described the structure mapping required to show how 324 minus 179 can be understood, as opposed to performed, (Halford & Boulton-Lewis, 1992). In performing that task, to understand how the concrete analog supports the subtraction algorithm the child must cognize a complex set of relationships of relational and system level mappings. The operations require more information to be processed at once than even most adults are capable of. When the complexity of the mappings for three digit operations is analyzed in this way it becomes clear why the loads required to make the mappings in multidigit addition and subtraction could be impossibly high for a young child who did not know all the lower level mappings. The same is true for 2 and 3 digit addition operations.

Sweller and colleagues have undertaken extensive research which has lead them to propose a cognitive load theory which predicts that tasks will be more difficult if there is redundancy in the information which must be processed or if attention must be split between two different sources of information (Tarmizi & Sweller, 1988; Ward & Sweller, 1990). On the basis of his theory and our results for subtraction (Boulton-Lewis, in press) it seems that perhaps the most difficult way to perform two or three digit addition, in terms of the load on processing capacity, is to use analogs, to support limited knowledge of place value, whilst trying to learn to use an algorithm. It is probable that if a child has a good understanding of basic addition as an operation, and understands place value, then mental strategies would make less of a processing demand. For example if a child were to see the problem 265+126 represented in written form, either horizontally or vertically, and reason mentally as follows she would need to hold and process less information at any one time than she would if she needed to make mappings from the concrete representations to the symbols in a written algorithm; 6 and 5 are 1 more than 10 so there's one unit, 9 tens and 3 hundreds, that is 391.

The explanation that is proposed then for the difficulties often experienced with concrete representations, and children's apparent reluctance to use them, is that previous analyses have taken insufficient account of the processing loads, over and above the processing load of the basic operation, that their use initially entails. This implies that children often will not derive the anticipated benefits of concrete representations because, if the analogs are not well known, then they make an extra demand on processing capacity and the processing load of the mapping can become too high for the cognitive resources they can apply to the task. In order to reduce the load it is important to ensure that the child understands the operation, the relations between quantity, numeration and place value, and any symbolic and concrete representations of the task.

CHILDREN'S ADDITION STRATEGIES

There is a large body of research which is concerned with describing and understanding young children's solutions to simple addition problems (cf. Carpenter, 1985; DeCorte & Verschaffel, 1987; Siegler, 1987). De Corte and Verschaffel (1987) compared their results, for strategies used by children in first grade to solve subtraction and addition problems, with those of Carpenter & Moser (1984). Carpenter and Moser's scheme for classifying children's solution strategies had two dimensions; first strategies were identified as additive or subtractive and then ordered according to the level of internalization; material (using objects), verbal (using counting), and mental (using known number facts). De Corte and Verschaffel identified similar levels of internalization.

There appears to be a remarkable amount of consistency in the literature regarding the strategies that young children use to solve addition problems. We classified the strategies according to whether they were material, verbal or mental (cf. De Corte & Verschaffel, 1987). After examination of our data we collapsed some categories, added others and used the codes for data analysis. In summary the strategies were as follows MATERIAL; CAA (counting all with analogs), CAJ (counting all in two sets with or without joining), COA (counting on from first or larger number), VERBAL; CA (counting all without analogs, COWA (counting on without analogs starting from the first or larger), MENTAL; RF (Recall of memorised facts), DF (Use of known or derived facts), OTHER; WAA (Written algorithm with analogs), WAWA (Written algorithms without analogs), PVA (Place value with analogs), PVWA (Place value without analogs), IM (Inappropriate methods), UNK (Indeterminate methods).

METHOD

Sample This consisted of 55 children, 18 in year 1, 19 in year 2, and 18 in year 3 in three suburban schools in Brisbane in low, medium and high socioeconomic areas.

Design Each teacher was asked to identify in advance those aspects of addition that she intended to teach and the representations and strategies that she intended to use. Two interviews with each child were conducted in term 2 and term 4 and separated by 10 weeks approximately to determine knowledge of content, and use of representations and strategies. All interviews with the children were videotaped and subsequently viewed, transcribed by the interviewer and then categorized and coded in consultation with the authors.

Test Items The test items were as follows; **1**. 3+1; **2**. 5+1; **3**. 2+0; **4**. 6+0; **5**. 4+3; **6**. 2+5; 7. 3+3; **8**. 4+4; **9**. 12+3; **10**. 21+4; **11**. 2+10; **12**. 34+5; **13**. 14+10; **14**. 9+7; **15**. 6+8; **16**. 12+21; **17**. 13+31; **18**. 17+5; **19**. 34+6; **20**. 121+324; **21**. 241+312; **22**. 265+126; **23**. 173+119; **24**. 143+281; **25**. 423+294. **26**. 166+254; **27**. 124+387. These required addition of 1, 0, doubles and other combinations within 10, addition of one digit to two digits with and without regrouping, and two and three digits with and without regrouping.

Materials These were selected on the basis of what the teachers said they intended to use, and included Multibase Arithmetic Blocks (MAB in base 10), sticks singly and in bundles of 10, counters, Unifix, the symbols " = " and " + " and the numerals for all problems written on cards, and paper and pencil for written calculations.

Procedure The child and the interviewer discussed the materials and their use. The child was shown the first appropriate addition task in horizontal form and asked to read it. If the child could not read the numerals they were read and discussed by the interviewer. It was suggested that he or she could use fingers, materials or paper and pencil to represent the operation and find the answer. The child was encouraged to talk about the process. Testing began at an appropriate item for each year level. Each child stopped after difficulty with three tasks in succession.

RESULTS

Teacher expectations and strategies Teachers said they followed the recommendations in the State curriculum. These will be summarized.

Children's responses A MANOVA for time by year was computed for the categories CAA, CAJ, CA, COA, COWA, RF, DF, WAA, WAWA, PVA, PVWA, IM and UNK. The overall year by time effect was significant (Pillai's trace =.94, F=2.5, DF=2,52, p=.001). The univariate analyses for year by time showed that differences for CAJ (counting all in two sets with or without joining), RF (recall of memorised facts), DF (use of known and derived facts) and PVWA (place value without analogs) were significant, F=4.9 (p=.01), F=10.3 (p=.000), F=5.9 (p=.005) and F=4.7 (p=.005) respectively. Tukey post hoc analyses were used to confirm honestly significant differences between groups early and late in the year in these categories. These differences are shown developmentally in Figures 1A, 1B, 1C and 1D. These will be discussed.

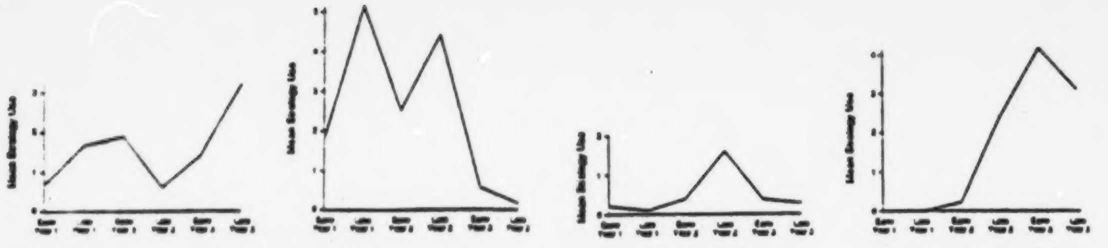

FIGURE 1A: Year by time effect for CAA FIGURE 1B: Year by time effect for RF FIGURE 1C: Year by time effect for DF FIGURE 1D: Year by time effect for PVWA

FIGURE 1 Significant differences for year by time

The univariate effects by year were significant for CAA (counting all with analogs), COA (counting on from the first or larger number), COWA (counting on without analogs), RF (recall of memorised facts), DF (use of known or derived facts), PVWA (place value without analogs), and IM (inappropriate methods). Tukey post hoc analyses indicated honestly significant differences at the 0.05 level for CAA (Fig 2A), for COA (Fig 2B), for COWA (Fig 2C), for RF (Fig 2D), for DF (Fig 2E), for PVWA (Fig 2F), and for IM (Fig 2G). There were no significant differences between years for use of written algorithms with or without analogs although there was an increase in their use in year 3. The developmental trends shown in Figures 2A to 2G show that children generally chose to use material strategies and count until year 2, to use the verbal strategies of recall of facts and derived facts until year 2, and then to use place value explanations (mental) or joining sets of objects and counting (material) as frequently at least as written algorithms with or without analogs in year 3 despite the fact that had been `taught' to use them.

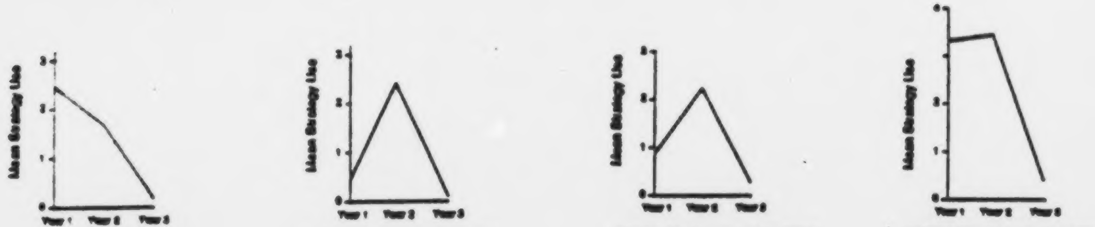

FIGURE 2A: Year effect for CAA FIGURE 2B: Year effect for COA FIGURE 2C: Year effect for COWA FIGURE 2D: Year effect for RF

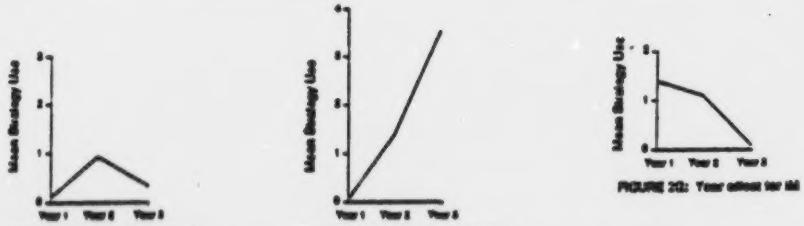

FIGURE 2G: Year effect for IM

FIGURE 2E: Year effect for DF FIGURE 2F: Year effect for PVWA

FIGURE 2 Significant differences by year

An analysis of strategy use by item confirmed the developmental trends obtained from the MANOVA. It also showed very clearly that children change strategy use to suit their confidence and perception of the difficulty of the task as proposed by (Siegler & Robinson, 1982; Russell, 1977).

The items on which children were incorrect 50% or more of the time were as follows 2+0, 12+21, 13+31, 121+324, 265+126, 173+119, 281+143, 387+124 . These errors and time of occurence will be discussed.

Most frequently children chose to use no analogs at all. The second most frequently chosen analog was fingers. There was limited use of MAB in years 2 and 3.

The oneway analysis of variance of categories by school showed no significant differences between schools.

DISCUSSION

The results indicate that children preferred to use verbal and mental strategies rather than formal algorithms and did not want to use analogs unless they could not perform the task in any other way. The results are encouraging in that children are making sense of the operations using their own strategies and apparently reducing the processing load by using no materials or materials only. They are disheartening if the objective is to have children relate their previous knowledge of addition to formal written procedures. Reasons for these results will be suggested and implications for teaching discussed.

REFERENCES

Boulton-Lewis, G. M. (in press) Young children's representations and strategies for subtraction. *British Journal of Educational Psychology.*

Carpenter, T.P. (1985). Learning to add and subtract: An exercise in problem solving. In E.A. Silver (Ed.) *Teaching and Learning Mathematical Problem Solving: Multiple Research Perspectives.* Hillsdale, N.J.: Erlbaum.

Carpenter, T.P. and Moser, J.M. (1984). The acquisition of addition and subtraction concepts in grades one through three. *Journal for Research in Mathematics Education*, 15, 179-202.

Case, R. & Sowder, J.T. (1990). The development of computational estimation: A neo-Piagetian analysis. *Cognition and Instruction, 7*(12), 79-104.

De Corte, E. and Verschaffel, L. (1987). The effect of semantic structure on first graders' strategies for solving addition and subtraction word problems. *Journal for Research in Mathematics Education*, 18(5), 363-381.

Dufour-Janvier, B., Bednarz, N. and Belanger, M. (1987). Pedagogical considerations concerning the problem of representation. In C. Janvier (Ed.), *Problems of Representation in the Teaching and Learning of Mathematics* (pp. 109-122). Hillsdale, NJ: Erlbaum.

Halford, G.S. (forthcoming). *Children's Understanding: The Development of Mental Models.* Hillsdale, N.J.: Erlbaum.

Halford, G.S. (in press). Analogical reasoning and conceptual complexity in cognitive development. *Human Development.*

Halford, G.S. (1988). A structure-mapping approach to cognitive development. In A. Demetriou (Ed.), *The Neo-Piagetian Theories of Cognitive Development: Toward an Integration.* Amsterdam, Netherlands: North-Holland.

Halford, G.S. and Boulton-Lewis, G.M. (1992). Value and limitations of analogues in teaching mathematics. In A. Demetriou, M. Shayer and A. Efkliades (Eds.). *Neo-Piagetian Theories of Cognitive Development: Implications and Applications for Education.* London: Routledge and Kegan Paul.

Hart, K. (1989). There is no connection. In P. Ernest (Ed.), *Mathematics teaching: The state of the art.* London: The Falmer Press.

Lesh, R., Behr, M. and Post, T. (1987). Rational number relations and proportions. In C. Janvier (Ed.), *Problems of Representation in the Teaching and Learning of Mathematics* (pp. 41-58). Hillsdale, NJ: Erlbaum.

Lesh, R., Landau, M. & Hamilton, E. (1983). Conceptual models and applied mathematical problem-solving research. In R. Lesh & M. Landau (Eds.), *The Acquisition of Mathematics Concepts and Processes.* New York: Academic Press.

Russell, R.L. (1977). Addition strategies of third-grade children. *Journal of Children's Mathematical Behaviour, 1 (2),* 149-160.

Siegler, R.S. & Robinson, M. (1982). The development of numerical understandings. *Advances in Child Development and Behavior, 16,* pp 241-312.

Sowell, E. J. (1989). Effects of manipulative materials in mathematics instruction. *Journal for research in Mathematics Education* , 20(5), 498-505.

Tarmizi, R. & Sweller, J. (1988). Guidance during mathematical problem solving. *Journal of Educational Psychology: General, 80,* 424-436.

Ward, M. & Sweller, J. (1990). Structuring effective worked examples. *Cognition and Instruction, 7,* 1-39.

CHANGING MATHEMATICS TEACHING: THE CASE OF CHRISTINE BROWN

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In this paper, we describe the case of a teacher, Christine Brown, as she incorporates new teaching approaches and organisational strategies within her classroom practice. Drawing on a set of interviews and observations, we analyse the effects of the experimentations with classroom organisation, management and teaching approaches carried out by Christine. We find that Christine made significant shifts in her practice that are consistent with constructivism. Furthermore, her classroom experimentations did result in a change of beliefs about how students might best learn mathematics.

Concerns in New Zealand about student achievement in mathematics have led to the preparation of a substantial mathematics curriculum statement, 'Mathematics in the New Zealand Curriculum' (Ministry of Education, 1992). The statement recommends teaching approaches which are consistent with constructivism. There is also recognition that, in order to implement this curriculum initiative, significant changes in classroom practices will be required among a large sector of the mathematics teaching community.

Claxton and Carr (1992) have conceptualised the process of teacher change as passing through a number of stages. They suggest that change may involve teachers moving from initial rejection of an alternative approach, to consideration of the ideas associated with the approach, a phase of experimentation and evaluation of the results of the approach (sometimes associated with feelings of disappointment that it has not worked as well as it might), and finally a phase in which the approach is owned and incorporated in their teaching repertoire.

THE CASE STUDY

Christine Brown is one of eighteen teachers who participated in a Teacher Development Project. The teachers set their own goals and were asked to reflect regularly on the success or otherwise of their day-to-day classroom practice. They were encouraged by each other and by the Project researchers to make adjustments to their classroom organisation, management, and teaching approaches as they saw fit. The teachers participated in regular sessions where they shared teaching ideas, approaches and strategies.

Christine had been teaching at a girls' secondary school for six years. She indicated that, although she was a mature teacher, she was, 'not that experienced'. She expressed the belief that students need to understand what they're doing and why they're doing it and said that she wanted her students to develop an attitude of enquiry.

Christine was observed in her classroom on five separate occasions. Three of the observed lessons are described and analysed in order that shifts in teaching practices can be documented.

Lesson One (February 1991): Teaching Geometry: Angle Sum of Polygons

Students sat in rows facing the front of the class. In this lesson, students were first required to draw a pentagon, measure each angle and then add up the five angles after first estimating the sum. They were asked to record their results in a table format given to them by Christine. When the students had finished, Christine called for answers from every student in turn, recording them without comment on the board as she went. When all the angle sums had been recorded, Christine suggested that, "A lot are close and a few might want to add their angle measures again". She went on to indicate whose results should be reconsidered. Other students were asked to repeat the exercise with a quadrilateral and then go on to complete the table for other polygons given on a worksheet. She pointed out that, "Triangles should be close to 180" before asking the class, "What do you think a quadrilateral should be?"

A review of this lesson suggests that the task set for the students is 'closed' rather than 'open' and for the most part, students were required to 'discover' what mathematical ideas the teacher 'had in mind'. It was Christine who drew each of the tables for recording data. It is questionable whether the students realised that the process of tabulating data provided a helpful way to later view the data in order that generalisations could be made with some ease. A constructivist teacher might have been more inclined to facilitate group discussions about good ways to present data so giving rise to a learning opportunity in which the useful strategy of tabulating data as an aid to pattern seeking might 'emerge'.

The evidence collected from these first observations suggests that Christine's learning model might best be represented as 'guided discovery' which, as suggested by Burkhardt and Fraser (1992, p 5), is 'really an alternative style of explanation'.

Future Directions

Christine used summaries of the observed lessons to help identify aspects of her teaching that she intended focussing attention on in the coming months. She subsequently set herself the goals of 'standing away from the board', encouraging students to take more responsibility for their own learning, extending her use of group work and using a mixture of teacher problems and student generated problems. In talking about 'standing away from the board', Christine indicated her intention for her lessons to become less teacher-centred and more student-centred. She explained her rationale for introducing group teaching.

> ... originally I used to have my class in single rows, and it was silly because ... and there were fifteen hands going up around the room. I've certainly used that quite a bit - you know - group setting. Single desks would be all right sometimes ... (Interview, May 1992)

Extracts from Christine's diary suggested that, while she plans for her teaching in the form of a simple yet structured agenda, she begins to comment on her role as a listener and observer during group activities, suggesting progress towards a more student-centred approach. While she identified student 'errors' that may have arisen from serious student misconceptions, she made no comment as to what she did about them at the time nor what strategies she might employ for overcoming them.

Lesson Two (April 1992): Teaching Equations and Graphs

Students sat in groups of four or five. The focus for the lesson arose out of a discussion between Christine and the researcher. Christine had been working on the beginnings of equations and the researcher suggested that this might be extended by looking at a graphical solution of pairs of simple simultaneous equations that involved a context familiar to the students. Making choices as to the best rate for babysitting among several such rates for different hours worked arose in the discussion so it was agreed that Christine would develop a lesson around this theme.

Christine presented the problem of choosing from three different babysitting rates: Smiths, who pay $10 basic and $2 per hour, Grants who pay $5 per hour, and Thompsons who pay $10 for up to 2 hours and $4 per hour after that. Each group was provided with sets of tables showing time and earnings for the students to complete, an incomplete literal rule for each rate, and a graph of the Smith's rate as an example. Graphs illustrating the other rates were to be drawn on the same set of axes.

The students worked collaboratively on this problem for 40 minutes while Christine travelled around the room checking the progress of individuals and groups. The students then went on to make up and graph their own suggestions for babysitting rates.

As part of her lesson wrap-up, Christine asked the students to arrange their equations in order of steepness and then relate steepness to pay rate. In a brief class discussion that followed, Christine drew attention to the comparative steepness of two graphs that had been drawn on the board by students.

Following the lesson, Christine indicated that she felt that the lesson had gone reasonably successfully. "I usually start graphs by plotting points - '4 up and 3 across'. Today's lesson seemed to be much more successful because it built on what they already knew. They didn't seem to have a problem at all. Plotting points the way I have done it in the past is kind of sterile - it isn't related to anything." In a conversation with the researcher, a group of students indicated that they enjoyed maths like this, "because it is more than using numbers and multiplying". They also reported that for them, the mathematics involved in this problem was, "Making a choice - solving a problem". In subsequent written reflections on the lesson, Christine noted that the students really liked making up their own problems but reported that she found it difficult to draw the threads together in her wrap-up. She would have liked to have used more of their own examples but just ran out of time.

The lesson reflects Christine's use of an inductive approach in which she tries to assist pattern seeking leading to generalising by firstly setting up tables of data for the students to work on. While the continued use of this approach, identified in each of the lessons observed previously, may suggest that Christine had made little variation to her general teaching approach, there are features of this lesson that suggest both development and change. The lesson had provided an opportunity for Christine to experiment with adjustments to her teaching in two areas. Firstly the lesson had focussed on a context that provided a real link with the students' own lives. And secondly, the interest generated from this real context had resulted in meaningful student interaction and collaboration as they made their decisions about the best rate.

The following comments made by Christine about this lesson illustrate the early development of Christine as a constructivist teacher.

> ... so from the problem which was one that I think they all related to pretty well, they actually learned a lot of things and they could see the point of drawing a graph, so we didn't sort of go, I didn't go and say this is how you draw graphs, this is an x-axis, this is a y-axis, and you know this is how you plot these points. We actually approached it from a place where drawing a graph would help us solve a problem. (Interview, May 1992)

Lesson Three (May, 1992): Teaching an Investigation Related to Volume

The main part of the lesson began with Christine relating a story to the class about a former colleague who embarked on a fund raising exercise at another school. The teacher, together with the school rugby team, had won a contract to deliver new telephone books throughout Auckland. But they had not worked out the amount of storage required for the books prior to delivery and were subsequently embarrassed as they began storing the books in every possible location at the school; the hall, the gymnasium and so on. Christine then went on to tell the class that she wanted them to see if they could find out how much room they would need; how many rooms (classrooms) etc, to store the Auckland phone books. Each student was then provided with a prepared handout with the following directions as to how to proceed.

1 Find out approximately how many numbers are listed in the book.
You will need to find out
- how many pages in the book
- how many columns in a page
- how many numbers in a column

2 Work out how many phone books would fit into this classroom?
You will need to:
- measure a phone book
- measure the floor of this room
- work out how many phone books would be needed to cover the floor
- measure the height of this room
- work out how many layers of books would fit

3 How many classrooms would be needed to store all the phone books?

Christine told the class that ... "at the end of the session everyone should have written down in their book all the important pieces of working". Christine then travelled around the class taking time to interact with each group. This involved, posing problems to focus students' ideas, explaining how to use equipment such as measuring tapes, explaining the steps the students should take, and asking students to explain their thinking and ideas. The following recorded interactions between Christine and a group of four students is illustrative of the teacher student interactions during this part of the lesson.

> *Christine:* The next thing you need to do is to work out how many phone books would be needed to cover the floor. OK imagine how many phone books are going to fit all the way across the room like that? Can you work that out?
> *Students*: Can we put the phone books on the floor?
> *Christine:* Well you could but you should be able to work it out from what you've measured. You should be able to work it out from the measurements that you have written down. Now think about it.
> *Students:* Shall we do it this way?
> *Christine:* Right, how wide is that? You've got it written down.

Students: 21
Christine: Alright. So how many 21's are going to fit across here? How wide is this? Should we change that into centimetres. What would that be in centimetres?
Students: 7,300.
Christine: No, that's 735. Alright, so what you've got to do now is figure out how many times 21 goes into 735. Go to it. And you can write that down and then you'll have to do the same thing the other way but remember the book goes this way.

Christine began her lesson wrap-up by asking each group for their response to the question, 'How many phone books fit into this room?' The responses from three groups were similar and around 70,000 while the other two groups significantly different. One of these reported about 100,000 while the fifth suggested 780,000. Christine suggested that the last group had got an extra zero and then went on to suggest that they take the number of phone books in a room to be about 70,000.

A similar whole class discussion ensued about the number of phone numbers in a book. Christine concluded from the group responses that a reasonable assessment might be about 400,000 and then went on to explain that the answer they were looking for, that is the number of rooms required to store all the phone books could be found by dividing 400,000 by 70,000. Christine explained that a quick way to do this was to cut out some zeroes leaving 40 divided by 7 so producing an answer of about six rooms. Finally, she explained that each person listed in the phone book would receive three books (one book of white pages and two of yellow pages). So altogether 6 x 3 or 18 rooms would be required to store the books prior to delivery.

The lesson was tape recorded and analysed minute by minute. An analysis of the lesson transcript reveals that, while Christine used a 'real' problem to focus the lesson, the lesson itself was dominated by Christine as she first set the scene, posed the problem, and then assisted students individually or in groups to follow the steps required to solve the problem. There was little opportunity for students to devise their own plans of action and to discuss those plans prior to making the necessary measurements and calculations. Of a 55 minute period, 34 minutes were taken up with organisation, posing the problem and explaining, while 21 minutes involved asking open questions and listening.

Christine had intended that her teaching during the lesson observed in May, 1992, (Lesson Three) would be largely 'hands off', that is, she would take on the role of facilitator rather than that of explainer. The task she set, that of finding the number of rooms to store a new edition of telephone books prior to delivery, had been carefully structured so as to, in Christine's view, enable the students to find a sensible answer to the problem by following a sequence of steps. As a result of the careful structuring, the task which was both interesting and challenging, seemed to degenerate into a set of teacher prepared exercises. It seemed that Christine, in her desire to keep a close control of the lesson and its direction, had provided the mechanism for the students to, in turn, control her. Their inertia as they sought to understand what was expected of them coupled with their uncertainty as to the requirements of each step, provided the impetus for Christine to give the individual attention that students so often expect of their teacher. Thus Christine found herself in the role of explainer, not just to one student or one group but rather to all groups and all students. The mismatch between what Christine had intended and what had actually occurred led to some personal frustration for Christine and it is likely that this frustration in turn led to her doubting the efficacy of this teaching approach. A factor here is the change required of students under the new curriculum. If their teachers continue to see them as receivers rather than constructors, it is likely that they will act that way, thus making the job of the beginning constructivist teacher more difficult.

TEACHER CHANGE - WHAT HAPPENED FOR CHRISTINE?

The stages associated with teacher change conceptualised by Claxton and Carr, (1992), seem to be clearly evident in the change in teaching approach that Christine adopted during the Project. Certainly Christine underwent a phase of experimentation and as she experimented, there were the feelings of disappointment that her approach had not worked out as it might have. Experimentation was evident in the last two lessons. Both lessons used a group format to promote more student-centred activity and both began with Christine setting the scene with a situation that related to the world of the students. Christine herself identified this aspect as her significant area of change.

> ... I've tried to get them more now to think about problems. Before I suppose I tended to teach by skill and then extend that into problems whereas I try now more to go the other way. To look at problems first and then ... into skills. (Interview, May 1992)

Although there were disappointments as she experimented, Christine was able to confidently explain newly acquired beliefs about how students best learn mathematics. Her excitement with the results of her 'babysitting rates' lesson, together with the clarity of the explanation of the approach she used as the basis for successful learning in mathematics, suggest strongly that not only did Christine participate in Claxton and Carr's experimentation phase but also began to reflect upon her teaching practice and clarify her own philosophy. This is consistent with the findings of Schon (1987), which suggest that clarification of the ideas and philosophy of a new practice occur after experimentation occurs and those of Ellerton, Clements and Skehan (1989), who found that teachers involved in an action research project became more conscious of what they did and as they became more confident, they developed as reflective practitioners.

Shortly after the end of her involvement in the Project, Christine was invited to make some additional reflective comments on her teaching. The comments illustrate some of the personal conflicts that confronted Christine as she sought to make adjustments in her teaching. Her comments also indicate how Christine has resolved some of the conflict.

> I have changed in that I am prepared to try different things, to allow noise and group activities and discussion, but I am not prepared to throw away the syllabus and 'go with the flow'. I am not secure enough to throw away all my props. I get upset when the environment seems 'messy' to me and when I feel that I'm not in control.
>
> I do think that my classes are happier and have a more positive attitude to maths though I'm not sure if they actually learn more. I mostly enjoy my teaching and like thinking up new ideas or adapting those I've got from courses or other teachers. I seem to need this stimulation. If I am bored, then my classes become bored too. I need to continue trying new and different things so that my enthusiasm is contagious.

REFERENCES

Burkhardt, H. & Fraser, R. (1992). An overview. In B. Cornu & A. Ralston (Eds.) *Science and technology education 44. The influence of computers and informatics on mathematics and its teaching.* Paris: UNESCO.

Claxton, G., & Carr, M. (1991). Understanding change: The personal perspective. In A. Begg (Ed.) *SAMEpapers 1991.* Hamilton: Centre for Science and Mathematics Education Research.

Ellerton, N. F., Clements, M. A, & Skehan S. (1989). Action research and the ownership of change: A case study. In N. F. Ellerton & M. A. Clements (Eds.) *School mathematics: The challenge to change.* Geelong: Deakin University.

Ministry of Education. (1992). *Mathematics in the New Zealand Curriculum.* Wellington: Learning Media.

Schon, D. A. (1987). *Educating the reflective practitioner.* London: Jossey-Bass.

INSTRUCTIONAL GAMES: A MEANINGFUL CONTEXT FOR LEARNING

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Consistent with a growing awareness of constructivism and child centred learning and teaching approaches, a research project is presented where instructional games were implemented into a classroom setting. The project allowed the children's mathematical concepts to be developed and their understanding consolidated and reinforced by creating an environment which used language as a bridge between the children's informal mathematical knowledge and abstract mathematical concepts.

> ""The most important single factor influencing learning is what the learner already knows. Ascertain this and teach him (her) accordingly"
>
> Ausubel, School Learning, 1968.

Many educators, researchers, and members of the general public are becoming increasingly more aware and accepting of child centred approaches to learning and teaching. As a result children are being encouraged to talk about and share their ideas and conjectures with peers and important others without fear of rejection or reprimand. A child's previous experience not only influences her/his acquisition of in-school knowledge, it influences the constructed realities of other learners.

Numeration skills continue to play an important role in both primary and lower secondary mathematics curriculums. This intuitive sense for number is vital for further mathematical learning and essential for future placement in the workforce (Carl, 1989; Resnick, 1989; Booker, Briggs, Davey & Nisbet, 1992). Consequently it becomes pertinent for schools to adopt sequential and meaningfully developed mathematics programs, especially within the areas of skill and concept. The *National Statement On Mathematics for Australian Schools* (1991) and Queensland's current P-10 syllabus documents (1987) reflect these concerns by linking conceptual understanding with procedural understanding via the delayed introduction of written algorithms and by encouraging the use of technology and problem solving activities in the classroom.

This numerical understanding can be achieved by engaging in sequentially planned activities that begin with the manipulation of concrete referents and gradually move toward pictorial or semi-symbolic representations which in turn evolve into mental images. This sequence is accomplished by making important links between a child's everyday language and more formal mathematical discourse, until abstract mathematical concepts finally emerge in the form of meaningful symbols.

Thus, knowledge is expressed and sustained by a diversity of negotiated mediums. For instance, language has long been a means of expressing and maintaining the mathematical concepts that enhance daily living. Consequently, the role of language in the learning of more formal mathematics is crucial because the informal language with which it begins facilitates meaningful learning and comprehension as well as the acquisition of new constructs. It is these new constructs which lay the groundwork for the understanding of further abstract mathematical concepts. This is strikingly affirmed in a journal entry of an eleven year old girl in a study conducted by Richards (1990):

> "Language helps our maths by being able to write, use words, use symbols, being able to read and listen. We share our ideas, challenge ourselves, maybe help others and correct them, be able to have our own opinion. It helps explain things, making discoveries, understanding things, setting goals for yourself, learning, knowing" (Page 14).

Support for the role language plays in the formation of conscious reasoning may be seen in the work of Vygotsky (1978). Similarly, work by van Oers (1992) and Voigt (1992) emphasise the importance of language in the development of shared mathematical understandings because mathematics is a human activity rather than a body of knowledge to be learnt alone. The development of environments that encourage children to discuss and justify their thoughts and conjectures is increasingly seen to be an important means of resolving conflicting viewpoints and establishing communally shared understandings (Vygotsky, 1978; Hand, 1990). This has also been prominent in recent constructivist perspectives on learning (Yackel, Cobb, Wood & Merkel, 1990):

> "Our goal is to have students develop mathematical concepts and relationships in ways that are personally meaningful. Although students construct their own mathematical understandings, they do not do so in isolation. Interactions with both other students and the teacher give rise to crucial learning opportunities" (Page 34 - 35).

THE ROLE OF INSTRUCTIONAL GAMES IN LEARNING

Play is a fundamental part of children's learning. It is their way of making sense of the world around them through situations which are engaging, familiar and non-threatening. Furthermore, games create a social setting that is rich with real world experiences. Games can be used at various stages of instruction, both to introduce new concepts and to consolidate established ideas and processes. Moreover, they allow children to use and understand mathematical language and social organisation in a purposeful way.

The essence of this paper comes from a research project designed to examine the viability of a school-based program using instructional games (Burnett, 1991). The program was introduced into a grade one classroom in a school on Brisbane's southside. At least one or both of the children's parents worked, either in their own business or as company employees. Each class within the school was formed heterogeneously, leading to a wide range of ability levels. While the school's mathematics program developed number understanding first, followed by the addition concept and basic facts a parallel development of measurement, time, patterning, graphing, construction, shape and money also occured.

The study used an ethnographic paradigm in conjunction with the philosophical theory of constructivism. This allowed the researcher and teacher involved to engage the class in an act of sense making that enabled the children to formulate multiple meanings and solve existing problems in both a social and formal setting (Confrey, 1987; Clements, 1991; Burnett, 1992). Use of video transcripts, journal entries and formative assessment brought out the social context as a whole rather then fragmenting it into many seemingly unrelated pieces. Furthermore it showed that when games and hands-on activities were combined in an environment with multiple learning styles and small group work, all contributed to the development of the children's numerical concepts. As a consequence instructional games were found to be of significant help in a child's construction of mathematical knowledge and the fostering of cooperative social skills (Burnett, 1991).

In particular, number knowledge was found to play a fundamental role for success in other areas such as money. The children were able to assist each other's learning by modelling their own understanding and were able to resolve the majority of tensions as they occured.

This following transcript highlights the negotiation of understanding between a grade one child and more knowledgeable other:

Child1: One, two, three, four (She places four 2 cent pieces on the ground in front of herself).
Grade7: How much is there? (Points to the coins Child1 has just counted).
Child1: That is four two's.
Grade7: What's the next one? (Pauses waiting for the child to respond). Seven. (Referring to the value of the object in the next question on the stencil). That's eight cents (Refers to the value of the coins that Child1 has just put out).
Child1: (Checks that she has put out eight cents). Take that one away. (Points to the fourth 2 cent piece. The Grade7 removes it). What's that? (Refers to the value of the three remaining coins).

The child is aware that she has eight cents in front of her but only requires seven. She realises that four 2 cent pieces is too much but seems unsure as to what she should do next:

Grade7: That's six, so you need one more. (Child1 finds a one cent coin and puts it next to the other coins she has in front of her. Grade7 and Child1 count the coins together). So count them up...Two, four, six,.......
Child1: Six, eight.
Grade7: No, that's only one. (Points to the 1 cent piece).

The grade 7 child, a group helper, reinforces the value inherent in our money system by modelling. This seemed necessary despite the grade one child's ablility to recognise the symbols on the coins:

Child1: Two, four, six, ...
Grade7: Seven.

Therefore it appears that number knowledge laid the way for success in other areas of mathematics, particularly with money as the children had to recognise the symbol on the coin, and then understand the patterning and value organisation inherent in our money and number systems.

The children were able to model their understanding for each other:

Child1: (Picks up a card from the centre of the gameboard). Hmm.
Child2: (Looks over her shoulder and points to the written algorithm on the card). You count six and two more.

Child2 reminds the first child of the counting-on strategy they had previously learnt. This allows child2 to obtain an answer to the algorithm which both agree upon.

Child1: Eight.
Child2: Yep.

And again in this next instance one child is able to share her/his understanding with another:

Child2: (Picks up a card and looks at it, he does not seem sure about what is written).
Child1: (Looks at the card child1 is holding and reads it for him). Thirty.
Child2: (Repeats child1). Thirty.
Child1: So it 's three tens.

Child2 was able not only to read the number which was written on the card for child1, but was also able to share that thirty is made up of three tens.

At times, power plays occurred within the groups. In this instance a decision about whether to tell another child a correct answer evolved:

Child4: 3 and 6 is.......
Child1: I know what it is. (Looks over at the card Child4 is holding).
Child2: So do I. (Children wait quietly for Child4's answer).
Child4: Is ......is 10.
Voices: (In unison). No.
Child8: No, you put it back.

Child8 reinforces the rules of the game by telling child4 to put the card back in the centre. She/he then tries to convince the other children in the group not to tell child4 the correct answer even though they have been encouraged to share their ideas and understanding with each other:

Child1: (Checks the answer by using three fingers to count on). Hey, do you want me to tell you?
Child8: No, don't tell him.
Child1: Yes........
Child8: No.
Child1: Well....he.....
Child8: You can't tell ......
Child4: Well I already know it.
Child7: 9.
Child3: It doesn't matter.

Initially the children were eased into cooperative groups in an attempt to shift them away from their self-centred, individualistic behaviour to being members of a whole class group. As the year progressed and their understanding of the mathematical ideas and reasons grew their interactions became increasingly autonomous. For instance, the children were able to reinforce the ideas of the games without overt direction from the teacher, researcher or grade seven children who came to help:

Child1: (Picks up the dice and throws it). Nine. (All the children watch intensely as she moves her counter along the board. Child5 claps her hands and cheers).
Child6: Yah, pick up a card.
Child1: (Reaches over to the cards in the centre of the board and takes the card on top. After she has looked at the written algorithm she turns it around for the other children to see what is written). Five. (The other children look and appear to agree. The card is placed in the discard pile on the board). Arr Harr I get another turn. (She picks `up the dice and throws it again). Six.
Child2: You don't, she doesn't get two turns! (Children all start to talk together then stop realising that nobody can be heard or understood).
Child5 & 6: (Say in unison nodding their heads). Yes she does.
Child5: (Points to the cards in the centre of the board). If she knows the answer to it she gets another turn.
Child6: If you don't know, if you don't know the answer.....(Child1 has moved her counter, Child6 looks at where she has stopped). Did you land on a star? (Some of the Children respond that she had not). Your go A____. (Child1 passes the dice to Child2). When it gets to J_____ (Points to Child3) and then to R_____ (Points to Child4) it will get to me (Points to himself).

Consequently rich interactions took place between the children that allowed them to share their ideas and understanding of mathematical concepts and the game's social structures. Their ability to negotiate these early mathematical meanings appeared to strengthen their knowledge and lay the groundwork for success in other areas of mathematics.

CONCLUSION

While the results may be seen to be exclusive to a specific group of children in their first year of formal schooling, the majority of ideas and resources are readily adaptable to any class or school. Instructional mathematics games when used in the social setting of a classroom can, and do, benefit the development of children's early numeration concepts as the children are able to reconstruct mathematics in a manner which is both meaningful and purposeful. This environment also encourages children to talk with peers and the teacher and work with materials in groups or alone as the need arises.

However it must be emphasised that it is not simply that the games provide the teaching: they are only a means of allowing learning. Rather, it is the discussion which stems from the games and the social context which seems to facilitate the learning. Thus, the implementation of similar programs into other school settings would appear viable especially as the teacher is able to individualise instruction and cater for individual strengths and weaknesses while the children gain important social skills at the same time.

ACKNOWLEDGMENTS

I would like to thank Stacey Grier and George Booker for their valuable comments and insights during the writing of this paper.

REFERENCES

Australian Education Council(1991). *A National Statement on Mathematics for Australian Schools.*

Ausubel(1968). School Learning.

Booker, G., Briggs, J., Davey, G., & Nisbet, S(1992). *Teaching Primary Mathematics.* Melbourne, Australia: Longman Cheshire.

Burnett, Lynn(1991). *The Implementation of a School-Based Program Using Manipulatives, Hands-On Activities and Instructional Games to Develop Early Numeracy Skills.* Unpublished Honours Thesis. Griffith University.

Burnett, Lynn(1992). Using Instructional Games to Construct Number. Understanding and Skills. In M. Horne and M. Supple(Eds.), *Mathematics: Meeting the Challenge.* Melbourne, Australia: The Mathematics Association of Victoria.

Carl, Iris(1989). Essential Mathematics For the Twenty-First Century: The Position of the National Council of Supervisors of Mathematics. *Mathematics Teacher,* 82 (5), 30-34.

Clement, J(1991). Constructivism in the Classroom. *Journal for Research in Mathematics Education.* , 22 (5), 422-428.

Confrey, J(1987). Working Group on Constructivism. In Bergeron, Herscovics & Kieran (Eds.), *Proceedings of the Eleventh International Conference for the Psychology of Mathematics Education Program..*

Department of Education(1987). *Years 1-10 Mathematics Syllabus.* Queensland, Australia.

Hand, B(1990). Constructivism: A New Catchword or a Useful Strategy?. *Prime Number.* 5 (2), 12-15.

Resnick, Lauren(1989). Developing Mathematical Knowledge. *American Psychologist,* 44 (2), 162-169.

Richards, Leah(1990). Measuring Things in Words: Language for Learning Mathematics. *Language Arts,* 6 (1), 14-25.

van Oers B(1992). Learning Mathematics as Meaningful Activity. *Paper presented at the 7th International Congress on Mathematical Education.* Quebec, Canada. August 1-23.

Voigt J(1992). Negotiation of Mathematical Meaning in Classroom Practices. *Paper presented at the 7th International Congress on Mathematical Education.* Quebec, Canada. August 1-33.

Vygotsky(1978). *Mind in Society.* Harvard Boston, Massichususs: University Press.

Yackel, Cobb, Wood, & Merkel(1990). Experience, Problem Solving, and Discourse as Central Aspects of Constructivism. *Arithmetic Teacher,* 38 (4), 34-35.

MULTIMODAL FUNCTIONING DURING MATHEMATICAL PROBLEM SOLVING

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This study investigated the interface between ikonic and concrete symbolic functioning by studying two different aspects of visual processing associated with mathematical problem solving. The first of these involved the use of mathematically based visual images and diagrams, while the second was concerned with incidental non-mathematical visual imagery associated with the problem story. Four groups of grade 10 students were studied, representing all combinations of high and low concrete symbolic ability and high and low ikonic ability. The results suggested that success at problem solving was related to concrete symbolic, but not to ikonic ability, and that the use of mathematically based visual images and diagrams was similarly related to concrete symbolic rather than ikonic ability. Incidental visual imagery associated with the problem story was, however, linked to individual differences in ikonic processing and was not related to concrete symbolic functioning.

There is increasing recognition that intuitive and imagery based processes play an important role in all levels of mathematical problem solving from those of the child in the early stages of mathematical development (Moses, 1982; Maher & Alston, 1989; Wheatley, 1991); through the non-conventional workplace mathematics of the adult with little formal mathematical training (Resnick, 1987; Scribner, 1984); to the novel creations of highly gifted mathematicians and scientists (Hadamard, 1954; Shepard, 1988). There is, moreover, a more general recognition of a relationship between imagery ability, and problem solving and creativity in areas outside of mathematics (Shaw & DeMers, 1986; Armbruster, 1989; Kaufmann & Helstrup, 1985). These findings suggest that the development of mathematically competent and creative students involves not only the traditional focus on computational and logical problem solving abilities, but also the development of associated ikonic skills at all stages in the developmental process.

Biggs & Collis (1991) provide a theoretical model of how ikonic processes might develop in interaction with concrete symbolic ones. Following Piaget and others, they describe the development of different modes of thought, and argue that the ikonic mode of functioning begins in early childhood, after the sensorimotor mode is well established. The ikonic mode, however, is not simply associated with early childhood thinking, to be replaced during school years by the concrete symbolic mode. Rather it continues to develop in power and complexity and to interact with other modes as they develop. Collis & Biggs (1991) argue that much of our more complex thinking and problem solving is multimodal and use of the ikonic mode in conjunction with the conventional logic of the concrete symbolic and formal modes, provides richness and flexibility.

While there has been considerable research focusing on the development of the concrete symbolic mode and a firm understanding achieved (Biggs & Collis, 1982; Campbell, Watson & Collis, in press), analysis of ikonic mode development and its interaction with the concrete symbolic is still exploratory and conclusions remain tentative (Collis & Romberg, 1991; Collis, Watson & Campbell, 1992; Watson, Campbell & Collis, in press). The present study was designed to investigate the interaction between ikonic and concrete symbolic modes during problem solving. This paper looks at one aspect of this, namely visual processing.

METHOD

Subjects

A group of one hundred Grade 10 students from two public high schools in upper socio-economic class suburbs of Hobart, Tasmania were screened using the ACER Mathematics Profile Series: Operations Test to provide a measure of concrete symbolic mathematical ability, and the visual imagery scale of the Betts Questionnaire on Mental Imagery (QMI; Sheehan, 1967) to assess one factor hypothesised to be important in ikonic mode functioning. Fifty students with the highest and lowest scores on the two tests were selected and four groups of 12 or 13 students each were formed, providing all four combinations of high and low concrete symbolic ability and high and low vividness of visual imagery.

Procedure

The above fifty students were interviewed and their responses tape recorded while they solved four problems. Two of these problems were novel while two were school-based; and two of the problems required the use of diagrams or visual images for their solution while the other two could be solved using non-visual concrete symbolic processes. Responses on the following three problems are analysed here to explore the way in which students produced and used visual images and diagrams:

Hungry Man: Three tired and hungry men went to sleep with a bag of apples. One man awoke, ate 1/3 of the apples, and then went back to sleep. Later a second man awoke, and ate 1/3 of the remaining apples, and then went back to sleep. Finally, the third man awoke and ate 1/3 of the remaining apples. When he had finished there were 8 apples left in the bag. How many apples were there originally?

Painted Cube: A cube that is 3cm by 3cm by 3cm was dipped in a bucket of red paint so that all of the outside was covered with paint. After the paint had dried, the cube was cut into 27 smaller cubes, each measuring 1cm on each edge. Some of the smaller cubes had paint on 3 faces, some on 2 faces, some on only 1 face, and some had no paint on them at all. Find out how many of each kind of smaller cube there were.

Drink-Driving: The blood alcohol readings of two drivers were recorded the morning after an accident. The readings were:

Mark	Wayne
6 hours after accident: **5 units**	**5 hours** after accident: **7.5 units**
8 hours after accident: **2 units**	**9 hours** after accident: **5.5 units**

Assuming a linear relationship answer the following:

(i) Who had the highest reading at the time of the accident?
(ii) When were their readings the same?

Two different aspects of visual functioning were studied:

(i) Mathematical diagrams and visual images associated with these;
(ii) Non-mathematical visual images related to the problem story.

RESULTS AND DISCUSSION

The data were analysed with reference to the following three questions, each of which will be analysed in turn.

Question 1: Is success at problem solving related to individual differences in either concrete symbolic or ikonic abilities?

A two way ANOVA was conducted to test for significant differences in the total number of problems solved correctly by students with high and low concrete symbolic ability, and high and low vividness of visual imagery. There was a significant effect due to concrete symbolic ability (F=29.16, d.f.=1, p<.0001), but no significant effects due to vividness of visual imagery or its interaction with concrete symbolic ability. These results indicate that individual differences in ikonic processing, as measured by the QMI, have no direct effect on problem solving success, while ability with basic mathematical operations is significantly related to problem solving ability.

Question 2: How are diagrams and mathematically based visual images used in problem solving; and is their use related to individual differences in concrete symbolic or ikonic abilities?

While solving the painted cube problem, almost all students, irrespective of their vividness of visual imagery, stated that they formed images of the cube. Evidence of the use of visualisation strategies by students with low as well as high vividness of visual imagery also came from the methods by which students correctly calculated how many cubes had no painted faces. Some calculated this mathematically, while others pictured the single cube in the centre of the 3x3x3 array, a solution requiring manipulation of visual imagery. There were no significant differences between students in different categories in the use of these strategies, which indicates that an understanding of the structure of the problem was more important than vividness of visual imagery in the use of the visualisation strategy.

In addition to using visual imagery, 84% of students drew one or more diagrams to facilitate problem solution. Of the eight students who drew no diagrams, three solved part or all of the problem correctly. These students all reported a clear image of the cube and counted smaller cubes on this image. All had either high concrete symbolic skills or high visual imagery skills. In contrast, the five students who drew no diagrams and who were unable to solve any part of the problem, all belonged to the low concrete symbolic, low vividness of visual imagery category and none reported a clear picture of the cube. These results suggest that a distinction can be made between students who do not draw diagrams because they are able to generate and use appropriate visual imagery to solve the problem, and students whose imagery and understanding of the problem are so poor that they are unable to produce useful images or diagrams. The former group all had either high concrete symbolic skills or high vividness of visual imagery, while the latter group were low in both, suggesting that students in the low, low category may have fewer resources available to them when solving such problems than students in the other three categories.

Analysis of the number and type of diagrams drawn by students in different categories also indicated that the production of mathematically-based images and diagrams was dependent upon an understanding of the mathematical structure involved (a concrete symbolic process) rather than upon a more general ability to produce vivid visual images (an ikonic process).

Similar conclusions were reached when the use of graphs in solving the drink-driving problem was examined. Students with high compared to students with low concrete symbolic ability were significantly more likely to mention or use a graph while solving part of this problem ($X^2=9.92$, df.=1, $p<.005$); or actually to solve part of the problem correctly using a graph ($X^2=6.64$, df.=1, $p<.01$). There were no significant effects due to vividness of visual imagery. These results suggest that while mathematically based images, diagrams and graphs may represent an interaction between ikonic and concrete symbolic functions, the concrete symbolic component would seem to be the more important in determining whether appropriately structured images and diagrams are produced.

Question 3: Do problem solvers form non-mathematical visual images relating to the problem story; are these related to individual differences in ikonic or concrete symbolic abilities; and do they play a role in problem solving?

Students were asked whether they had any images relating to the story or its characters while solving the drink-driving and hungry man problems. Students with high compared to those with low vividness of visual imagery were significantly more likely to report such imagery on the drink-driving problem ($X^2= 15.24$, df.=1, $p<.001$); or to report clear imagery on the hungry man problem ($X^2=6.35$, df.=1, $p<.05$). There were no significant differences due to concrete symbolic ability.

The results suggest that, unlike the use of mathematically based visual images and diagrams, the production of incidental visual imagery connected to the problem story is linked to individual differences in ikonic functioning as measured by the QMI, rather than to concrete symbolic ability. Such visual imagery may play a useful role in problem solving. Comments from students in all categories suggest that visual images and stories related to the problem make maths problems more interesting. They also help in understanding the structure of the problem and in assessing the reasonableness of the answer.

CONCLUSION

The results of this study suggest that success in both novel and school-based mathematical problem solving is related to basic concrete symbolic abilities and not to individual differences in ikonic processes. In addition, while mathematically based visual images and diagrams undoubtedly have an ikonic component, their production and use are primarily dependent on a concrete symbolic understanding of the structure of the problem. This provides an example of how ikonic processes, in this case visual imagery, may be used by the concrete symbolic mode for its own logical deductive purposes. In contrast, incidental visual imagery associated with the problem story appears more closely tied to individual differences in ikonic ability and to be unrelated to concrete symbolic functioning. While the value of mathematically based visual images and diagrams in problem solving is well accepted, it is not clear whether incidental, non-mathematical visual imagery has a useful function. According to student reports, however, such visual imagery appears to motivate students, and to help them clarify the story structure and assess the reasonableness of their answers. As such it may provide another problem solving strategy for teachers to foster, much in the way described by Moses (1982), which will facilitate problem solving, especially among students experiencing difficulties. Watson et al. (in press) demonstrated that ikonic support may be particularly helpful when conceptual understanding is weak, but tends to disappear once the problem is well understood.

REFERENCES

Armbruster, B.B. (1989). Metacognition in creativity. In J.A.Glover, R.R. Ronning & C.R. Reynolds (Eds.), Handbook of creativity (pp.177-182). New York: Plenum Press.

Biggs, J.B. & Collis, K.F. (1982). Evaluating the quality of learning: The SOLO Taxonomy. New York: Academic Press.

Biggs, J.B. & Collis, K.F. (1991). Multimodal learning and the quality of intelligent behaviour. In H.A.H. Rowe (Ed.) Intelligence: Reconceptualization and measurement (pp.57-75). Hillsdale, NJ: Erlbaum.

Campbell, K.J., Watson, J.M. & Collis, K.F. (in press). Volume measurement and intellectual development. Journal of Structural learning and Intelligent Systems.

Collis, K.F. & Biggs, J.B. (1991). Developmental determinants of qualitative aspects of school learning. In G. Evans (Ed.), Learning and teaching cognitive skills (pp.185-207). Melbourne: ACER.

Collis, K.F. & Romberg, T.A. (1991). Assessment of mathematical performance: An analysis of open-ended test items. In M.C. Wittrock (Ed.), Cognition and instruction (pp.82-130). Hillsdale, NJ: Erlbaum.

Collis, K.F., Watson, J.M. & Campbell, K.J. (1992). Multimodal functioning in novel mathematical problem solving. In B. Southwell, B. Perry & K.Owens (Eds.) Proceedings of the Fifteenth Annual Conference, Mathematics Education Research Group of Australasia (pp.236-243). Richmond: MERGA.

Hadamard, J. (1954). The psychology of invention in the mathematical field. Princeton, NJ: University Press.

Kaufmann, G. & Helstrup, T. (1985). Mental imagery and problem solving: Implications for the educational process. In A.A. Sheikh & K.S. Sheikh (Eds.), Imagery in education. New York: Baywood.

Maher, C.A. & Alston, A. (1989). Is meaning connected to symbols? An interview with Ling Chen. Journal of Mathematical Behaviour, 8, 241-248.

Moses, B. (1982). Visualization: A different approach to problem solving. School Science and Mathematics, 82, 141-147.

Resnick, L.B. (1987). Learning in school and out. Educational Researcher, 16, 13-20.

Scribner, S. (1984). Studying working intelligence. In B. Rogoff & J. Lave (Eds.), Everyday cognition: Its development in social context (pp.9-40). Cambridge, M.A: Harvard University Press.

Shaw, G.A. & DeMers, S.T. (1986). The relationship of imagery to originality, flexibility and fluency in creative thinking. Journal of Mental Imagery, 10, 65-74.

Shehan, P.W. (1967). A shortened form of Betts questionnaire upon mental imagery. Journal of Clinical Psychology, 23, 386- 389.

Shepard,R. (1988). The imagination of the scientist. In K. Egan & D. Nadaner (Eds.), Imagination and education. New York: Teachers College Press.

Watson, J.M., Campbell, K.J. & Collis, K.F. (in press). Multimodal functioning in understanding fractions. Journal of Mathematical Behavior.

Wheatley, G.H. (1991). Enhancing mathematics learning through imagery. Arithmetic Teacher, 39, (1), 34-36.

ACKNOWLEDGMENTS

This paper is based on research funded by a 1992 Australian Research Council Grant (ARC.Ref.No.AC903 1914). The authors wish to acknowledge the assistance of Jo Jordan in collecting the data.

TRIADIC SYSTEMS IN EDUCATION: CATEGORICAL, CULTURAL OR COINCIDENCE

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Recent research in education, and mathematics education, in particular, has led to the identification of independent categorizing systems intended to mirror the structures found in such diverse fields as teacher professional development (Barnett, 1992); student writing in mathematics (Clarke, Stephens, & Waywood, in press); and student acquisition of calculus knowledge (Frid, 1992). There are particular characteristics of these categorizing systems which display a tantalising similarity:

- *Contextual similarity - the common location of all three studies within educational environments;*
- *Structural similarity - the "three-valued" (triadic) structure of all three categorizing systems;*
- *Conceptual similarity - categories in each system resemble each other in the nature of their conceptual distinctions.*

This degree of similarity suggests that each categorizing system is an independent manifestation of a more fundamental triadic system (TRIADS). This paper examines the characteristics of these triadic systems and makes comparison with other systems (or analytical frameworks) found in the research or theoretical literature, in an attempt to establish the significance of the degree of conceptual similarity found in the categorizing systems employed in mathematics education. It is proposed that cognitive sophistication be identified with personally contextualized knowledge rather than with formally abstracted knowledge. TRIADS is proposed as a robust structure having relevance in a variety of educational contexts. It is also proposed that conceptual similarities between the first two levels of TRIADS and Skemp's (1976) diadic structure for mathematical understanding support the addition of a third level, to be called Contextual Understanding.

As with other fields, the research endeavour in education is a search for order, pattern and structure. Schoenfeld, among others, has bewailed our lack of "explanatory frames and methods that do *simultaneous* justice to both the cognitive and social aspects of what takes place" in mathematics classrooms (Schoenfeld, 1991, p. 2, emphasis ours). Perhaps it is the perceived need to partition behaviour into affective and cognitive that has made the conceptualization of structure unnecessarily difficult. Beliefs, for instance, lie within the overlap of the cognitive and the affective, and are misrepresented by an analysis which locates them in one domain only. Clarke (1992a) has argued that reductionist approaches to educational research frequently deny the fundamental interrelatedness and socially-situated nature of educational constructs. By way of example, in discussing student adjustment to secondary school mathematics, Clarke (1992b) proposed a single model of "Transition" that could apply to all aspects of behaviour, both cognitive and affective. Certainly, one technique to reveal structure is to study behaviours in diverse contexts, arguing that commonalities across diversity imply fundamental underlying structures. It is the assertion of this paper that such a similarity exists across the behaviour domains of student writing in mathematics, the acquisition of calculus knowledge, and teacher professional development. A case will be made in the following discussion for the generality of the triadic structure proposed for a variety of educational contexts.

Three Empirical Triadic Structures

The archetypal triadic structure (designated "TRIADS") which is being proposed in this paper emerged from comparison of three empirical triadic structures. These three structures, their characteristics, and some details of their empirical origins are outlined in Table 1 and in the following discussion.

Calculus Students: Sources of Conviction

The part of the research study (Frid, 1992) presented here focuses on the nature and role of calculus students' convictions regarding the validity or truth of calculus interpretations and problem responses and the ways students construct their calculus conceptualizations. The term *sources of conviction* is used to refer to how one determines mathematical truth and validity.

The research was a naturalistic study involving three undergraduate calculus classes located at three different post-secondary institutions in Western Canada. Included were a large university and two small private colleges. Task-based personal interviews with 17 students were the method of inquiry into the nature and role of students' *sources of conviction* and manner of construction of calculus conceptualizations. The 12 primary problems given to students asked them to identify, describe, interpret, explain, or apply limit and derivative concepts and they included open-ended as well as relatively focused tasks. The interviews also incorporated relevant personal questions related to students' perceptions of calculus and the learning of calculus, study practices, ways of determining "correctness" and attitudes towards calculus.

Interview data revealed the existence of three groups of students who differed in their *sources of conviction*. These groups were named Collectors, Technicians, and Connectors. The names reflect the nature and role of the groups' *sources of conviction*. Salient characteristics of each of the three groups will now be outlined.

Collectors

Students who from their *sources of conviction* are classified as Collectors display *sources of conviction* that are generally external in nature, in that they reside in statements, rules and procedures presented by the teacher or textbook. They do not generally reside in what students have construed for themselves. The students construct their mathematical knowledge by assembling isolated, relatively unconnected mathematical statements, rules and procedures. Thus, a Collector's calculus conceptualizations can be said to be a "collection" of statements, rules and procedures. Although the student might validly apply calculus knowledge, the student does not claim to know personally whether particular pieces of mathematics are valid or correct. Rather, the student relies on others to determine validity or correctness. These other individuals are perceived by the student to be people for whom calculus is understandable and meaningful.

Technicians

Technician students display a mixture of internal and external *sources of conviction*. Their external *sources of conviction* are similar to Collectors' in that they are based on knowledge of calculus statements, rules and procedures. However, Technicians differ from Collectors in their perception and use of these statements, rules and procedures. Technicians see calculus as a logical organization of statements, rules and procedures and they employ this organization as a technique for thinking about and applying calculus concepts. What therefore most distinguishes Technicians from Collectors is that Technicians display knowledge of how calculus statements, rules and procedures fit together into a logical whole. This logical whole thereby becomes a calculus "technology" in that it is a science or method for thinking about and applying calculus. Technicians can therefore be viewed as skilled users of the application of calculus techniques. Technicians' *sources of conviction* are based upon statements, rules and procedures organized into a coherent, structured set.

Connectors

Connectors generally display *sources of conviction* that are internal in nature. Their *sources of conviction* reside largely in ideas and techniques they perceive to make sense. That is, Connectors view calculus knowledge as something of which they can gain personal understanding and use. They speak of approaching their calculus learning in terms of aiming to understand, make sense of and flexibly think through and apply ideas and techniques. In this way Connectors use their internal *sources of conviction* to construct calculus conceptualizations of which they feel personal understanding. Their conceptualizations are displayed as a network of "connections" between various aspects of calculus and between calculus and themselves.

Student Writing: Stances to Knowledge

This study of student writing was undertaken through the analysis of the mathematics journals of 150 students randomly selected from a school population (Years 7 to 12) of over 500 and through questionnaire completion (Clarke, Waywood and Stephens, in press). During 1988 and 1989, an evaluation was conducted of the consequences of student journal use in school mathematics. The data pertinent to this paper were those concerned predominantly with associated student beliefs and with the nature of student writing related to school mathematics. Consultation with school staff, perusal of a sample of student journals, and the selective interviewing of a cross-section of pupils led to the construction of a questionnaire which, after field testing, was administered to all students in Years 7 to 12. The questionnaire examined student use of journals and their perceptions of the purpose of journal communication and its contribution to their learning of mathematics. Students' conceptions of the nature of mathematics and of mathematical activity in schools were also addressed. While questionnaires were administered to every student, a sample of 150 students, a random sample of 25 at each year level, was chosen for statistical analysis. Three questionnaires were administered ('Mathematics', 'Journals - Part A' and 'Journals - Part B', in that order) and the sample selection procedure ensured that all students at a particular year level, who had completed all three questionnaires, had the same chance of appearing in the sample.

In this study, three *modes of student writing* were identified empirically: Recount, Summary and Dialogue. The categorization of student journal writing, validated by a "blind review" process, was matched with student questionnaire responses regarding the mechanics and the perceived purposes and value of journal use, together with student conceptions of the origin of mathematical ideas, and student perceptions of school mathematics and classroom mathematical activity. Structural consistencies suggested that the triadic structure identified in student writing could also be found in student views regarding the purpose of their journal writing and regarding the nature of mathematical knowledge. The findings of this study suggest that:

- When students write in the *Recount* mode, they see mathematical knowledge as something to be described;
- In the *Summary* mode, students are engaged in integrating mathematical knowledge, now conceived of as a collection of discrete items of knowledge to be collected and connected;
- When writing in the *Dialogue* mode, students are involved in creating and shaping mathematical knowledge, which has now become personalised and purposeful.

The structure of student writing identified by Clarke, Stephens & Waywood (in press) was robust, triadic and hierarchical. The three modes of student writing also appeared to signify three levels of sophistication of mathematical thought. The hierarchical nature of the structure became evident in the analysis of the relationship between mode, student year level, and length of experience with journal completion. A progression from recount to summary to dialogue could be identified, and this progression was associated with the duration of the experience of journal writing rather than simply with year level. The association of each writing mode with a stance towards mathematical knowledge arose from comparison of textual categorizations and questionnaire responses. Those characteristics of the triadic structure pertinent to this paper are shown in Table 1.

Mathematics Teachers: Locus of Authority and Learning Goals

A study by Barnett and Sather (1992) documented the transitions of mathematics teachers who participated in a professional development program based on case discussions. The purpose of the study was to identify changes in teachers' thinking and beliefs about student errors and misunderstandings. Data from pre- and post-interviews were analyzed according to nineteen codes identifying such things as whether a teacher capitalized on student errors as catalysts for discussion or concealed student errors from public scrutiny. Three clusters of codes emerged from the data, and a model was constructed to frame those clusters. The model identifies three levels of progression with regard to teachers' orientations toward authority and their learning goals. The levels appear to be hierarchical and indicate a general progression from conventional to more reformist points of view. They are briefly characterized below:

In level 1, the teacher believes that the exclusive authorities in the classroom are the teacher and the text. The learning goal is reception and retention.

In level 2, students are given some voice, but the final authority resides with the teacher or the text. The learning goal is for students to understand and make connections amongst ideas.

In level 3, students are the primary authorities. The learning goal is critical analysis and construction of knowledge.

Once the model was constructed, a second analysis was performed to find out if teachers changed levels during the course of the case discussion intervention. In this analysis, each teacher was assigned to a separate level for their pre-intervention and post-intervention interviews. Researchers made their assignments independently by examining their original coding results and by rereading transcripts. The classifications were agreed upon by both researchers for all twenty teachers.

Although the sample is small, the findings suggest that the three levels in the model may be developmental. This is conjectured because each teacher made either no change in level from pre- to post- intervention or advanced from a lower level to a higher level. No teacher moved from a higher level to a lower level. Although the data for this study were focussed on teachers' thinking and beliefs about student errors and misunderstandings, the emergent triadic structure may be indicative of fundamental underlying beliefs that could be identified in other domains of teaching as well.

Table 1. Empirical Triads

Alternative Models			Alternative Conceptual Frameworks			TRIADS Conceptual Similarities			
Sandra Frid - Calculus knowledge	Clarke, Waywood & Stephens - Student Writing	Barnett - Mathematics teaching (professional development)	Source of conviction (Frid)	Stances to knowledge (Clarke/ Waywood/ Stephens)	Locus of authority and learning goals (Barnett)	Perspective	Process	Based in	Located
Collectors	Recount	Explain, simplify	Rules attributable to external authority	Externalized knowledge as facts	Teacher/text is exclusive authority - Goal: reception and retention	External (Predetermined)	Without logic	Unexamined Beliefs	Decontextualized
Technicians	Summary	Discuss, relate	Rules attributable to coherence of logic	Recognition of a system of relationships	Students consulted, teacher/text is final authority. Goal: connection and understanding	Systemic	Logical	Reason	System is the only context
Connectors	Dialogue	Debate, evaluate	Internalized and personalized knowledge	Internalized and personalized knowledge - relativity of truth	Student is final authority. Goal: critical analysis and construction	Personal	Interpretative	Purpose	Contextualized

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The first six columns of Table 1 set out the characteristics of the three empirical triadic structures which prompted this paper. The last four columns set out the characteristics of the proposed archetypal TRIADS structure in terms of Perspective, Process, Base, and Location. The next question to be addressed concerns whether TRIADS has relevance in areas beyond the three studies which provide its empirical base.

Other structures

The significance of the emergent similarity in the above structures can be seen in a further comparison with other structures in current and recent educational literature. The contribution and relevance of TRIADS can be delineated by comparison with structures in educational and other contexts.

Piagetian developmental models

The SOLO taxonomy (Biggs & Collis, 1982; Biggs & Collis, 1990) is representative of developmental approaches to modelling cognitive capability, which have built upon the work of Piaget. With other neo-Piagetians, Biggs and Collis usefully distinguish mode of functioning from levels of structural complexity within a mode.

> With some reservations, it seems that the neo-Piagetian theorists ... would agree that there are two phenomena involved in determining the level of an individual's response to an environmental cue; the mode of functioning, which is determined by the level of abstraction of the elements utilized, and the complexity of the structure of the response within that mode.
> (Collis & Romberg, 1989, pp.7-8)

TRIADS is compatible with the notion of levels as an hierarchical sequence of structural complexity, which is not physiologically based (Collis & Romberg, 1989, p. 13), however the process of progressive refinement of notions of knowledge and associated authority which is modelled in TRIADS places an emphasis on the personalization of knowledge which does not align well with neo-Piagetian models which describe the highest level of sophistication as "Extended Abstract".

Vygotsky's stages of conceptual thought development

Vygotsky's (1962) stages of conceptual thought development form a triadic, hierarchical structure. As children organize their environment by abstracting and labelling perceived qualities of perceived phenomena their conceptual thinking passes through three stages: 1. Arbitrary clustering of experiential phenomena; 2. Organizing experience into complexes using learned modes of organizing and naming phenomena; and, 3. Making personally-derived abstractions.

This interpretation of Vygotsky's third stage melds the personal and the abstract, bringing the categorization scheme in line with TRIADS.

Morality and the development of faith

Kohlberg's stages of moral development (Kohlberg, 1963) are in substantial correspondence with TRIADS. In Kohlberg's picure, Pre-conventional Morality is identified with the existence of an external source of punishment and reward, Conventional Morality invokes societal "laws" as the governing principles, and Post-conventional Morality places priority on an individual's personal sense of "Good", "Just" and "Right". Fowler's theory of faith development (1981) draws upon Kohlberg's work and parallels Kohlberg's stages in the context of the development of religious faith. The correspondence with TRIADS is clear, and can be attributed to similar concerns with the location of authority.

It is possible to find other triadic categorizing structures (for instance, Ernest's (1991) categorization of ideologies into Utilitarian, Purist and Social Change bears some resemblance to the key elements of TRIADS), however the value of TRIADS will ultimately lie with its explanatory and predictive capacity, and it is with a discussion of this that this paper concludes.

Discussion and concluding remarks

Two issues relating to TRIADS warrant discussion:

- the association of the personal with the highest level of sophistication; and,
- the generality of the triadic structure.

Each of these is discussed briefly below.

The Personal or the Abstract

The three studies which have led to the development of TRIADS shared a concern with the way mathematical knowledge was conceptualized. On the surface, any emergent theoretical structures would be located within the cognitive domain. It appears, however, that the correspondence between TRIADS and other theoretical structures is most precise in areas related to *belief*. In fact, the specific conflict which TRIADS establishes with models of cognitive development concerns the tension between the identification of the extremely personal and the extremely abstract as the highest level of sophistication.

Ultimately, theoretical models must be judged according to their utility. Our advocacy of TRIADS as an appropriate and useful model can be supported by reference to previous attempts to understand cognition and, in particular, to understand understanding. The notion of "situated cognition" (Lave, 1988) focussed attention on the contextualized and social nature of learning. Schoenfeld (1992) built upon the notion of cognitive apprenticeship.

> The apprentice tailors are apprenticing themselves into a community, and when they have succeeded in doing so, they have adopted a point of view as well as a set of skills - both of which define them as tailors.
>
> (Schoenfeld, 1992, p.341)

In the context of mathematics learning, Schoenfeld identified this "point of view" with something called a "mathematical disposition", and this in turn was associated with, among other things, "a predilection to quantify and model" and with "the habit of seeing phenomena in mathematical terms" (Schoenfeld, 1992, p.341). The process which commonly has been associated with this inclination in the past has been that of abstraction of some mathematics from its embeddedness in some problem context. We would suggest rather, that the relevant process should be seen as the learner's previous contextualization of mathematics in an increasing diversity of situations, and that it is this access to multiply-contextualized representations of the relevant mathematics in problematic situations which determines the level of sophistication of the mathematical learning. This personal contextualization of mathematics, which we identify with effective learning, precedes the individual's encounter with a problem solving situation. This view of learning shares some of the features of Sweller's (1989) association of the development of problem solving expertise with effective schema acquisition. It is from this perspective that TRIADS identifies the personal and the contextual with mathematical sophistication.

If the proposed transcendent triadic structure is to be taken seriously and subjected to some sort of test, then one appropriate proving ground would be its application to structures which do not take a triadic form. The descriptive framework first proposed by Skemp (1976) for the categorization of mathematical understanding can be seen to include only the first two of the three categories proposed in this paper. Skemp identified *Instrumental* and *Relational* as categories of understanding and distinguished them as "rules without reasons" and as "knowing both what to do and why". These two categories are entirely consistent with the External and Systemic perspectives proposed in this paper (Skemp, 1992). A consequence of the acceptance of TRIADS would be the identification of a third level of student understanding with the characteristics of the Personal perspective. We propose that Skemp's original framework be extended to include a level with the designation *Contextual* . This level of understanding would be characterized by an ability to articulate mathematical structures in a form that specified a mathematical procedure (for instance) in terms of the context (personal, social, and cultural) in which the mathematics has relevance and meaning. Skemp's formulation of understanding is incomplete in that the Instrumental and Relational forms are appropriately decontextualised, but that the additional dimension of contextually-located personal meaning is omitted entirely.

A student with *Contextual Understanding* knows how to do the mathematics, why it is done that way, and what purpose the mathematics might serve. In practice, the student restricted to instrumental understanding can only mimic taught procedures. The student restricted to relational understanding should be able to solve problems calling for similar mathematical procedures. A student with contextual understanding can employ the mathematical procedure in novel situations in a variety of contexts. It is in this third level of understanding that we find the complex schemas needed for mathematical problem solving. Table 2 sets out TRIADS in relation to Kohlberg's stages of moral development, and the reconceptualized structure for Skemp's (1976) model of understanding.

Table 2. Comparison of TRIADS with Kohlberg (1963) and Skemp (1976)

Alternative Models			Conceptual Frameworks		TRIADS Conceptual Framework			
TRIADS (Clarke, Frid & Barnett, 1993)	Moral Development (Kohlberg, 1963)	Mathematical Understanding (Skemp*, 1976)	Moral Development (Kohlberg, 1963)	Mathematical Understanding (Skemp*, 1976)	Perspective	Process	Based in	Located
EXTERNAL	Pre-Conventional Morality	Instrumental Understanding	Punishment and Reward	Rules without Reasons	External (Predetermined)	Without logic	Unexamined Beliefs	Decontextualized
SYSTEMIC	Conventional Morality	Relational Understanding	Societal "Law and Order"	Knowing both how and why	Systemic	Logical	Reason	System is the only context
PERSONAL	Post-Conventional Morality	*Contextual Understanding**	Personal principles	*Personally contextualized knowledge**	Personal	Interpretative	Purpose	Contextualized

* Skemp's (1976) model of understanding has been augmented to align it with TRIADS.

Categorical, cultural or coincidence

Three alternative explanations can be identified for the degree of correspondence noted above, and each has implications for associated research in education. These alternatives are:

- A single underlying triadic system (TRIADS), of which other triads can be seen to be manifestations within a particular domain of human endeavour, and which should be seen as ultimately informing research in all domains and the understanding of which should constitute the ultimate goal of much educational research (see Eco & Sebeok, 1988);
- A culturally-determined characteristic of the way we categorize our world: three-valued categorizing systems reflecting an inherent inclination for triadic structures which we impose upon all experience - a cautionary reminder that our categories at bottom tell us more about ourselves than they do about the world of our experiences (see Lakoff, 1987);
- A phenomenological coincidence, in which case our understanding of the various categorizing structures is not enhanced by their similarity, and research in the various domains can and should proceed independently of each other (see Clarke, 1992a).

If the first alternative constitutes the best explanation, then an understanding of the characteristics of the underlying "primary triad" (TRIADS) should inform our understanding of the various "secondary triads" both within their individual domains of relevance, and through their common relationship with the primary triad and, consequently, with each other. In this case research should explore the possibility that the study of the behaviour of individuals in one domain could suggest likely characteristics of the behaviour of those individuals in other domains. Such research would have the potential both to inform our understanding of human behaviour and to compel a reconceptualisation of what constitutes an analytic domain. It could emerge that domains such as mathematical behaviour, religious belief and linguistic expression are far more closely interrelated than might have been suspected, and that the study of one can usefully inform the study of the other.

If it could be demonstrated that triadic structures are a consequence of the way people categorize their experiential world, then significant research effort should be expended towards cross-cultural studies of such triadic categorizing systems. The results of such research would inform the interpretation of explanatory frameworks for human behaviour in educational and other settings.

An examination of the many categorizing systems employed in mathematics education, and in education in general, may serve to demonstrate that human behaviour is domain specific, and that an understanding of an individual's mathematical behaviour is neither informed by nor related to an understanding of their behaviour in other domains. In this case, research should be conceived in domain-specific terms and a major goal would be to determine the bounds and characteristics of the hypothesized domains, such as mathematical behaviour.

The research summarised in this paper, and the examples cited above, suggest that triadic structures are in frequent use in educational contexts. Moreover, many of these have strong conceptual similarities. It is our claim that TRIADS is a robust, hierarchical system with the capacity to describe and even explain a wide variety of developmental phenomena in educational, and possibly other, contexts. The implications of the acceptance of TRIADS are profound, particularly with respect to the development of problem solving expertise and the future directions for research into mathematical behaviour.

REFERENCES

Barnett, C. & Sather, S. (1992). Using case discussions to promote changes in beliefs among mathematics teachers. Paper presented at the annual meeting of the American Educataional Research Association, San Francisco.

Biggs, J.B. & Collis, K.F. (1982). *Evaluating the quality of learning: The SOLO taxonomy.* New York: Academic Press.

Biggs, J.B. & Collis, K.F. (1990). Developmental learning and the quality of intelligent behaviour. In H. Rowe (Ed.), *Intelligence: Reconceptualization and measurement.* Melbourne: ACER.

Clarke, D.J. (1992a). Finding structure in diversity: the study of mathematical behaviour. Paper given at the Research Pre-session of the 70th Annual Meeting of the National Council of Teachers of Mathematics (USA), Nashville, Tennessee, March 30 - 31, 1992.

Clarke, D.J. (1992b). The Challenge of Secondary School Mathematics.' Paper given at the 1992 conference of the International Congress of Mathematics Education, Quebec, August.

Clarke, D.J., Waywood, A., & Stephens, M. (in press). Probing the structure of mathematical writing. Accepted for publication in *Educational Studies in Mathematics.*

Collis, K.F. & Romberg, T.A. (1989). Assessment of mathematical performance: An analysis of open-ended test items. Manuscript.

Eco, U. & Sebeok, T. A. (Eds) (1988). *·The Sign of Three: Dupin, Holmes, Peirce.* Bloomington: Indiana University Press.

Ernest, P. (1991). *The philosophy of mathematics education.* London: Falmer Press.

Fowler, J. (1981). *Stages of faith.* Melbourne: Dove.

Frid, S. (1992). Undergraduate calculus students' language use and sources of conviction. Unpublished doctoral dissertation, University of Alberta, Canada.

Kohlberg, L. (1963). The development of children's orientation toward moral order, 1: Sequence in the development of moral thought. *Vita Humana, 6,* 11-13.

Lakoff, G. (1987). *Women, fire and dangerous things.* Chicago: University of Chicago Press.

Lave, J. (1988). *Cognition in practice.* Boston: Cambridge University Press.

Schoenfeld, A. (1991). On paradigms and methods: What do you do when the ones you know don't do what you want them to do? Paper presented at the annual meeting of the American Educational Research Association, Chicago, April, 1991.

Schoenfeld, A. (1992). Learning to think mathematically: Problem solving, metacognition, and sense-making in mathematics. Chapter 15 in D. Grouws (Ed.), *Handbook for Research in Mathematics Teaching and Learning,* New York: Macmillan, 334-370.

Skemp, R.R. (1976). Relational Understanding and Instrumental Understanding. *Mathematics Teaching,* No. 77, December, 1976, 20 - 26.

Skemp, R.R. (1992). Informal communication. Breakfast conversation at the 1992 annual conference of the Mathematics Education Research Group of Australasia, University of Western Sydney, Nepean campus, July, 1992.

Sweller, J. (1989). Cognitive technology: Some procedures for facilitating learning and problem solving in mathematics and science. *Journal of Educational Psychology, 81*(4), 457-466.

Vygotsky, L.S. (1962). *Thought and language.* Cambridge, Mass.: MIT Press.

SOCIAL JUSTICE AND THE MATHEMATICS CURRICULUM: AN EVALUATION OF ONE ATTEMPT TO DEVELOP AN INCLUSIVE CURRICULUM

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The Junior Secondary Mathematics Resource Schools Project was established with a central purpose: "To enhance and improve the mathematical capability of all students." Its realisation was dependent on the collective expertise and enthusiasm of the project co-ordinators on the six Project schools, their associated "key-teachers" and other staff, and the co-ordination and support available to the Project as a whole. The research design for the evaluation of the J.S.M.R.S.P. acknowledged the existence of distinct communities of interest with respect to the Project. This paper reports those aspects of the study most likely to be of interest to the research community: that is, the study design and the student and teacher outcomes that can be associated with the implementation of this innovative curriculum and the associated teacher professional development. At this stage it appears that Project teachers are reporting a growing satisfaction with their participation in the Project and a growing awareness, understanding, and endorsement of the Project's goals. Student outcomes suggest that the emerging inclusive curriculum is succeeding in both cognitive and affective areas at least as well as other more conventional curricular practices.

The Junior Secondary Mathematics Resource Schools Project (J.S.M.R.S.P) is a three year project (1991-3) set within Years 8-10 of schooling. It is run and fully funded by the Education Department of South Australia and, since the beginning of 1992, has been incorporated as part of the Focus School Program. Focus Schools are expected to develop exemplary teaching and learning pedagogy in their focus area and to document this in ways that other schools can use. This involves preparation of print resources and a commitment to collaborate with other schools through leadership of local networks.

The project's aim of improving students' experiences and outcomes was to be achieved through attention to the learning needs of all students and, in particular:

- the development of mathematical knowledge and skills appropriate for the 1990's;
- the promotion of a positive attitude toward mathematics;
- an appreciation of the wide application and contribution of mathematics to the lives of all people.

The J.S.M.R.S.P. was established through the allocation of resources to six schools. These resources took the form of a cash allocation to support buying materials, teacher release and travel, and the funding of a project coordinator in each of the schools. In order to establish a project "team" in each of the schools further personnel funding enabled two or three teachers in each school to be appointed as Key Teachers and given a small amount of release time for project duties.

\Two sets of forces operated in the process of selecting the project schools. These were the need to:

1. have one in each of the then Education Areas into which the state was divided (four in metropolitan Adelaide and two country Areas)
2. choose schools such that each could take a special focus on a different aspect of educational disadvantage and/or targeted group.

This second criterion was linked to the school's setting or client group. Special focus areas were Aboriginal students, students in poverty, isolated students, girls, students from non-English speaking backgrounds and students with special learning needs and this provided a subtext for the general developmental work being done in the school.

It is worthy of note that a proven track record or particular emphasis on mathematics as part of the school's background and practice were not used as criteria.

Operations of the project

The J.S.M.R.S.P. operates in three concurrent phases:

Phase 1 - in the six J.S.M.R.S.P. schools
Phase 2 - in selected networks of state secondary schools which form the outreach program of the S.M.R.S.P.
Phase 3 - in the Focus School Program.

Phase 1 - the developmental phase

Phase 1 consisted of the developmental work done in each of the J.S.M.R.S.P. schools. Project schools identified different needs, priorities and issues and this gave the Project its originality and diversity. At the same time a strong sense of collaboration helped establish a common operational framework. Coordinators met for sharing and planning for two days each term. Less frequent meetings of project school principals and of key teachers from all of the schools were important factors in building a strong cooperative base.

The major focus of this phase was the development and documentation of inclusive and exemplary junior secondary mathematics programs which reflected and incorporated current political, pedagogical and resource aspects. An extensive and detailed Project Evaluation was undertaken, and it is the data from the evaluation of this phase of the project which provides the basis for this paper.

Phase 2 - the outreach phsse

J.S.M.R.S.P. schools have a commitment and responsibility to share programs, practices and resources. Schools which were able to meet criteria necessary to become a member of the J.S.M.R.S.P. outreach program had to formally apply and be selected on the merit of their application. In 1992 twenty-two schools were accepted, while in 1993 this number had grown to thirty-three.

Phase 3 - the dissemination phase

The general Focus School Program work of the J.S.M.R.S.P. takes two forms - preparation of publications based on the Project and a broader professional development program for schools and teachers outside of the outreach program. Publications address issues such as using the *National Statement* (AEC, 1991) in curriculum construction, assessment, managing resources and exemplary lessons. As the Project moves into its final stages more documentation, much of it resulting from evaluation activities, will become available.

STUDY DESIGN

The evaluation was structured around three purposes. These three purposes required distinctively different data collection procedures.

Purpose 1. *Informing progressive action*

It was essential for the efficient implementation of the Project that information be collected which would inform Project Co-ordinators in the short-term of the local realisation of the Project's goals. Such data was also intended to provide on-going documentation of the developmental processes associated with the Project. The choice of data collection techniques for Purpose 1 reflected the principle that the most efficient progress towards teacher change would occur where the information likely to inform teaching practice was collected by the individuals required to act on that information.

Purpose 2. *Focussed goals and site-specific outcomes*

Some of the goals of the project were best documented in the form of site-specific case studies. The amalgamation of these case studies under themes which reflected the concerns of the project: inclusive curricula or community involvement, for instance, have the potential to inform the practices of other schools which share a particular concern. The purpose of an evaluation with respect to these goals was realised through site-specific case studies collated in monographs combining related themes.

Purpose 3. *Accountability and the achievement of general Project goals.*
The overall goals of the Project over all sites could be summarised as "the development of school mathematics programs which will improve the mathematical capability of all students, together with professional development to facilitate the implementation of these programs in Project Schools and elsewhere". As such, the success of the Project could be judged by the extent to which these general goals were achieved.

Any educational project finds itself enmeshed in a web of multiple accountability: to a central administration, which may also be the funding body; to regional administrations with their own priorities and concerns; and to participant schools, students, teachers and communities. All these groups will have their own notions of the purpose of the Project, their own feelings of ownership, and their own demands. These multiple accountabilities can be aligned with three levels of responsibility: the Education Department, the Region, and the School Community. This project, in particular, was conceived in a manner which invoked regional and local concerns and encouraged feelings of project ownership at all levels. To be of value the evaluation needed to provide information in a manner that would inform action at each level.

DATA COLLECTION

Purpose 1 - Formative evaluation to inform local implementation

Working from a draft of a Vision Statement of the Project's goals, Project coordinators and key teachers translated general statements such as "Students will be confident in using and applying mathematics in a range of contexts" into a more detailed listing of "Intended outcomes for students". This listing identified several "indicators" associated with each of the outcomes from the vision statement, and linked these to "implications" for teaching. This document provided the structure for Purpose 1 data collection. While the document derived from the original outcomes in the Project Vision Statement, each site elaborated the document in ways that reflected the local concerns of the site, and the professional readiness and capacity of the participant staff to give practical meaning to the Project's goals.

Three strategies were suggested by which teachers might monitor their implementation of the School Indicators/Implications document. The strategies proposed were selected as combining maximum information with minimal additional teacher workload. The strategies were: *annotated classlists*; the *IMPACT* student self-assessment procedure; and, *student work portfolios* (Clarke, 1989). As the Project evolved, other data collection methods were employed. In particular, student journals were used on several sites. Sites tended to use methods that could be integrated easily into instruction and which also served local assessment purposes. In addition, Site Co-ordinators initially maintained a *work-log* in which they recorded an outline of each day's activities. These work-logs provided a record of the implementation of the Project, and were intended to inform the subsequent implementation of the Project on other school sites, by providing details of the role of the School Co-ordinator.

Documents relating to Purpose 1 are Project Implementation Documents and did not form part of the data base for this paper. The best accounts of Purpose 1 implementation will be available in the case studies to be titled: *The Process of Change*. These case studies will provide the clearest documentation of the process by which the Project was realized on each site.

Purpose 2 - Issues-based data collection

The realisation of some Project goals were most usefully documented in ways that reflect site characteristics and concerns. For instance, the development on each site of mathematics programs which acknowledge and accommodate the needs of specific disadvantaged groups was best documented in site-specific terms. Project outcomes of this type took the form of collections of site-specific case studies, intended to offer teachers in other schools suggestions concerning the most effective ways in which particular issues or groups might be accommodated within a school mathematics curricululm.

Purpose 3 - Overall Project Evaluation

Student data collection

Project goals were characterised by certain key terms or phrases. One such term was "inclusive". The Project Evaluation collected data on the participation rate of identifiable student groups: Girls; NESB students; Aboriginal students; Economically-disadvantaged students; and, Special Education students. This data took the form of enrolment in specific mathematics classes and class attendance.

Student confidence was another key term. A survey instrument was developed for the measurement of student attitudes and, in particular, student confidence with respect to mathematics.

Student perceptions of the classroom environment were monitored twice yearly in all participant classrooms on all sites. Items from the *Learning Environment Inventory* (Anderson, Walberg & Fraser, 1982) were adapted to form a suitable instrument, administered in terms 2 and 4 each year.

For the purposes of the Project Evaluation, **student cognitive outcomes** of the Project were measured using an instrument of open-ended tasks (the Common Assessment Tasks - CAT 1 and CAT 2, see Sullivan and Clarke, 1991; Clarke, 1993). This instrument was supplemented by an adaptation of the ACER PAT test, revised by the Project Team to improve local validity, and intended to measure conventional learning outcomes.

Teacher data collection

A **Skills Audit instrument** was developed on which teachers were asked to indicate their level of professional development with respect to specific teaching practices: their level of skill in the specific practices; their willingness to employ them; their experience in their use; their confidence in their use; and any related comments. This instrument was administered to participating Project teachers each year, commencing in term 4 of 1991. One goal was to detect the influence of length of participation in the Project on teacher skill acquisition.

In addition, changing **teacher beliefs about effective mathematics teaching** and their educational priorities were monitored through interviews. Interviews were conducted by the Project co-ordinator each year. A suitable interview protocol, employing a card-sorting technique, was devised for this project.

An annual teacher questionnaire, to be administered in term 4 of each year, measured **participant teacher satisfaction and concerns** with the Project.

RESULTS

This paper draws upon Purpose 3 data collection over the first two years of the project.

Teachers

From analysis of the data, it appeared that Project teachers were particularly sensitive to the role of discussion and communication in their classrooms. Current practice was sometimes seen as different from the practice associated with effective teaching. These differences took two forms. Current practice was seen to give less value than that associated with effective teaching in the areas of: students value other students' ideas; closed tasks; computers; teacher poses challenging questions; students ask questions and initiate discussions; students clarify and justify ideas; students use a variety of tools to solve problems and to communicate. Alternatively, current practice was seen as overvaluing particular approaches, as in the case of: individual work; conventional mathematical terms; teacher explains and demonstrates; students work on textbook problems.

Analysis of teacher responses to the Skills Audit instrument distinguished skill in a particular teaching strategy from experience and confidence in its use, and willingness to employ it in the classroom. The high mean level of willingness on every item suggested a teacher sample committed to classroom experimentation, with a willingness to innovate. Areas in which teachers reported low levels of skill and confidence included the negotiation of the curriculum with students, alternative assessment strategies, and accessing community resources.

Participating teacher concerns with the J.S.M.R.S.P. project in 1991 centred on a perceived lack of definition of their role and on what was seen as an excessive workload associated with involvement in the project. The 1992 Interim Evaluation Report highlighted this concern. Teacher satisfaction data collected in late 1992 when compared with 1991 data showed either stability or improvement on every measure of teacher satisfaction with the project. A significantly greater proportion of teachers reported that they had "learned a lot from Project training and development sessions" in 1992 than was the case in 1991. A significantly improved understanding by teachers of the project's goals was also evident from a comparison of 1991 and 1992 data. Role definition and workload, while still of concern, were not as significant as in 1991.

Students

Students within the study were characterized with respect to five dichotomous variables: Participation in project classrooms; School card possession (a measure of socio-economic disadvantage); Aboriginality; Non-English Speaking Background; Gender. The following statistically significant differences were evident in both 1991 and 1992 data analyses:

1. Project students reported significantly higher levels of satisfaction with their mathematics classrooms than did non-project students.
2. Project students were significantly less likely to perceive their classroom environment as difficult than were non-project students.

With respect to cognitive outcomes:

1. Project students performed at least as well as non-Project students at the same year level on a test of conventional mathematical knowledge.
2. Project students performed at least as well as non-Project students on five short open-ended mathematics tasks.
3. Project students were significantly more successful on an extended problem solving task than were non-Project students at the same year level.

It appears that students are in no way disadvantaged by their participation in Project classes and that significant positive outcomes of project participation can be identified in both cognitive and affective domains.

CONCLUDING DISCUSSION

The Junior Secondary Mathematics Resource Schools Project was established with a central purpose: "To enhance and improve the mathematical capability of all students." This was an ambitious goal. Its realisation was dependent on the collective expertise and enthusiasm of the project co-ordinators on the six Project schools, their associated "key-teachers" and other staff, and the co-ordination and support available to the Project as a whole. A distinctive aspect of the Project was the extent to which the interpretation of the Project's goals was *intended* to be site-specific. While all schools shared the common central purpose, each school offered a particular perspective on the teaching and learning of junior secondary mathematics. This perspective reflected the social, cultural and academic characteristics, and the interests, strengths and priorities of each school. The notion of an "inclusive curriculum" which pervaded Project documents took on very distinctive meanings at each school.

A more specific detailing of the Project's goals placed emphasis on "equity and social justice", teacher professional development in the areas of "teaching and learning processes" and "resource management", "parent and community participation", and "fully documented, well trialled junior secondary mathematics programs". These goals provided the structure for the Project and the broad parameters for the project evaluation.

The research design for the evaluation of the J.S.M.R.S.P. acknowledged the existence of distinct communities of interest with respect to the Project. This paper reports those aspects of the study most likely to be of interest to the research community: that is, the study design and the student and teacher outcomes that can be associated with the implementation of this innovative curriculum and the associated teacher professional development.

At this stage it appears that Project teachers are reporting a growing satisfaction with their participation in the Project and a growing awareness, understanding, and endorsement of the Project's goals. Student outcomes suggest that the emerging inclusive curriculum is succeeding in both cognitive and affective areas at least as well as other more conventional curricular practices. Evaluation of the Project's impact on increasing the successful participation in school mathematics of educationally disadvantaged groups will be undertaken once data collection is complete. It is possible, at this stage, to justify some degree of optimism, since increased student success and satisfaction seem to be emerging characteristics of mathematics classrooms within the J.S.M.R.S.P.

REFERENCES

Anderson, G.J., Walber, H.J., and Fraser, B.J. (1982) *Assessment of Learning Environments: Manual for the Learning Environment Inventory*. Perth: Western Australian Institute of Technology.

Australian Education Council (AEC) (1991). *A national statement on mathematics for Australian schools.* Carlton, Victoria: Curriculum Corporation.

Clarke, D.J. (1989) *Assessment Alternatives in Mathematics*. Canberra: Curriculum Development Centre.

Clarke, D.J. (1993). Open-ended tasks and assessment: The nettle or the rose. Paper presented to the 1993 Research Presession of the Annual Conference of the National Council of Teachers of Mathematics, Seattle, USA, March 29 - 31, 1993.

Sullivan, P. and Clarke, D.J. (1991). *Communication in the Classroom: The Importance of Good Questioning*. Geelong: Deakin University Press.

MODELLING TEACHER CHANGE

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The process by which classroom teachers change their practices and their knowledge and beliefs about the teacher's role and about their subject is fundamentally a learning process. The purpose of this paper is to outline the characteristics of a new model of teacher change. It must be emphasized that this model deals with a process of change; it is not a model of the instructional process. In this model, the learning aspects of teacher change lead us to characterize the process as "teacher professional growth". In modelling teacher professional growth, our concern is solely with change in each of the four domains which encompass the teacher's world and the mechanisms by which change in one domain leads to change in another. Central to this new conception of teacher professional growth is the significance accorded to the mediating processes of reflection and enactment, by which change in one domain is translated into change in another.

The process by which classroom teachers change their practices and their knowledge and beliefs about the teacher's role and about their subject is fundamentally a learning process. This learning process can be called "professional growth". The purpose of this paper is to outline the characteristics of this new model of teacher change. The research from which this model was developed was conducted in the course of the evaluation of a professional development program for secondary mathematics teachers (the "Active and Reflective Teaching In Secondary Mathematics" (ARTISM) program) (see Clarke, Carlin & Peter, 1992). The model of professional growth which forms the subject of this paper can be shown to be consistent with earlier models of teacher professional development (for example, Guskey, 1985). The challenge in developing the new model was to incorporate the research-substantiated features of existing models, while also modelling aspects of teacher change not addressed elsewhere. In particular, the possibility of teacher change in the absence of external inservice input had not been recognized explicitly in any previous model, despite its acknowledged occurrence (Sparks & Loucks-Horsley, 1990).

It must be emphasized that this model deals with a process of change; it is not a model of the instructional process. In modelling teacher growth, our concern is solely with change in each of the four domains which encompass the teacher's world and the mechanisms by which change in one domain leads to change in another. Central to this new conception of teacher professional growth is the significance accorded to the mediating processes of reflection and enactment, by which change in one domain is translated into change in another.

THE EVOLUTION OF MODELS OF TEACHER PROFESSIONAL DEVELOPMENT

In the 70s and early 80s, teacher professional development was identified with inservice activity. A common view of the process of professional development as it was then conceived is displayed in Figure 1 (derived from Guskey, 1985). This model of professional development took staff inservice activity as the sole stimulus for teacher professional growth. The inservice programs which were based on Figure 1 sought to change teacher knowledge and beliefs, on the assumption that these changes would lead to a change in classroom practices, and ultimately result in improved student learning.

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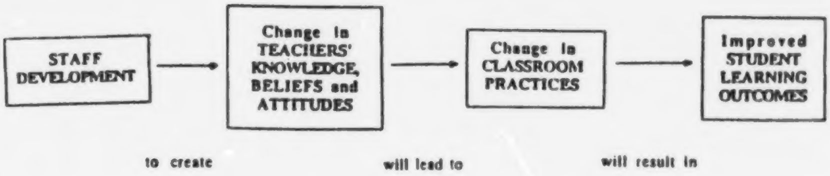
STAFF DEVELOPMENT
Change in TEACHERS' KNOWLEDGE, BELIEFS and ATTITUDES
Change in CLASSROOM PRACTICES
Improved STUDENT LEARNING OUTCOMES
to create
will lead to
will result in

Figure 1. An early model of change in response to staff development

Guskey (1985) offered a reconceptualisation of the model of Figure 1 in which the principle and immediate consequence of inservice activity was seen to be change in teachers' classroom practice. It was suggested that most teachers defined their success in terms of the learning of their students, rather than in terms of their own actions or other factors (Lortie, 1975). This model retained the linear character of the earlier model, but resequenced the elements to locate student learning outcomes as a mediating element in a process whose goal was change in teacher beliefs and attitudes (Figure 2).

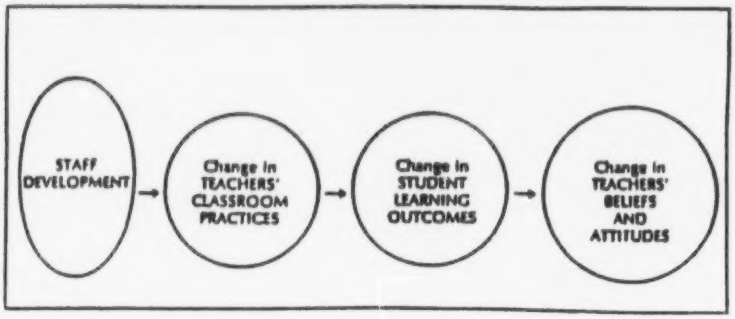
STAFF DEVELOPMENT
Change in TEACHERS' CLASSROOM PRACTICES
Change in STUDENT LEARNING OUTCOMES
Change in TEACHERS' BELIEFS AND ATTITUDES

Figure 2. Guskey's (1985) model of the process of teacher change

Clarke's (1988) cyclic model of staff development (Figure 3) introduced two new elements into the modelling of teacher change:

- First, teacher professional devolopment was conceived as an on-going learning process, building upon successive changes in a spiral of professional growth. In this view, each stage was mediated by factors peculiar to the particular teacher's situation. As a result, the nature of each teacher's professional development was a unique product of their personal history and their present situation;

- Second, it would be possible to join the cycle at any point: a chance question or an improvised activity may demonstrate the inadequacy of previous methods and lead to a reassessment of beliefs about good teaching; or participation in a professional development program might raise the unexpected possibility of attractive new methods. In this model, it was also possible to leave the cycle at any point should the mediating factors provide insurmountable obstacles. For instance, a new approach might be successful in achieving goals which are not valued by the school community, and the professional development subside for lack of support, recognition or encouragement.

This view of professional development employs the same elements as the linear model developed by Guskey (1985, see Figure 2), and embodied a picture of the teacher as learner, constructing personal professional practice from inservice activities and classroom experiences in a way which reflects the teacher's personal history and present situation.

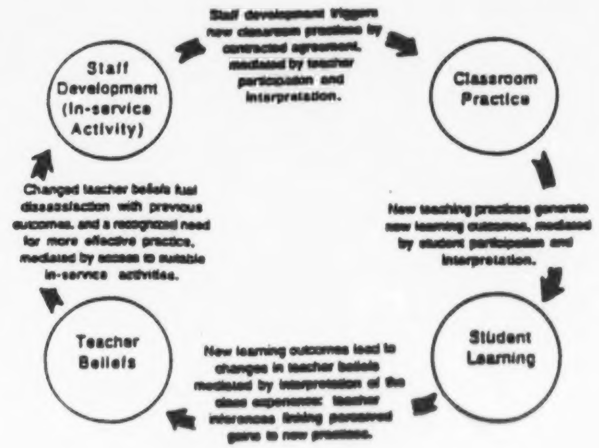
Staff Development (In-service Activity)
Staff development triggers new classroom practices by contracted agreement, mediated by teacher participation and interpretation.
Classroom Practice
New teaching practices generate new learning outcomes, mediated by student participation and interpretation.
Student Learning
New learning outcomes lead to changes in teacher beliefs mediated by interpretation of the class experience: teacher inferences linking perceived gains to new practices.
Teacher Beliefs
Changed teacher beliefs fuel dissatisfaction with previous outcomes, and a recognized need for more effective practice, mediated by access to suitable in-service activities.

Figure 3. Clarke's (1988) cyclic model of professional development

A DYNAMIC MODEL OF TEACHER PROFESSIONAL GROWTH

The data arising from the evaluation of the ARTISM program informed the refinement of this cyclic model of professional development. Analysis of the research evidence arising from this study (Clarke, Carlin & Peter, 1992) revealed a multiplicity of possible teacher change pathways. Recognition of this multiplicity challenged the sequential nature of all previous models. It also became clear that some elements within existing models were too narrowly conceived: Inservice activity was not the only form of external stimulus for teacher professional growth; student learning outcomes were not the only consequences of changed teaching practice which led to changes in teacher knowledge and beliefs; and, most importantly, the occurrence of teacher change independent of external stimulus or support had to be accommodated in any comprehensive model of teacher professional growth. The model shown in Figure 4 incorporates these additional factors, while retaining all previous models as particular cases or pathways.

The model invokes two distinct categories of construct: *analytic domains* and *mediating processes*.
The four analytic domains which characterize the model are:

- The Personal Domain - *Teacher Knowledge and Beliefs*
  Teaching practice is in large part the enactment of individual teacher's knowledge and beliefs regarding their subject, effective instruction, student learning, and the socio-political environment of the school setting. The Personal Domain is concerned with the knowledge and beliefs underlying practice.
- The Domain of Practice - *Classroom Experimentation*
  The enactment of teacher knowledge and beliefs takes the form of classroom practice. Where the classroom situation is perceived as a problematic or challenging one, teacher classroom practice becomes classroom experimentation. We would assert that this experimentation is always present to some degree.

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- The Domain of Inference - *Valued Outcomes*
  Those professional outcomes to which the teacher attaches value constitute the mediating domain by which classroom experimentation is translated into changed teacher knowledge and beliefs. These valued outcomes may include student learning, teacher satisfaction, teacher planning effectiveness and efficiency, reduced teacher classroom stress, and increased student and teacher classroom enjoyment.
- The External Domain - *Sources of Information, Stimulus or Support*
  Teacher classroom experimentation and teacher reflection may both be stimulated by external sources. These external sources might be an inservice program, professional reading, faculty meetings, or informal conversations with colleagues.

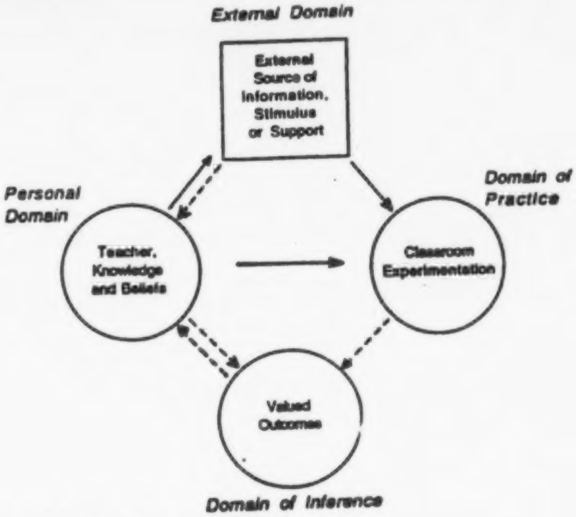
External Domain
External Source of Information, Stimulus or Support
Personal Domain
Domain of Practice
Teacher, Knowledge and Beliefs
Classroom Experimentation
Valued Outcomes
Domain of Inference

Figure 4. The Clarke-Peter model of professional growth
(solid line = *enactive* mediating process; broken line = *reflective* mediating process)

The mediating processes translate growth in one domain into another. These mediating processes can be classified as being either *enaction* or *reflection*. The term "enaction" has been chosen to distinguish the translation of a belief or a pedagogical model "into action" from simply "acting". Acting occurs in the Domain of Practice, and each action represents the enactment of something a teacher knows, believes or has experienced.

On-going professional growth
The refinement of the cyclic model was one outcome of the ARTISM evaluation study. In the refined model (Figure 4) it is assumed that recognition by teachers of Valued Outcomes arising from new practices will fuel a change in Teachers' Knowledge and Beliefs. Teacher dissatisfaction with previous established classroom practices can be seen to be a basic belief change. This change may prompt some teachers to engage in classroom experimentation with alternative teaching approaches. This model differs from many other models of professional development in its recognition of the possibility of Classroom Experimentation and on-going teacher professional growth in the absence of inservice activity (see Figure 5).

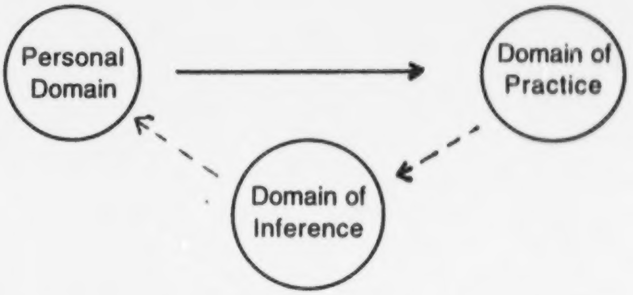

Figure 5. Professional Growth in the absence of External Influences
(solid line = enactive mediating process; broken line = reflective mediating process)

> Teachers learn many things on their own. They read professional publications, engage in discussions with colleagues, and experiment with new instructional strategies, among other activities. All of these occur with or without the existence of a formal staff-development program.
>
> (Sparks & Loucks-Horsley, 1990, p. 235)

The role of inservice activity

As Sparks and Loucks-Horsley (1990) make clear, teacher experimentation may be informed by conversations with colleagues or by professional reading. Teaching is a creative activity, and some teachers will engage in classroom experimentation informed solely by their personal "wisdom of practice" (Shulman, 1987, p. 11). However, it may be that a teacher lacks either the expertise or the knowledge of possible alternatives required to engage in effective experimentation. The role of inservice activity can then be seen as stimulating and supporting *informed* teacher experimentation. The schematic model of professional growth shown in Figure 4 has at least two distinguishing characteristics: its non-linear nature; and the essential recognition that professional growth is an inevitable and continuing process which will occur to some extent whether or not it is informed by inservice activity or, indeed, by any External Sources of Stimulus, Information or Support. These external sources, where present, could include demonstration lessons, structured reflective forums, informal discussion with colleagues, or professional reading.

The effect of teachers' experience of such external sources may be realized through either the process of enactment or of reflection (see Figure 6). Change in teacher classroom experimentation arises from the enactment of models of pedagogical practice provided by such external sources. Change in teacher knowledge and beliefs is a consequence of teacher reflection on the examples, values, research findings, and the shared wisdom of practice provided by such external sources.

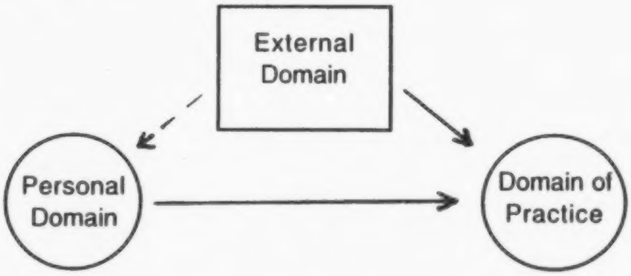

Figure 6. The effect of External Sources of Information, Stimulus or Support.
(solid line = enactive mediating process; broken line = reflective mediating process)

Teacher professional growth must be viewed in the same way as any other form of learning. As learners construct cognitive models of past experiences and test these against new experiences, so teachers experiment with new classroom practices and refine these acccording to their perceived association with valued classroom outcomes. Central to this process of experimentation and refinement is teacher reflection on valued outcomes and their relationship to the classroom experimentation.

The role of teacher reflection

Figure 4 suggests that professional growth involves a continual and complex cycle of action and reflection. The only pathway to the Domain of Practice (Classroom Experimentation) is via a mediating process involving enaction. By contrast, change in Teacher Knowledge and Beliefs can only arise within the model as a consequence of reflection. Ultimately, Teacher Knowledge and Beliefs are translated into practice via a mediating process of enaction. Professional growth is conceived as residing in the combination of the Personal Domain of Teacher Knowledge and Beliefs and the Domain of Practice (Classroom Experimentation). Since the only avenues to change in Teacher Knowledge and Beliefs are mediated by teacher reflection (see Figure 7), the provision of opportunities for structured reflection should be given the highest priority in any professional development enterprise.

For some teachers a consequence of the process of reflection was a marked change in knowledge and beliefs relating to the teaching of mathematics. As Schulman (1986) and Bromme (1992) have noted, change in teacher knowledge and beliefs can be classified in different ways. It is important to recognize that change in teacher knowledge can take distinct forms. While most professional development enterprises appear to have as their goal change in "pedagogical content knowledge" (Shulman, 1986), it is important to recognize the possibility of change in any of the following: Content knowledge; Curricular knowledge; Pedagogical knowledge; or Pedagogical content knowledge

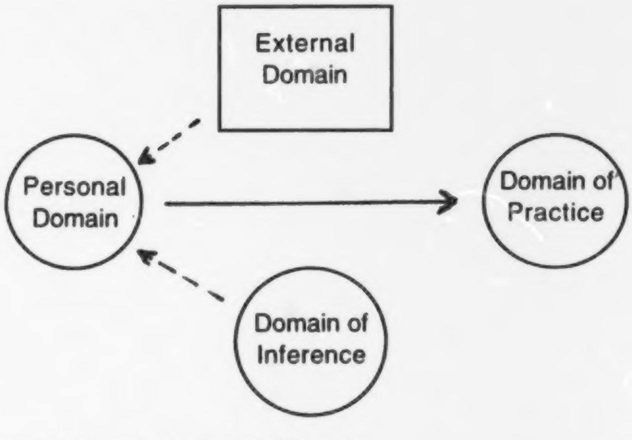

Figure 7. Change in the Personal Domain
(solid line = enactive mediating process; broken line = reflective mediating process)

Consequences of change in teacher knowledge and beliefs

Change in Teacher Knowledge and Beliefs can have three consequences:

a. Recognition of a need for professional growth can be enacted through participation in professional development programs, through professional reading, or the initiation of discussion with colleagues; that is, the teacher seeks information, stimulus or support from some external source;

b. Change in teacher knowledge or beliefs can be enacted through classroom experimentation;

c. Change in teacher knowledge or beliefs can take the form of a restructuring of the teacher's value system and lead to a reassessment of existing outcomes, mediated through reflection.

These three change pathways are illustrated in Figure 8.

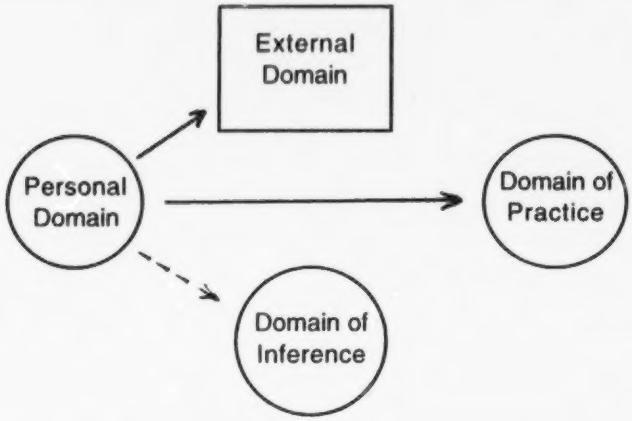
External Domain
Personal Domain
Domain of Practice
Domain of Inference

Figure 8. The consequences of change in teacher knowledge and beliefs
(solid line = enactive mediating process; broken line = reflective mediating process)

Teacher experimentation is the principal operationalized consequence of the changes in knowledge and beliefs, as well as being a principal stimulus for reflection. From this perspective the role of teacher classroom experimentation within professional growth is a central one.

Classroom experimentation
Teacher classroom experimentation is seen as the critical catalyst for teacher professional growth. Figure 9 makes it clear that the role of teacher classroom experimentation as a component of professional growth can take three forms, each corresponding to the relationship between Classroom Experimentation and one of the other three analytic domains.

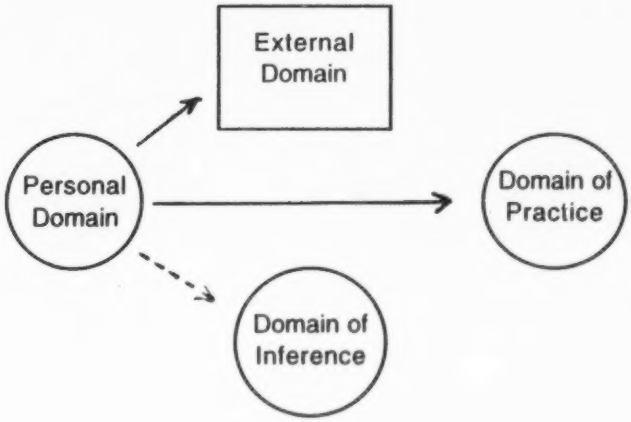
External Domain
Personal Domain
Domain of Practice
Domain of Inference

Figure 9. The role of the Domain of Practice within Professional Growth
(solid line = enactive mediating process; broken line = reflective mediating process)

The most obvious form was teacher classroom experimentation as the *informed mimicry of advocated innovative practice* arising from a teacher's participation in an inservice program.

The second function of Classroom Experimentation within teacher professional growth is as a *stimulus for reflection*. This reflection process mediates between Classroom Experimentation and the teacher's Valued Outcomes. It is within this Domain of Inference that the teacher judges the success or otherwise of the new practices.

To this point, the change sequence: Inservice, Classroom Experimentation, Valued Outcomes, Knowledge and Beliefs reproduces Guskey's linear model of teacher change (1985, p. 58). An essential feature of the Clarke-Peter model, however, is the recognition that change can occur without the mediation of any inservice activity through the *operationalization of Teacher Knowledge and Beliefs*. In particular, where Teacher Knowledge and Beliefs changes in response to the inferred success of new practices, further Classroom Experimentation can occur as teachers put into practice their changed beliefs regarding effective mathematics teaching without the further intervention of additional inservice activity (see Figure 5). This progressive refinement of teaching practices establishes individual teacher ownership of the new strategy and the associated beliefs. It was clear from the data collected in the ARTISM study that the process of teacher professional growth was marked by adaptation of advocated practices, rather than simply their adoption. Figure 5 suggests that this adaptation occurred as a process of cyclic refinement.

CONCLUSIONS

The purpose of this paper is the explication of a new model of teacher professional growth. The data from which the model was derived (Clarke, Carlin & Peter, 1992), and which have served to illustrate the various relationships, also led to the following propositions:

- Previous models of teacher change have given inadequate recognition to the complexity of the process of teacher professional growth, leading to inadequate and simplistic descriptions of the relationship between action and reflection;
- Teachers are always experimenting, and the role of an inservice program is to inform and stimulate their experimentation;
- It is in the combination of the Personal Domain and the Domain of Practice that a complete description emerges of the individual teacher. This emergent picture sets out what the teacher knows, what the teacher believes, and what the teacher does;
- Classroom experimentation should be seen as the contextual catalyst for professional growth, and the outward evidence of teacher change;
- Reflection is the key mediating process by which teacher experience changes teacher knowledge and beliefs;
- Enaction is the mediating process by which change in knowledge or beliefs is translated into changed practice;
- Changes in teacher beliefs regarding the efficacy of new practices are mediated by the teacher's inferences linking the new practices to valued outcomes. These valued outcomes will inevitably reflect the teacher's existing conception of the goals of instruction, and of acceptable classroom practice; that is, the teacher's knowledge and beliefs.

The structure of the Clarke-Peter model (see also Clarke & Peter, 1993) and the findings of this study have significant implications for future teacher professional development programs. Recognition of the complexity of professional growth in a form which models possible growth pathways allows the developers of inservice programs and other professional development enterprises to anticipate and encourage all avenues to professional growth. The location of key analytic domains and mediating processes within the model highlights the particular elements which might most usefully form the components of an inservice program.

REFERENCES

Bromme, R. (1992). *Der Lehrer als Experte: zur Psychologie des professionellen Wissens*. Bern: Huber.

Clarke, D. J. (1988). Realistic assessment. In D. Firth (Ed.), *Maths counts - who cares?*, Parkville: Mathematical Association of Victoria, pp. 187 - 192.

Clarke, D. J., Carlin, P., & Peter, A. (1992). *Professional development and the secondary mathematics teacher: A case study*. Research Report No. 6. Oakleigh: Mathematics Teaching and Learning Centre, Australian Catholic University.

Clarke, D. J., & Peter, A. (1993). *A model of teacher professional growth.* Oakleigh: Mathematics Teaching and Learning Centre, Australian Catholic University.

Guskey, T. R. (1985). Staff development and teacher change. *Educational Leadership, 42*(7), 57 - 60.

Lortie, D.C. (1975). *Schoolteacher: A socialogical study.* Chicago: University of Chicago Press.

Shulman, L.S. (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher No. 15*, 4 - 14.

Shulman, L.S. (1987). Knowledge and teaching: Foundations of the new reform. *Harvard Educational Review, 57*(1), 1 - 22.

Sparks, D. & Loucks-Horsley, S. (1990). Models of staff development. In W. R. Hewson (Ed.), *Handbook of research on teacher education* (pp. 234 - 250). New York: Macmillan.

THE INSTRUCTIONAL IMPACT OF CHANGES IN ASSESSMENT

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If the use of assessment as a catalyst for systemic reform in mathematics education is to be justified, then research is required which links changed assessment practices with instructional consequences. Based on the Victorian experience, this study provides the first systematic investigation of this hypothesised "ripple effect". This paper reports the results of the document analysis which comprised the first stage of the study. The analysis of documents relating to the mathematics curriculum, teaching practice, as well as to assessment and reporting in mathematics from eleven Victorian high schools revealed extensive adoption of the distinctive features of the Victorian Certificate of Education (VCE) Study Design (VCAB, 1990), and its multi-component assessment scheme. The document analysis charts the impact of these changes in nomenclature on the structure and practice of the mathematics curriculum in Years 7 to 10. The first stage of the study strongly confirms the impact of changed assessment practices in Years 11 and 12 on curriculum policy and practice, and on how mathematics is taught and assessed throughout secondary school.

Assessment is being conceived both in Australia and internationally as a catalyst for systemic reform in mathematics education. Experiences in Britain and Australia support this view; where it is suggested that changes in mathematics assessment at the senior secondary level have been associated with extensive changes in instructional and assessment practices throughout the school system. This curricular "ripple effect" has yet to be demonstrated by any systematic study.

The recent history of mathematics assessment has been primarily directed towards increasing the degree of correspondence between the intended curriculum, the taught curriculum, the learned curriculum, and the assessed curriculum. In the past, countries such as Australia in attempting to assess the taught curriculum have limited their assessment to timed and written tests, where the major focus has been on facts or skills. It is becoming increasingly evident that exclusive reliance on this form of testing is inappropriate: misrepresenting mathematics, at odds with contemporary curricula, misleading in the information it provides teachers, and potentially destructive in its effects on some learners (Clarke, 1992).

The alternative vision of mathematical behaviour, which can be found in contemporary curriculum documents (AEC, 1990; NCTM, 1989), demands an alternate vision of assessment. The realization of such a vision requires tasks which provide pupils with the opportunity to engage in mathematical thinking in a variety of forms and contexts, and at different levels of sophistication.

Attempts to model mathematical behaviour in assessment systems have recently led to the development of multiple-component systems, such as the senior secondary mathematics assessment in most Australian states. In the Victorian Certificate of Education, which forms the focus of this study, students complete a multiple-choice skills test, an extended answer analytic test, a 10-hour "Challenging Problem", and a 20-hour "Investigative Project" (Victorian Curriculum and Assessment Board, 1990). Each of these components is weighted equally in its contribution to the final grade, and all are intended in their totality to model mathematical behaviour in a variety of

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This project benefitted from funding provided by the National Centre for Research in Mathematical Sciences Education, University of Wisconsin - Madison.

contexts and forms. Similar innovations in assessment have been introduced in other Australian States and Territories.

Application of the "ripple effect" argument is predicated on a belief that curriculum, teaching practice, and forms of assessment in the junior and middle secondary years will replicate practices arising from the implementation of changed assessment at the senior secondary level. However, the question remains as to whether teachers of junior and middle secondary mathematics find in this expanded range of assessment practices and the instructional correlates an opportunity to increase the effectiveness of their teaching, or simply additional administrative and organizational burdens, which increase their workload in ways that ultimately reduce rather than enhance the quality of their instruction.

In their evaluation of the VCE implementation after three years, Northfield et al. (1992) report that:

> Almost without exception teachers of Year 7 to 10 indicated that there were changes in work practices of junior students as a result of the VCE. Many schools were introducing the terminology of VCE. Work requirements were being specified commonly in Years 9 to 10. Some extreme cases have seen work requirements and even CATs [Common Assessment Tasks] introduced as far down as Year 7, although one suspects that, in many instances, this is little more than renaming an end of term test as a CAT.
>
> (Northfield et al., 1992, 31-32)

Our study suggests that the extent of change in instruction and assessment practices is much more than the "nominal change" (Romberg and Price, 1984) implied by Northfield et al. In all schools participating in this study the end of term tests were no longer the only form of assessment. In all schools, assessment through substantial problem solving activities or investigations had become a significant component of a much more elaborate assessment scheme than had been in operation prior to the introduction of the VCE. In some schools, end of term tests had disappeared altogether, although the assessment of conventional mathematical skills and applications continued through various forms of continuous (that is, progressive) assessment, alongside the assessment of problem solving and investigations.

THIS STUDY

The research discussed in this paper comprises a study into the instructional effects in Year 7 to 10 mathematics classes of changed assessment practices in the senior secondary years. It is anticipated that the findings and the methodology employed in this study will be applied in similar studies in other Australian states and that the study itself could be replicated in the United States.

Sample

Schools were selected by a process of theoretical sampling to include the following characteristics: rural and metropolitan; government and non-government schools; and social demographic characteristics (including ethnicity and language). The variation in Victorian schools with respect to these characteristics has been encompassed effectively in a sample of eleven schools. Only schools providing instruction for Years 7 to 12 were included in the sample.

Method

This study used interviews, questionnaires and the analysis of relevant documents. The intention is to obtain sufficient data to adequately portray current practice on each of the school sites, and also to document the recent origins of current practice, and the immediate consequences of the introduction of the VCE.

Three sources of data have been used: documentation of school curriculum policy and practice, including work samples from teachers and students; interviews; and a questionnaire. In this paper we report the results of an extensive analysis of curriculum documents provided by participating schools. The method by which this document analysis was undertaken is set out below.

Document characteristics

The following sources of information were requested from the participating schools in order to indicate significant aspects of mathematics teaching in Years 7 - 10 which could be associated with changed assessment practices in senior mathematics:

- course planning documents (eg course outlines, syllabuses)
- information to parents/students regarding course options and requirements
- sample assessment tasks
- sample of student assignments and project work
- school report forms
- teaching notes
- other school developed support material for teachers

In addition, school communications with the researchers provided further valuable data on the practical implementation of assessment-related curricular changes.

Content analysis

Each document was analyzed with respect to the occurrence of key terms (words or phrases). For the purposes of this study, a term was considered significant if it met either of two criteria:

- the term represented a change in practice
- the term matched a corresponding term found in VCE documentation

Lists were compiled of such significant terms, grouped initially according to the source document. Within the compiled lists of terms, it was possible to distinguish three categories of curricular change:

1. Change in nomenclature (terminology)

For inclusion in this category, the use of a term in a school document was taken to represent change at the level of description, without necessarily implying a consequent change in practice.

2. Change in structure

Some terms could be taken as indicative of new structures for assessment and instruction. For example, the specific inclusion of investigative projects and problem solving tasks as separate components of school-based teaching and assessment represented a structural change in the school mathematics curriculum. The degree of change at the level of teacher or student practice could not be inferred necessarily from the structural inclusion of such elements.

3. Change in practice

Terms such as "predict", "model", "draft and re-draft", and "format" (in relation to the writing of students' reports of extended projects or problem solving activity) indicated changes in both teacher and student classroom practice.

RESULTS OF THE DOCUMENT ANALYSIS

Documented change

The results of the document analysis can be reported as indicative of change in four key areas: Policy; Curriculum and Teaching; Assessment; and Reporting. Analysis of the occurrence of "Work Requirements" in the various documents is used in the following discussion as an illustrative example both of the methodology used in this document analysis and of the manner in which elements characteristic of VCE mathematics assessment find their embodiment in each of these four key areas.

Changes in Policy

The VCE *Mathematics Study Design* frames a curriculum in terms of specific Work Requirements. These prescribe the sorts of activities in which students must engage for the satisfactory completion of classwork. In their course planning for Years 11 and 12, teachers are required to allocate significant proportions of classtime to all of the work requirements. There is also a direct correspondence between the work requirements and the components of assessment. In mathematics, three Work Requirements are specified: Investigative Projects; Problem Solving and Modelling activities; Skills Practice and Standard Applications.

In their policy documents for Years 7 to 10, all eleven participating schools had adopted the VCE terminology of "Work Requirements" with respect to extended mathematical projects, problem solving, and skills practice and standard applications. Documents such as the "Year 7 Information Booklet and Course Details" from one school specifically listed as "work requirements in order to achieve satisfactory completion of a unit" such activities as: "set exercises", "project work", "problem solving", "a written summary of work covered", and a "self assessment".

Changes in Curriculum and Teaching

Within course documents produced by the mathematics departments, these work requirements were elaborated for students and teachers in very specific terms strongly resembling those used in VCE curriculum documents. One school mathematics department included "Problem Solving Teaching Notes" with subsections such as "Some suggestions on how to teach problem solving", and "Problem Solving Student Notes" aimed at clarifying the nature of mathematical problem solving activity and the demands of non-routine mathematical tasks.

Changes in Assessment

Changed assessment practices were evident in:

- the tasks which comprised a school assessment package;
- what students were expected to do in those tasks; and
- how the tasks themselves were assessed.

All schools made specific mention of work requirements in assessment-related documents. In all schools, the assessment of problem solving and projects, together with the assessment of standard mathematical skills and applications, had been incorporated into the formal structures of assessment and reporting.

There is overwhelming evidence from the data collected in this study that, as in the VCE, assessment explicitly included reports of problem solving and projects. These activities are substantial, ranging from short one-page reports or summaries of a problem solving activity in Year 7 to more extended reports in Years 9 and 10. In these Years, the procedures for developing a report, and the suggested format for reports, more closely resemble those required for the VCE. At one school "Consultations and evidence of drafting" were required to be recorded by students in Years 9 and 10 submitting reports of problem solving or investigative project work.

A key feature of the Investigative Project CAT in the VCE is the use of the project report format, which sets out not only the headings under which the report should be prepared but also key mathematical elements that are to be included. The project report format at one school specified key mathematical considerations to be included in the various sections. For example, in the section labelled "Conclusions" we find the following:

> In this section, the results of the project should be brought together, evaluated and related to the stated aims. The conclusions should include discussion of the more important limitations of the investigation and possible further investigation suggested by the results.

In preparing a report of a problem solving activity, students at this school are required to present not only a solution, but to give explicit attention to a statement of the problem, the solution process which they have followed, and to carefully justify both the solution and the steps which have led to it. The similarity with VCE assessment requirements is obvious and quite deliberate.

In assessing reports of problem solving activities and investigative projects, teachers almost without exception have adapted the criteria for assessment of the corresponding components in the VCE. The number of separate criteria is frequently reduced, and are sometimes tailored to suit a specific problem solving assignment. For example, in the assessment of a Year 8 project on Pick's Rule, eight criteria were used, each being graded, as in the VCE, as Not Shown, Low, Medium and High. The use of the VCE coding Not Shown, Low, Medium and High appeared to be quite general among the schools sampled.

Changes in Reporting

The recognition by schools that their reporting practices had changed was evident in the information conveyed to parents through school handbooks about the content of student reports. The significant changes in reporting appeared to include:

- report format;
- the nature of the student activities reported upon; and,
- the manner in which the information was conveyed.

The most common practice in report format was to provide separate assessments for each student on Skills and standar applications, Projects, and Problem solving; that is, for precisely those work requirements specified within the VCE.

One school reported to parents on "Group investigation and oral reports to class". Evaluative and numerical grades were provided on the "Oral presentation", the "Prepared written presentation", "Research undertaken - depth, breadth and appropriateness", "Explanations and answers to questions", and "Use of mathematics and conventions" in the pro-forma employed in reporting to parents. This explicit acknowledgement of aspects of student mathematical activity which had not previously formed a part of school communication to parents can be explicitly linked to the VCE.

The expression "Satisfactory completion" was frequently found in school reports. Letter grades were combined with descriptive assessments. Often, skills and applications were reported separately for the different content areas covered during the term or semester. A new assessment vocabulary was in evidence, and this vocabulary could be traced directly to VCE terminology and practice.

A new vocabulary

The introduction of the VCE involved, to some extent, the adoption of a new language by which mathematical activity could be described. Terms like "mathematical modelling", "justification", and "draft-redraft" had not previously been current in either school curriculum documents or in the day-to-day discourse of classroom teachers. Each of these new terms required not just the assimilation of a new label for an existing activity or practice, but in many cases represented a new agenda for classroom practice and assessment. The new language was concerned as much with process as with product. At the level of the VCE, teachers had no choice but to learn how to use this new language. There was, however, no obligation upon teachers to employ the same language to describe or assess the less sophisticated mathematics of Year 7. Analysis of school documents revealed that terms derived from the language of the VCE were used to describe and guide student mathematical activity at all levels of secondary schooling, from Year 7 onwards.

Significant terms in general use included: explain and discuss; find a model; predict; give a justification; starting point or scenario; research proposal; planning document; and evaluation of conclusions. These terms accord value to a broader conception of what is to know and do mathematics. It is a non-trivial consequence of the VCE to find reference to these higher-order processes consistently occurring in relation to junior secondary mathematics.

IMPLICATIONS OF THE STUDY

Implications and interpretations

Northfield et al. (1993) have argued that "academic schools" would be more likely to be concerned with some of the assessment procedures and whether the VCE could extend the more able stiudents (cf p. 81). Indeed, it could be argued that changed assessment procedures and practices within the VCE might be confined to changes in assessment in Year 7 to 10, with very little flow-on to day-to-day curriculum and teaching. Among the schools studied, the procedures used to assess problem solving activities and investigations did reflect assessment criteria used for these components of VCE mathematics, usually modified according to year level and background of students. used. In all cases, however, there was an explicit link between assessment and teaching, with substantial classroom time being spent on problem solving activities and investigations, in explaining to students how to approach these tasks, and how to prepare a written report of their mathematical activity.

In the study coordinated by Madaus and West (1992) into "The influence of testing on teaching Math and Science in Grades 4-12" the views of 2229 teachers were sought. The minimal reported influence of standardized tests and textbook tests contrasted with reports of "test pressure and test-oriented instruction" associated with mandated "high stakes" testing programs (Madaus & West, 1992, p. 7). The Madaus and West study based its conclusions on questionnaire data and an analysis of tests representative of those cited by respondents. The study identified differences in the prevalence of "test-oriented instruction" in different classrooms. Questionnaire analysis in the next stage of this study will identify whether such differences occur between Victorian schools and classrooms. Interviews with teachers will help to detail the values, beliefs and motivations behind any such differences and, where appropriate, relate these to the influence of the VCE.

Some terms found in contemporary writings in mathematics education, such as "problem posing" or "journal writing", were not employed in VCE curriculum or assessment documents. The absence of these terms from the school documents analyzed suggests that the documented changes can be attributed to the VCE rather than to a response by teachers and schools to more general changes in the professional literature or in the educational climate. The consequent hypothesis is that unless a term or a practice receives the explicit sanction of inclusion in high stakes assessment it is unlikely to influence school policy or classroom practice.

Anticipating Stage Two: Questions to be addressed.

In the next stage of the project, comparison will be sought from all respondents with regard to experiences prior to and during the introduction of the VCE. In particular, the study will document respondents' perceptions of the changes in administration, curriculum planning, classroom instruction, and assessment in the junior and middle secondary years, which are perceived as being linked to the requirements of VCE mathematics.

The detail of changes in practice, which have been inferred from the document analysis, will be reassessed through questionnaire and interview. The stage two analysis will also determine the extent to which the documents provided exemplified typical classroom practice in a school. It is important to establish the strength of the link between changes in instruction and assessment and knowledge of the practices required by the VCE. In this respect, we need to ascertain whether the translation of VCE terminology and practice into junior secondary mathematics classrooms is more pronounced for those teachers who have had experience teaching the VCE.

> As with any new change, different people embrace it at different rates and some of our staff use almost no project/problem solving activities at years 7 - 10.
>
> (Mathematics Coordinator at one of the study schools)

As noted, the absence from school documents of certain terms and practices, endorsed by the academic and research community, but not given prominence in the VCE, will provide an important key through interviews into an understanding of the priorities, criteria and values by which teachers decide which innovative practices to implement and which to ignore. It is quite possible, of course, that it is only through such dramatic change in assessment practices that the contemporary reconceptualisation of school mathematics (cf. AEC, 1992; NCTM, 1989) has been brought to the attention of classroom teachers.

If the use of assessment as a catalyst for systemic reform in mathematics education is to be justified, then research is required which links changed assessment practices with instructional consequences. Based on the Victorian experience, this study provides the first systematic investigation of this hypothesised "ripple effect".

REFERENCES

Australian Education Council (AEC) (1991). *A national statement on mathematics for Australian schools.* Carlton, Victoria: Curriculum Corporation.

Clarke, D.J. (1992). The role of assessment in determining mathematics performance. Chapter 7 in G. Leder (Ed) *Assessment and learning of mathematics.* Hawthorn: Australian Council for Educational Research, 145 - 168.

Madaus, G.F. & West, M.M. (1992). *The influence of testing on teaching math and science in grades 4 - 12.* Chestnut Hill: Boston College.

National Council of Teachers of Mathematics (NCTM) (1989). *Curriculum and evaluation standards for school mathematics.* Reston, VA: NCTM.

Northfield, J., Brown, J., Corrigan, D., Gill, M., Lewis, L., Loughran, J., Redley, L., Shannon, C. & Slattery, M. (1992). *Understanding the challenge: An evaluation of the VCE implementation after three years.* Clayton, Victoria: Monash University.

Northfield, J., Brown, J., Corrigan, D., Gill, M., Loughran, J., Shannon, C., Slattery, M. & Winter, R. (1993). *The challenge continues: An evaluation of the VCE implementation after four years.* Clayton, Victoria: Monash University.

Romberg, T.A., and Price, G. (1983). Curriculum implementation and staff development as cultural change. In Gary A. Griffin (Ed.) *Staff Development.* Eighty-second Yearbook of the National Society for the Study of Education. Chicago, Ill.: NSSE.

Victorian Curriculum and Assessment Board (VCAB) (1990). *Mathematics Study Design.* Carlton, Victoria: VCAB.

INFLUENCES ON THE CHANGING ROLE OF THE MATHEMATICS TEACHER

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This case study research investigated changing teacher roles associated with two teachers' use of innovative mathematics materials at grade six level. Using daily participant observation and interviews with the teachers and the project staff member responsible for providing in-school support, a picture emerged of changing teacher roles and of those factors influencing the process of change. One teacher demonstrated little change in either espoused beliefs or observed practice over the course of the study. The second teacher demonstrated increasing comfort with posing non-routine problems to students and allowing them to struggle together, without suggesting procedures by which the problems could be solved. He also increasingly provided structured opportunities for student reflection upon activities and learning. Major influences on this teacher's professional growth appeared to be the provision of the innovative materials and the daily opportunity to reflect on classroom events in conversations and interviews with the researcher.

THE TEACHER'S ROLE IN A REFORMED CLASSROOM

With increasing interest in curriculum and professional development programs designed to encourage mathematics teachers to reconceptualize their roles, two needs have emerged. First, there is a need for clear descriptions of the role of teachers in such "reformed" classrooms. Second, there is a need for detailed descriptions of the process of teacher change, with an emphasis on those factors which are seen to exert an influence on that process. The major question addressed by this study was the following:

> What are the factors likely to influence the process of changing teacher roles as they implement innovative mathematics curricula?

A six-component categorization of the teacher's role was developed, drawing on seven projects or studies that had many features in common with the vision of mathematics education reform (Cobb, Wood, & Yackel, 1990; de Lange, van Reeuwijk, Burrill, & Romberg, in press; Fennema, Carpenter, & Peterson, 1989; Lampert, 1988a; Middle Grades Mathematics Project 1988; Schoenfeld, 1987; Stephens & Romberg, 1985). This categorization appears in Table 1, and provided a "conceptual framework" for the study (Eisenhart, 1991). There is considerable evidence in the research literature that beliefs and practice are dialectically related, and the structure of the table reflects this relationship. It is also recognized that teachers can espouse certain beliefs not in harmony with observed practice (Thompson, 1992). It is not claimed that the six components are an exhaustive list, and not all of the studies placed equal emphasis on each aspect. However, the

Table 1

Components of the Role of the Teacher in a Reformed Classroom and Related Beliefs about the Teaching and Learning of Mathematics

Components of the Role (what the teacher does)	Related Beliefs About the Teaching and Learning of Mathematics
1. The use of non-routine problems as the starting-point and focus of instruction, without the provision of procedures for their solution	Students can solve non-routine problems without first being taught a procedure

2. The adaptation of materials and instruction according to local contexts and the teacher's knowledge of students' interests and needs	Mathematics needs to be studied in living contexts which are meaningful and relevant to students, including their languages, cultures, and everyday lives
3. The use of a variety of classroom organizational styles (individual, small-group, whole-class)	Differences in mathematical tasks and preferred learning styles of individuals demand variety in classroom organization
4. The development of a "mathematical discourse community," with the teacher as "fellow player" who values and builds upon students' solutions and methods	An atmosphere of conjecture and justification of mathematical ideas enhances learning Teachers should be open about their own struggles with mathematical problems Students' solutions and methods provide the basis of discussion of problems
5. The identification and focus on the big ideas of mathematics	Important mathematical ideas are not confined to specific procedures in isolated content areas, but rather mathematics is seen as an integrated whole, in which the processes of problem solving, reasoning, and communication are central
6. The use of informal assessment methods to inform instructional decisions	Observing and listening to students provides a "window" into their thinking which can be used to plan further instruction

components did provide a useful framework for studying the role of the teacher in this study, and in particular the changes in this role. One consequence of this study was the augmentation of this framework.

THE FOCUS OF THIS STUDY

This was a study of teacher change. A case study approach was used in studying two grade six mathematics teachers, Ms Bartlett and Mr Martin, from a school in a midwestern United States town. The teachers had taught at the school for around 20 years, most of this time at grade six level. During the course of the study, the teachers participated in a professional development program, part of which involved teaching a six-week unit of work consisting of non-routine problems built around the theme of the discovery of a bone belonging to a "mystery person". The unit of work used in the study was one from *Maths in Context* , a curriculum development project based at the University of Wisconsin. The professional development program was designed by project staff to reflect principles from the literature identified as important in facilitating teacher's professional growth (see Clarke, in press). The teachers were supported in the process of teaching the unit by regular visits from a project staff member, Ms Saunders, a graduate student.

Participant observation and interviews with the teachers were used to build a picture of their roles, their perceptions of these roles, and the process of change in these. As well as observing both teachers on 36 occasions, I also observed their participation in team meetings and inservice sessions, and interviewed them six times each. The decision to work with two teachers provided the opportunity to document the ways in which different individuals perceive and respond to very similar opportunities for, and impediments to professional growth. The two teachers were part of a team of four who worked closely together in planning their curriculum.

In the same way that reform documents describe student learners as active constructors of mathematical meaning in a social context, in this study teachers too were viewed as learners who actively create knowledge through experience, and through reflection on that experience, in large part through communication with other teachers (Gunstone & Northfield, 1988).

While the study attempted to answer the main research question stated on page 1, the sub-questions relevant to the context in which this study occurred are the following:

1. In using a unit of work based largely on non-routine problems situated in real contexts, in what ways does the role of the teacher change?
2. What factors influence the process by which the identified changes occur, and what is the nature of these influences?

Data were collected inside and outside classrooms in several phases over a period of seven months. These phases corresponded to a short period before teachers began the first *Maths in Context* unit, an intensive period of work during the teaching of the first unit, and further (but less intense) data collection during the time between the teaching of the first and second *Maths in Context* unit (around four weeks). Analysis of the data enabled me to describe the process of teachers changing their role, and the varying influences of a range of factors on that process. This analysis was greatly enhanced by the use of a software package, *Hyperqual* (Padilla, 1990), which enabled the coding and sorting of interview and observational data.

THE CHANGING ROLE: THE TWO TEACHERS

Ms Bartlett

The teaching of the innovative curriculum unit proved to be a fairly stressful experience for Ms Bartlett. There were many apparent reasons for this, some related to her teaching role, others of a more personal nature. For the six weeks during which the unit was taught, Ms Bartlett had a maximum of 45 minutes for mathematics, and for four of these weeks Ms Bartlett had to leave the classroom with her students in order to assist them with computer studies. She found this scheduling arrangment most inconvenient, and also found assisting with the computer class a difficult challenge (Field notes, 9/9/92).

The small growth that I observed (and my observations were confirmed in conversations with Ms Bartlett) was not so much in Ms Bartlett's *role* in the classroom, but rather in her stated commitment to certain *beliefs* associated with that role. Ms Bartlett identified many areas in which she believed she had changed in the two or three years prior to the study (including problem solving and the use of cooperative group work), but the experience of teaching the unit and the related staff development activities appeared to lead to little change in the short-term in Ms Bartlett's classroom practice.

Mr Martin

Although Mr Martin saw himself as having changed his teaching gradually over the previous five years (like Ms Bartlett, he mentioned more problem solving and group work as major changes in his teaching prior to the study), the teaching of the measurement unit, in Mr Martin's terms, provided "a better vehicle" than he had ever had before. He enjoyed the unit, was excited by what his students had done with it, and was already planning how he would teach it the following year.

In the time during which I observed Mr Martin *prior* to his involvement in the *Maths in Context* project, he had placed considerable emphasis on the importance of making the tasks that he presented as straightforward for students as possible. This was usually achieved by anticipating any difficulties students were likely to have, and either removing them or offering a procedure by which they could be overcome. This action on his part was later to be described by him as "short-circuiting" students' thinking.

The major area of professional growth that I observed over the course of the study was in Mr Martin's increasing willingness to "step back" and let students struggle with problems, resisting the pressure to curtail mathematical exploration by the provision of his method of solution. For Mr Martin, then, this component of the teacher's role became "to tell or not to tell." This area of growth was confirmed by Mr Martin during numerous brief conversations and interviews. He referred often to the constant desire that he felt to help students to avoid difficulties by laying out a path for them, a desire that he increasingly resisted over the course of the measurement unit as he saw the power of student solutions, given minimal guidance:

> I am aware of that now, more so than before of letting kids struggle with those decisions themselves rather than having me lay it out for them. I think what happens is teachers, at least I do, I have part of my own feeling of success or failure related to whether I think kids are successful. So maybe I

> do too much to insure a certain sort of shallow success. If I have to hand it to them perhaps it's really not success, but it could be interpreted as that anyway.
> (Interview, Mr Martin, 10/9/92)

During conversations with Mr Martin, he identified inconsistencies between his beliefs about teaching and learning and his classroom practice, particularly in the area of "basic skills" and assessment techniques. These inconsistencies did not appear to be resolved during the course of the study, but it appeared likely that they would continue to be addressed by Mr Martin over time, given appropriate opportunities to discuss them.

In considering classroom practice as providing insight into a teacher's beliefs about how mathematics should be taught (allowing, of course, for the kinds of inconsistencies discussed by Thompson, 1992), the context in which Mr Martin was teaching seemed to determine his approach. When using the measurement unit, his approach had many of the features of a classroom built around social constructivist learning principles (Cobb, Wood, and Yackel, 1990), and therefore in harmony with the entries in Table 1. However, when he was using other curriculum materials, the focus tended to be on introducing procedures which students were then expected to follow. The implications of this contrast will be discussed later.

A seventh component of the teacher's role--the facilitation of student reflection on activity and learning. In the time prior to and following the teaching of the measurement unit, Mr Martin seemed to spend only a small amount of time discussing the purpose of the day's mathematical activity. However, during the course of the measurement unit, Mr Martin increasingly took time to encourage students to reflect on their recent experiences. Questions such as "why did I get you to do that activity today?", "What were you supposed to learn from creating that table?", and "Why would someone want to know their surface area anyway?" became more and more common as the unit proceeded.

We discussed the need that Mr Martin saw for the pace of classroom activity to slow down to give students a chance to think about what they had been doing and what they had learned:

> I'm conscious of what went on the day before, or the week before even, and I attempt to bring that in because I get the feeling in my past . . . there can be this kind of busy stream of consciousness sort of phenomena, too. If I'm a kid and I'm sitting out there, it would seem to me like all these facts and activities are being run past me on 78 when I'm a $33\frac{1}{3}$ guy.
> (Interview, Mr Martin, 9/17/92)

Mr Martin used terms like "treadmill" and "roller coaster" to describe the frantic pace at which students move from content area to content area during the school day, and he stressed the importance that he attached to slowing down the pace. Mr Martin's emphasis during the reflection time in mathematics was related to his clear understanding of the big mathematical ideas of the measurement unit. He would invite students to suggest the mathematical purpose of the lesson and the specific activities within it, and help students to form connections with other mathematical content and between the content and their experiences. However, his attempts to allow for this reflection time were evident only during the measurement unit, and not particularly at other times.

For most of the measurement unit, Mr Martin had the flexibility to extend the mathematics lesson by up to 30 minutes. This extra time was used to either enable students to finish their group measuring or writing tasks, or to provide a kind of "debriefing" session on what had transpired on that day. This flexibility of time proved to be a major influence on his professional growth, as it provided further opportunity for him to see the quality of student work on the assigned tasks.

FACTORS THAT INFLUENCED THE PROCESS OF CHANGING ROLES

1. The reform movement in general. Documents such as the *NCTM Standards* paint a picture of the "reformed classroom." Such documents and the general climate of reform can provide impetus to changing teacher roles. *The Standards* were seen by teachers as validating "gut feelings" about teaching.

2. The principal and school community. The principal and school community can encourage innovative programs in schools by financial, affective, and other forms of support. The principal made innovation in mathematics a school priority, and his support was reflected in financial commitment to resources and inservice participation, the hiring of new staff, and by regular visits to classrooms.

3. Internal support personnel. The school mathematics coordinator and other internal support staff can contribute to the professional growth of teachers in a variety of ways. In this study, the mathematics coordinator saw her role as connecting interested teachers with appropriate resources and personnel, and working with teachers in classrooms. Such support varied considerably from teacher to teacher.

4. The spirit of collegiality, collaboration, and experimentation. There is a strong emphasis in the literature on the value to the professional growth of teachers of an atmosphere in the school. The school featured in this study was proud of its involvement in innovative practice in mathematics, and most teachers were comfortable with the expectation that they be constantly trying new approaches, and sharing both planning and reflection with their colleagues.

5. The grade level team of teachers. Teachers at a particular grade level can benefit from joint planning, and can support each other in the use of innovative materials or teaching approaches. The four grade six teachers in this study worked closely together in planning and teaching.

6. Innovative curriculum materials. As teachers and students use innovative curriculum materials, teachers may experience many positive benefits, e.g., student thinking may be exhibited in unanticipated ways, or students and teachers may feel more positive about the subject area. Both teachers were impressed with how the students reacted to the curriculum materials and with the quality of students' work. One teacher took a far less directive role as he observed the quality of unaided student thinking.

7. The inservice program. Inservice programs have the potential to give teachers access to knowledge from research and from other practitioners, and provide opportunities for sharing experiences with teachers engaged in similar struggles. However, the two teachers claimed that they received little benefit from their participation in the four inservice sessions, largely because other participants were perceived by the two teachers to be "at a stage they were at" two or three years previously.

8. External support personnel. The research literature is clear on the importance of support in "the actual process of use," whether by external or internal personnel. This support may be cognitive or affective. The project support person had 16 staff with which to work, and was consequently able to spend only a small time with the two study teachers. The support was largely *affective*.

9. The researcher as audience and critical friend. The role of critical friend can theoretically be taken by any of a number of people involved in a new initiative (e.g., mathematics coordinator, external project staff, other teachers). The presence of the researcher and the discussions about classroom events necessary for data collection (both formal and informal), provided a vehicle for reflection by the teachers.

10. Outcomes valued by the teacher. Student learning has been shown by other research to have an impact on teachers' professional growth. However, other outcomes can be valued by the teacher, such as quality student work with innovative materials and positive student reactions. Such outcomes have the potential to influence teachers' commitment to innovative approaches.

11. Day-to-day conditions under which teachers work. Teaching conditions can prove a major impediment to the professional growth of teachers. Although the teachers had many positive working conditions (e.g., considerable joint planning time and within-classroom assistance for students with learning disabilities), the grade six teaching schedule was "packed" for one of the teachers.

12. Teacher knowledge. The depth of a teacher's content knowledge and the way it is organized has been shown to affect the way in which it is presented to students. One teacher perceived herself to be lacking mathematical knowledge and had difficulty in presenting the tasks for students in a way that might have fostered such connections. The unit used by the teachers had 13 different mathematical topics within it, and this added to the difficulty. The other teacher saw the connections between these different topics, and took considerable additional time to encourage students to reflect on their experiences and to start to make appropriate connections.

SOME ASPECTS WORTHY OF FURTHER CONSIDERATION

1. In terms of content knowledge, the logical question that arises from this study is the following: "How can a teacher who lacks a "network of big ideas and the relationship among those ideas and between ideas, facts, and procedures" (Lampert, 1988b, pp. 163-164), develop these things?
2. In projects focusing on reformed notions of mathematical teaching and learning, greater energy needs to be directed to the selection and preparation of support staff, whether this preparation occurs in the context of mathematics education or in more general settings. If this means that less effort is given to preparing staff to lead inservice work, then this study (coupled with the existing research on the process of change) would indicate that the pay-off may be worth it.
3. Thompson (1984, 1992) observed that the extent to which experienced teachers' beliefs about the teaching and learning of mathematics are consistent with their classroom practice depends largely on their tendency to reflect on their actions. It would seem that the role that the researcher played in facilitating reflection could conceivably have been carried out by the mathematics coordinator or by another teacher, assuming that time commitments permit, appropriate preparations are made, and the focus aspects of changing teacher role are mutually established by the participants.

REFERENCES

Clarke, D. M. (in press). Ten key principles from research on the professional development for mathematics teachers. In D. B. Aichele (Ed.), *Professional development of mathematics teachers* (National Council of Teachers of Mathematics 1994 Yearbook). Reston, VA: NCTM.

Cobb, P., Wood, T., & Yackel, E. (1990). Classrooms as learning environments for teachers and researchers. In R. B. Davis, C. A. Mayer, & N. Noddings (Eds.), *Constructivist views on the teaching and learning of mathematics* (pp. 125-146). Reston, VA: National Council of Teachers of Mathematics.

de Lange, J., van Reeuwijk, M., Burrill G., & Romberg. T. A. (in press). *Learning and testing mathematics in context: The case : Data visualization.* Madison, WI: National Center for Research in Mathematical Sciences Education.

Eisenhart, M. A (1991, October). *Conceptual frameworks for research circa 1991: Ideas from a cultural anthropologist; Implications for mathematics education researchers.* Paper presented at the Psychology of Mathematics Education Conference, Blacksburg, Virginia.

Fennema, E., Carpenter, T. P., & Peterson, P. (1989). Teachers' decision making and cognitively guided instruction: A new paradigm for curriculum development. In N. F. Ellerton & M. A. Clements (Eds.), *School mathematics: The challenge to change* (pp. 174-187). Geelong, Australia: Deakin University Press.

Gunstone, R. F., & Northfield, J. R. (1988, April). *Inservice education: Some constructivist perspectives and examples*. Paper presented at the annual meeting of the American Educational Research Association, New Orleans, Louisiana.

Lampert, M. (1988a). *The teacher's role in reinventing the meaning of mathematical knowing in the classroom*. East Lansing, MI: The Institute for Research on Teaching, College of Education.

Lampert, M. (1988b). What can research on teacher education tell us about improving quality in mathematics education? *Teaching and Teacher Education, 4*(2), 157-170.

Merriam, S. B. (1988). *Case study research in education: A qualitative approach*. San Francisco, CA: Jossey-Bass.

Middle Grades Mathematics Project. (1988). *The Middle Grades Mathematics Project: Good mathematics--taught well.* (Final report to the National Science Foundation). East Lansing, MI: Mathematics Department, Michigan State University.

Padilla, R. V. (1990). *HyperQual Version 3.0* [Computer Program]. Chandler, AZ: Author.

Schoenfeld, A. H. (1987). What's all this fuss about metacognition? In A. H. Schoenfeld (Ed.), *Cognitive science and mathematics education* (pp. 189-215). Hillsdale, NJ: Lawrence Erlbaum.

Stephens, W. M., & Romberg, T. A. (1985, March). *Reconceptualizing the teacher's role.* Paper presented to SIG/RME at the annual meeting of the American Educational Research Association, Chicago.

Thompson A. (1984). The relationship of teachers' conceptions of mathematics teaching to instructional practice. *Educational Studies in Mathematics, 15*, 105-127.

Thompson, A. G. (1992). Teachers' beliefs and conceptions: A synthesis of research. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 127-146). New York: Macmillan.

AN EXPLORATION OF STUDENT RESPONSES TO THE MORE DEMANDING KÜCHEMANN TEST ITEMS

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Understanding symbolic notation is usually considered crucial to the study of mathematics. One significant aspect that has emerged from research into this understanding involves the meanings students give to pronumerals (see for example, Collis (1975), Pegg and Redden (1990)). Küchemann (1981), in particular, has identified six different interpretations for the meaning attributed to letters by junior secondary students. These interpretations have been grouped into four levels representing a hierarchy of understanding.

The purpose of this study was to explore students' responses to three of the more demanding Küchemann test items which were identified after an initial sample of 278 students were given the entire set of test items. Twenty one students were then interviewed and the reasons for their responses to these questions were analysed. Students' responses and associated reasons could be divided into two distinct categories, depending upon the mental processes involved in answering these questions. The distinguishing factor was found to be the ability to take into account the 'range of possibilities and limitations' associated with relationships involving letters.

Fundamental to the study of algebra (in terms of generalised arithmetic) is the meaningful interpretation of its symbolism. For many students, however, this conceptualisation merely consists of the manipulation of letters without any real understanding of what the letters actually represent.

Identifying levels of understanding in relation to the meanings given to symbols by students has been the subject of many studies (see for example Küchemann (1981), Booth (1984), Pegg and Redden (1990), Coady and Pegg (1991), Quinlan (1992)). In all of these studies it has been generally accepted that the highest level of understanding is that of interpreting a letter as a variable. Interpretation at this level requires higher-order mental functioning, where thought processes are not confined to seeing relationships but rather on focusing on the nature of the relationships. One aspect of this would occur when allowances can be made for any 'possibilities' and consequent 'limitations' inherent within the governing principles of the problem/question.

Initally, 278 first year tertiary mathematics students were scored on the Küchemann test items. An analysis of these results revealed that those questions involving the consideration of possibilities relating to the value of a pronumeral proved very demanding in terms of cognitive functioning. These questions were:

* Question 1. *Which is larger, $2n$ or $n + 2$? Explain.*

* Question 11. *What can you say about c if $c + d = 10$ and c is less than d?*

* Question 12. $L + M + N = L + P + N$ *is*
always sometimes never true?
(when)
Circle your answer.

(* These question numbers refer to those on the original test paper.)

It was decided to consider, in depth, the range of responses to these three questions. In particular, the following issues were addressed:

1. Is there any consistency among groups of responses?
2. If these different groups of responses exist, what are their characteristic features?

This work extends the ideas offered by Küchemann as it is an attempt to explore the subtleties in the answers offered by students.

DESIGN

On analysing all 278 written test responses, the percentage of students incorrect on each of the abovementioned three questions were 79%, 64% and 50% respectively. In order to examine the reasons for these poor performances, twenty one students were then interviewed.

During the interview, each student was asked to attempt each of the questions previously mentioned while verbalising their thoughts. If further clarification of their reasoning was needed, 'how and why' questions were asked by the interviewer.

RESULTS AND DISCUSSION

Two distinct categories of responses could be identified for each of the questions. Broadly, these were:

Category a) responses gave no indication of an awareness of any underlying condition restricting the variable, i.e., reasoning was confined to manipulating the terms in the given system; and,

Category b) responses indicated that account was taken of various conditions, i.e., 'possibilities' were allowed for and the associated 'limitations' were determined.

An analysis of the responses to each question in each of these categories now follows.

Which is the larger, $2n$ or $n + 2$? Explain

Category a) responses to this question concentrated on the operations, multiplication and addition, resulting in the conclusion "2n". Sometimes this conclusion was verified by the substitution of one positive value of 'n'. Even if the interviewer probed, *"is this true for all values of n"*, the overriding consideration was still on the operations. The following extract characterises this type of response:

I: What is your answer to question 1?
J: *2n.*
I: Why?
J: *Because it is multiply ..... $2 \times n$, whereas the other is only plus.*
I: And is 2n larger for all values of n?
J: *Yes it would be. For example if $n = 3$, $2 \times 3 = 6$, but $3 + 2 = 5$, and 6 is bigger than 5.*

Category b) responses to this question were characterised by the realisation that in order to determine the larger of the two expressions, the value of n needed to be considered. This was usually the first statement made by the student. This was then followed by the substitution of one, two of three values of n which were all in the vicinity of $n = 2$. Students making one substitution only, invariably chose $n = 1$, generalising this to "$2n < n + 2$ for $n < 1$" or chose $n = 3$ leading to the conclusion "$2n > n + 2$ for $n > 3$". Where two substitutions were made, values of n less than 1 and values greater than 3 led to the same conclusions as mentioned previously. These responses do not necessarily preclude the substitution $n = 2$ which may have occurred 'in the student's head'. The following is typical of the highest level of reasoning shown in this question:

I: How do you do question 1?
M: *Relating the two variables between the two expressions, the value of n determines whether or not 2n or n + 2 is larger. For example, if $n = 1$, 2n would be smaller than n + 2 whereas in the example of say $n = 3$, 2n would equal 6 and n + 2 would equal 5 so therefore the variable n determines whether 2n is larger or smaller than n + 2.*
I: Could we write down a statement that qualifies which would be the larger?
M: *If n is less than 2, 2n equals ..... , if n is less than 1, 2n is less than n + 2* [writes down $2n < n + 2 \quad n < 1$]; *if $n = 2$, 2n equals n + 2*, [writes down if $n = 2$, $2n = n + 2$], *[if] n is greater than 2, 2n is greater than n + 2* [writes down if $n > 2$, $2n > n + 2$].

This student's response indicates that a complete overview of the problem was well within his/her grasp and accurate conclusions were drawn.

What can you say about c if $c + d = 10$, and c is less than d?

Category a) responses to this question were typified by the manipulation of symbols, similar to those techniques used when solving equations. Hence the most common response here was "$c = 10 - d$". Others in this category did try to incorporate the second part of the question ('c is less than d') which resulted in the expression "$c < 10 - d$". For example:

I: How could we work out an expression involving c?
E: *c plus d equals 10* [writes $c + d = 10$], *c is less than d* [writes $c < d$] .... could we write it like that? [writes $c = 10 - d$, $c < 10 - d$]
I: c is less than d and c is less than 10 - d, are they the same statement?
E: *No.*
I: What do we have to do now?
E: *I think maybe we should plot some numbers in.*
I: OK lets try that.
E: *Because it equals 10, we have to choose numbers less than 10......I really don't have any idea.*

Category b) responses again showed an understanding that some underlying condition on the value of c was needed. All students' responses in this category mentioned c as having a "boundary" value or that c was the "half-way" mark. Attempts to identify the answer can be considered within three groups of responses.
The first group in this category chose one value of c usually $c = 4$. After this value was seen to satisfy the conditions in the question, no further values were substituted and "$c = 4$" represented the conclusion drawn.

Responses still characteristic of this category but representing a slightly higher level of understanding, again concentrated on values of c, but a systematic list such as "$c = 0, 1, 2, 3, 4$" or "c goes from 0 - 4" was given. Under prompting, students were usually able to provide the verbal statement 'c must be less than 5'.

The third group were able to determine the critical value of c, after which a conclusion was drawn, as the next abstract shows:

I: What can you write down about c?
C: *First of all, because c is less than d, I actually write it down graphically* [writes $c < d$]. *Now because you have $c + d = 10$, that* [points to $c < d$] *means d must be greater than c. So for any number c, c is less than d so therefore c is less than 5* [writes $c < 5$].
I: Why 5?
C: *Because you have 10 as a number and because d is greater than c, then c cannot be greater than 5 because 5 is the half way mark, so you have 5 + 5, that will be 10, but it says c is less than d, then it cannot be 5 but it could be 4.9 plus 5.1 as d.*

$L + M + N = L + P + N$ is

always sometimes never true
(when)
Circle your answer.

While this question is similar in nature to the other two questions, in that there existed an underlying constraint on the variables, this constraint was not of the same style. Certainly here, a range of possibilities needs to be considered, but it is more general in nature and hence, not as difficult as the first two questions. That is, students need only to be able to accept that even though M and P can vary over a range of values there are situations in which equality is a possibility.

Category a) responses formed two groups. The first group answered this question with 'never', occasionally supported with a statement such as "$M \neq P$". Prompting from the interviewer suggesting the possibility of M and P having the same value did not alter this conviction. For example:

I: What is your answer to question 12?
F: *Never.*
I: Why?
F: *As M cannot equal P.*
I: But what if M and P had the same value, say 5?
F: *If they had the same value, then the same letters would be used, not different ones.*

The second group in this category showed responses similar to those used when solving equations, which happened to result in the correct solution. This procedure involved crossing off the L's and N's from both sides, which leaves $M = P$. Again, this is indicative of students working within a restricted or closed system related to the students' empirical reality.

Category b) responses indicated that the *values* of M and P must be the same if the statement is to be true, but no actual substitutions were made. This is highlighted in the following extract:

I: How do you do Question 12?
P: *With N being on both sides that means that those two are equal because they would be constants or they would have the same value, being the same letter...the other two* [meaning the L's] *would have the same value, so it would depend on the M and P, their values so it would be sometimes...it would depend on the values of M and P.*
I: Can you make some statement about the values of M and P? Can you write down anything further there?
P: *For the righthand side to equal the lefthand side, M would have to equal P for all values.*

CONCLUSION

Higher level interpretation of a letter and the cognitive processes associated with these has been the focus of this study. The concept of a variable which is generally considered to be an advanced interpretation requires the development of quite complex reasoning skills, different from those needed, say, when a pronumeral is interpreted as a generalised number. Empirical data, as well as quantitative data, from this study has supported this view. An analysis carried out on the original 278 written responses to the three Küchemann test items which differentiated these levels of thinking revealed that only 9% of students could correctly answer all three questions, whereas 35% of students were wrong on all three questions.

This study has also highlighted a number of issues which relate to the mental development associated with high level cognitive skills. These can be summarised as follows:

1) A response which appears limited to working within a given system is indicative of a lower order processing skill. Manipulation of symbols without regard to potential restrictions is one such example.

2) The ability to account for possibilities and the consequent limitations, however, suggests the presence of higher order functioning.

3) There appear to be important subcategories to higher order functioning.

Finally, this study has provided insight into how students' responses to certain algebra questions might be evaluated. In particular, it offers a useful framework from which to consider why certain items are seen as more difficult. Hopefully, once more is known, effective teaching strategies can then be established so that higher order responses might be within the grasp of more students.

Acknowledgement
Our thanks to Dr Dietmar Küchemann for the use of his items.

REFERENCES

Coady, C and Pegg, J.E. (1991). An Investigation of Basic Algebraic Understanding in First Year University Students. *A paper presented to the 14th Annual MERGA Conference, University of Western Australia, Perth, July.*

Collis, K. (1975). *A Study of Concrete and Formal Operations in School Mathematics. A Piagetian Viewpoint,* Australian Council for Educational Research, Australia.

Küchemann, D. (1981). 'Algebra' in *Children's Understanding of Mathematics, 11-16*, K. Hart (Ed.). Alden Press, Oxford, London.

Pegg, J. (1992). *Teaching Understanding of a Variable.* A paper presented to the Second Mathematics Teaching and Learning Conference *From Numeracy to Algebra*, Centre for Mathematics and Science Education, Queensland University of Technology, Brisbane, September.

Pegg, J. & Redden, E. (1990). Procedures for & Experiences in Introductory Algebra. *Mathematics Teacher,* 43(5), 386-391.

Quinlan, C. (1992). *View of Symbols and Achievement in Algebra.* A paper presented to the Second Mathematics Teaching and Learning Conference *From Numeracy to Algebra*, Centre for Mathematics and Science Education, Queensland University of Technology, Brisbane, September.

MENTAL COMPUTATION STRATEGIES FOR ADDITION AND SUBTRACTION ALGORITHMS

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The changing nature of modern technological society and the role of mathematics in its functioning is resulting in calls for increased emphasis on mental computation and computational estimation and reduced emphasis on pen-and-paper algorithms. There is strong evidence that children begin primary school with many original and creative strategies for the operations, strategies that support mental algorithms and computational estimation. However, the procedures in traditional pen-and-paper algorithms appear to inhibit these spontaneous strategies and the early treatment of pen-and-paper algorithms may be wrongly placed in modern mathematics syllabi. This paper identifies and describes the different spontaneously derived cognitive strategies for one-, two- and three-digit mental computation in addition and subtraction used by 130 children within a longitudinal study to examine changes in knowledge and use of strategies over Years 2 and 3 of traditional pen-and-paper algorithm instruction.

There have been changes to the scope and sequence of mathematics in recent Australian syllabus documents to delete and defer some pen-and-paper algorithms and give greater focus to calculators, mental computation and estimation. Experience with materials before introducing pen-and-paper recording has been stressed. Computations with large numbers have been left for the calculator. However, unlike some initiatives overseas (e.g. Shuard, 1991), mastery of pen-and-paper algorithm procedures has tended to remain the chief goal of operations while mental computation, estimation and calculator usage have followed the pen-and-paper algorithm instruction.

The procedures in the traditional pen-and-paper algorithms for addition and subtraction appear to contradict those for mental algorithms. When adding the traditional algorithm requires: (1) the two numbers to be split into ones and tens; (2) the ones and tens to be added separately, with the ones added first and any tens formed added in with the tens; and
(3) the two sums combined for the answer. On the other hand, efficient mental algorithms tend to do all or some of the following: (1) hold one of the numbers unsplit; (2) add the tens before the ones; and (3) add the numbers cumulatively, modifying the answer as the calculation proceeds, e.g. 25+37=55+7=62. Mental algorithms appear to differ from traditional algorithms in terms of what they do to the numbers and in the order in which they add tens and ones.

There is strong evidence that children begin primary school with many original and creative strategies for the operations (e.g. Carpenter & Moser, 1984; Ginsburg, 1977). In particular, children have been able to invent cognitive strategies for the addition and subtraction algorithms and these strategies have been closer to the mental than the pen-and-paper procedures (e.g. Bebout, 1990; Cobb & Merkel, 1989). Evidence has existed for many years that strategies improve performance (Flavell, Beach & Chinsky, 1966) and that invented strategies enhance learning (Flavell, Friederichs & Hoyt, 1970). There is also evidence that such cognitive strategies can be taught (e.g. Lindquist, 1987).

Therefore, there is cogent argument that instruction in pen-and-paper algorithms may be wrongly placed in Australian mathematics syllabi, in that it may diminish the use of invented strategies and therefore act against the needs of children in a technological society (Sowder, 1990). There are strong doubts as to the need for formal pen-and-paper algorithms at all and support for restricting recording procedures to informal child-chosen techniques (Shuard, 1991).

This paper reports on an ARC funded two-year longitudinal study across years 2 and 3 of children's spontaneously-derived cognitive strategies for one-, two- and three-digit mental addition and subtraction and the effect of instruction from the traditional pen-and-paper algorithms on these strategies. It describes a range of strateges identified and the changes in strategy use across the two years.

METHOD

Subjects

The subjects of the study were approximately 65 girls and 65 boys from 3 state primary schools and 3 private primary schools, representing a variety of abilities and socio-economic backgrounds.

Instruments

The instrument used was a 'mixed cases' interview (Ginsburg, Kossan, Schwartz & Swanson, 1983). The children were presented with a sequence of one-, two- and three-digit addition and subtraction tasks to solve mentally. The sequence moved: (1) from addition to subtraction; (2) from one-digit to two-digit to three-digit numbers; (3) from non-renaming to renaming situations; (4) from a real world problem and pictorial presentation form to symbolic presentation form (horizontal to vertical); (5) from take-away to missing addend to comparison concepts of subtraction; and (6) from strategy-friendly to non-strategy examples (e.g. 26+49 to 26+47). The sequence of addition and subtraction examples was designed so that when children began to fail at mental addition, the appropriate level at which to begin the subtraction examples was evident.

Procedure

The children were interviewed six times between the beginning of year 2 and year 4. They were withdrawn for their classroom and interviewed in a vacant room within the school. The interviews lasted approximately 20 minutes and were all videotaped.

For the different areas, addition, subtraction, real world and symbolic, the children were given more difficult examples until they were unable to provide a strategy or unwilling to attempt the example. The interview procedure was designed to challenge not threaten the child.

RESULTS

Analysis

The videotapes were transcribed into protocols and the children's solution behaviours categorised. These categories of performance were then related to similar findings from the literature (e.g. Carpenter & Moser, 1984; Ginsburg, 1977) and to analyses of expert bahaviour, and competence categories involving the presence of cognitive strategies hypothesised to explain the behaviours. The analysis focused on findings: (1) across children for the various example types; (2) within children across the different example types; and (3) across the six interviews for individual children and the cohort as a whole.

Strategies

In the first interviews in year 2, most children used counting strategies even for reasonably sized two-digit examples. This caused many errors as children were unable to keep track of the numbers they were counting. However, some children had efficient strategies for keeping track of these numbers, breaking them into smaller parts, and were able to complete reasonably sized two-digit examples with accuracy. In addition, a small percentage (about 10%) exhibited sophisticated mental computation strategies and a few students were able to use these strategies to efficiently answer three-digit examples with speed and accuracy.

From the results of these initial interviews, two predispositions to mental computation and six strategy clusters were identified. The first predisposition was associated with visualisation, some children appeared to have a strong ability to visualise numbers, while others did not appear to use it. The second predisposition was associated with three primary ways the children perceived the examples: (1) through number, breaking down the numbers as the first step; (2) through strategy, activating a strategy as the first action; and (3) through process, focusing on the larger picture to obtain an overall idea of what the item required of them. The six strategies were: (1) simplified understanding, choosing to do something simpler but incorrect (e.g. adding instead of subtracting); (2) counting, simple counting procedures even with two-digit numbers, counting on and back, by ones, twos, fives, etc.; (3) basic fact strategies, near doubles, near ten and think addition, used mainly with the one-digit sections of the computations; (4) using tens, adding numbers using tens first in some way (e.g. 34+27 is 34+20=54 and 54+7=61 and 34+27 is 30+20+11=30+31=61); (5) pen-and-paper procedure, the classical pen-and-paper procedure of the ones first and then the tens (e.g. for 34+27, 4+7=10+1 and 30+20+10=60, equalling 61); and (6) estimation-recall, situations where the children appear to recall, guess or estimate the answer in one step.

However, in the later interviews in year 3, the above categorisation was unable to classify many solutions for three-digit examples and classified many interview-different solution behaviours in the same category. Furthermore, in behaviour, the students separated into three groups: (1) a section which after two years of instruction were still counting by ones and still not able to compute two digit addition and subtraction mentally; (2) a group which used the pen-and-paper algorithm mentally; and (3) a smaller group who efficiently used sophisticated mental computation strategies on three-digit examples. Therefore, to meet this situation, a more complex categorisation was constructed based around the following three perspectives.

1. Approach

   This covered how numbers were used and included the sub-categories simplistic, partitioned, place value R->L (lowest place value first), place value L->R (highest place value first), and wholistic (e.g. adding 98 in one step).
2. Process

   This covered how the operation process was achieved and included the sub-categories simplistic, separated place values, aggregation (e.g. 38+15=38+5+5+5), separated place values and aggregation (e.g. 38+25=38+20+5), compensation-undoing (e.g. 23+98=23+100-2), compensation-levelling (e.g. 23+98=21+100), and benchmark (e.g. 41+59=110).
3. Calculation

   This covered "basic-fact" style strategies and included the sub-categories counting (all, on, back, in groups, with materials, with fingers, etc.), near doubles, near tens, inverses (e.g. think addition), relating to known fact, and memory.

These categories enable complex solutions to be categorised in detail. For example, solution 244+359=544+59=600+3=603 would be categorised as place value L->R, benchmark and near tens, while solution 244+359=240+360+4-1=603 would be categorised as wholistic, compensation-undoing and relating to known fact.

CONCLUSIONS

The analysis of the two years of data collected from the six interviews is not yet complete, however some conclusions are evident. Firstly, many children lack even the most basic understanding of concepts and procedures even after years of instruction. Secondly, many children are very creative with numbers, manipulating them to suit the purpose of the task. Thirdly, children's use of strategies is complex. Many children do seem to be able to use a variety of strategies, choosing the most appropriate for the task in hand. However, they tend not to vary in predisposition or approach, using different specific strategies within a fixed framework. Fourthly, a small but significant number of children can compute effectively and efficiently before instruction. Finally, pen-and-paper instruction is appearing to have a strong influence on students' strategy choice for mental computation.

REFERENCES

Bebout, H.C. (1990). Children's symbolic representation of addition and subtraction word problems. Journal for Research in Mathematics Education, 21, 123-131.

Carpenter, T. P. & Moser, S.M. (1984). The acquisition of addition and subtraction concepts grades one through three. Journal for Research in Mathematics Education, 18 (4), 11-28.

Cobb, P. & Markel, G. (1989). Thinking strategies: Teaching arithmetic through problem solving. In P.R. Trafton & A.P. Shulte (Eds.). New directions for Elemetary School Mathematics (NCTM 1989 Yearbook). Reston, VA: National Council of Teachers of Mathmatics.

Flavell, J.H., Beach, D.R. & Chinsky, J.H. (1966). Spontaneous verbal rehearsal in a memory task and a function of age. Child Development, 37, 283-299.

Flavell, J.H., Friederichs, A.H. & Hoyt, J.D. (1970). Development changes in memorization processes. Cognitive Psychology, 1, 324-340

Ginsburg, H.P. (1977). Children's Arithmetic: The learning process. New York: Van Nostrand.

Ginsburg, H.P., Kossan, N.E., Schwartz, R & Swanson, D. (1983). Protocol methods in research on mathematical thinking. In H.P. Gindsburg (Ed.). The Development of mathematical thinking. New York: Academic Press.

Lindquist, M.M. (1987). Strategic teaching in mathematics. In B. Jones, A. Paliscar, D. Ogle & E. Carr (Eds.). Strategic teaching and learning: Cognitive instruction in the content areas. Alexandria, VA: Association for Supervision and Curriculum Development.

Shuard, H. (1991). Calculators, children and mathematics. London: Menthuen

Sowder, J.T. (1990). Mental computation and number sense. The Arithmetic Teacher. 37 (7), 18-20.

STUDENTS' UNDERSTANDING OF THE MATHEMATICAL CONCEPTS OF EQUAL AND EQUIVALENCE

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The equivalence relation exhibited as an equals sign is misunderstood by students who consider it to mean `do something', to indicate the location of the answer and to act as a separator symbol. This presentation reports on a cross-sectional study of students in years P, 2, 4, 6, 8 and 10 to explore understanding of equals and equivalence, conceptions of same and different and knowledge of the properties of equivalence and approaches to the equals sign. Students appeared to predominantly have an operator (`do something') understanding of equals, and to exhibit differing understandings of equals in situations such as $2+3=?$, $2+3=5$ *and* $x+3=5$, *of the three properties (reflexivity, symmetry and transitivity) and of the two approaches (static and dynamic).*

Mathematics has been categorised by Scandura (1971) as having only three foci: things; relations or relationships between things; and operators, transformations or changes between these things. A relation that is very important in mathematics is equivalence. Equivalence obeys three properties: (a) reflexivity, the relationship relates a thing to itself, i.e. A is related to A; (b) symmetry, if A is related to B, then the opposite direction is also true, i.e. B is related to A; and (c) transitivity, if A is related to B and B is related to C, then A is related to C.

The most used equivalence relation in mathematics is 'equals'. The concept of equality is based on the concepts of `same' and `different'. Same and different do not appear to be subject to any major misunderstandings, even by students (e.g. Fischer & Beckey, 1990). However, the equivalence relation exhibited in the equals sign is an elusive concept for students to understand. There is a persistent idea that, rather than expressing a symmetric and transitive relation, the equals sign means 'do something' (Ginsburg, 1982), indicates the location of the answer (Denmark, Barco & Voran, 1976; Kieran, 1992), or is a separator symbol (Kieran, 1992). This misunderstanding appears to continue at the secondary and tertiary levels (e.g. Behr, Erlwanger & Nichols, 1976; Clements, 1982). Clements found that students confused the syntax of the language with the the with semantics of the algebraic sentence, writing $6S=P$ for there being six students for each professor. In addition, there appears to be lack of attention to the two different ways equality can be approached: (a) in static terms as 'balance', e.g. $2+3$ balances 5; and (b) in dynamic terms as 'transformation', e.g. $2+3$ changes 2 to 5 by adding 3.

Students' understanding of the equals sign

Behr, Erlwanger & Nichols (1980) studied 6 to 12 year-old students' understanding of the equals sign in open number sentences. They argued that, to adults, the equals sign in sentences such as $2+4=6$ is, intuitively, an abstraction of the notion of sameness and, on a more sophisticated level, an equivalence relation. They further argued that sentences with no plus sign (e.g. $3=3$), or more than one plus sign (e.g. $2+1=2+1$), do not suggest an action to adults; rather, these sentences are seen to "require a judgement about their truth-value" (Behr, Erlwanger & Nichols, 1980; p.14). They found that students:
(a) understood the equals sign in number sentences such as $2+4=$ as meaning that something had to be done; (b) did not see $3+2=2+3$ in terms of sameness, but rather as an action by restating the sentence as $3+2=5$ or $5=2+3$; (c) would not accept the equals sign in sentences without it being preceded with one or more operation sign; (d) had "an extreme rigidity" about written sentences and a tendency to perform actions rather than reflect; and (e) did not "change in their thinking about equity as they get older" (p.15). Other researchers (e.g. Ginsburg, 1982; Kieran, 1992; MacGregor, 1991) have also noted that students do not tend to view the equals sign as 'the same as', but rather as an action to be performed (an `operator'). As Baroody and Ginsburg (1983; p. 199) found:

> "... chidren expect written (horizontal) equations to take a particular form: an arithmetic problem consisting of two (or perhaps more) terms of the left, the result on the right, and in between, a connecting ("equals") symbol (e.g. 3+2=5). Children tend to reject equations such as 13=7+6, 6+4=3+7, and 8=8, equations that do not adhere to the typical form and easily lend themselves to an operator interpretation of equals.".

Research typically indicates that viewing equals as an operator sign persists through primary school (Baroody & Ginsburg, 1983). Moreover, a restricted understanding of equals may continue into secondary and tertiary education and may affect mathematics learning at these levels. For instance, if equals is not viewed as a relational sign, algebra solution strategies (such as adding identical elements to each side of an equation to simplify the expression on one side) may not be meaningful. Another example is MacGregor's (1991) finding that the reversal error commonly made in Clement's (1982) Student-Professor problem, was due to the inappropriate use of the equals sign.

This paper describes a cross-sectional study which intended to expand on the work done by Behr, Erlwanger and Nichols (1976), and explored students' understanding of the mathematical concepts of equals and equivalence in years P, 2, 4, 6, 8, and 10. It explores the extent of students' conceptions about same/difference and the equals sign. It extends these explorations to students' understanding of the properties of equivalence, and knowledge of the static and dynamic approaches to the equals sign.

METHOD

Subjects

The subjects in the study were chosen by their teachers to represent the range of abilities in their classes and included: (a) six students, four girls and two boys, from a private inner-city preschool; (b) eighteen students from an inner-city private catholic primary school, three boys and three girls from each of years 2, 4, and 6; (c) twelve students from an inner-city private catholic secondary school, three girls and three boys from each of years 8 and 10.

Instrument

The instrument used was the "mixed cases" focused clinical interview, an approach to data gathering based on talk aloud as well as traditional interview techniques. Students were interviewed whilst working on the following four sets of equals and equivalence tasks: (a) the notions of same and different - the students were asked to identify same and different for a variety of topic areas; (b) formal understanding of the equals sign - the students were asked what the sign meant in different situations; (c) the equivalence properties - the students' understanding of reflexivity, symmetry, and transitivity were explored; and (d) static and dynamic approaches - the students' meanings of equals as a balancing relationship and as a transformation or change were explored. The tasks involved the students working in situations and contexts appropriate to their year level: counters, blocks, pattern cards and drawings in the early years, and equivalent fractions, similar shapes and number and algebra statements and sentences in the later years. Some novel equality situations were considered in the later years.

Procedure

The students were interviewed at their school in a spare room. The interviews were video-taped. The focus of analysis was the ideas of equals and equivalence held by the students in terms of their present mathematical level.

RESULTS AND DISCUSSION

The video-tapes were transcribed into protocols and analysed for commonalities. From these, categories of understanding that might explain behaviour were inferred.

The students' reactions to the four sets of tasks varied. Some tasks were completed with what appeared to be good understanding, including those relating to same and difference (in accord with Fischer & Beckey, 1990) and to the equivalence property of transitivity. Other tasks were not often completed satisfactorily and appeared to be little understood, particularly those relating to the equivalence property of transitivity and to the transformation approach to equals.

However, behaviours were not always able to be interpreted in a straightforward manner. For instance, although the results with respect to same and difference showed that students could identify same and different, in many cases this only occured under questioning. It showed that, for these students, the idea of considering two things are the same, when this means the same in some attributes while being different in other attributes, in not a natural or easily accepted understanding.

These subtleties in understanding were also evident in meaning of equality. Similar to the literature (Behr, Erlwanger & Nichols, 1980; Ginsburg, 1982; Kieran, 1992; MacGregor, 1991), the equals sign was predominantly seen as an action, e.g. "2+3= leaves something to be done while in 2+3=5 it has been done". This action orientation led to some interesting coclusions from students, e.g. seeing 5=2+3 type examples as being "the wrong way around". However, there were some indications that students see the equals sign as meaning something slightly different in examples such as 2+3=5, 2+3=? and x+3=5. The subtlety in these distinctions appeared to increase with age. For many students, the equals sign in 2+3=5 is seen as the sides being the same while in 2+3=? as a direction to work something out. Older students would often say that the sign means the same thing in all situations but then proceed to assign a different meaning to different situations, e.g. "It is the same, but you have to find the answer in this situation (2+3=?)".

When the equals sign was considered in other sitations, e.g. equivalent fractions, similar shapes, the quality of the students' responses appeared to depend on their familiarity with the context. In the novel situation, used only for the older students, where I+I was made equal to XX for the reason that four popsicle sticks (or four straightlines) were used to make either side, students appeared to have real difficulty coping with having an equals sign in this statement

For the equivalence properties, transitivity appeared to be well understood, symmetry reasonably understood and reflexivity not recognised. When faced with a situation, where a design card or a number is placed in a collection and students are asked to find something in the collection that is the same as that card/number, they would rarely pick up the starting card or number.

Equality relationships were commonly seen in terms of balance and students were skillful in returning ralationships to balance when they were unbalanced. Equality as transformation was less familiar and older students appeared unable to interpret examples such as x+3=5 in terms of change and reversing change, even when an example was worked through with them.

REFERENCES

Baroody, A. J., & Ginsburg, H. P. (1983). The effects of instruction on children's understanding of the equals sign. Elementary School Journal, 84 (2), 199-212.

Behr, M., Erlwanger, S., & Nichols, E. (1980). How children view the equals sign. Mathematics Teaching, 92, 13-15.

Clement, J. (1982). Algebra word problem solutions: Thought processes underlying a common misconception. Journal for Research in Mathematics Education, 13, 16-30.

Denmark, T., Barco, E., & Voran, J. (1976). Final report: A teaching experiment on equality (PMDC Tech. Rep. No.6). Florida State University (ERIC ED144805).

Ginsburg, H. (1982). Children's Arithmetic: How they learn it and how you teach it. Austin: Pro-ed.

Kieran, C. (1992). The learning and teaching of school algebra. In D. A. Grouws (Ed.), Handbook of research on mathematical teaching and learning (pp. 390-419). New York: McMillan.

MacGregor, M. E. (1991). Making sense of algebra: Cognitive Processes influencing comprehension. Geelong: Deakin University Press.

Scandura, J. M. (1971). Mathematics: Concrete behavioural foundation. New York: Harper and Row.

RELATING SOCIAL-INTERACTION ROLES AND METACOGNITIVE FUNCTIONING IN MATHEMATICS PROBLEM SOLVING

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This conference paper discusses possible relationships that may exist between social-interaction roles and children's metacognitive functioning. The paper reports on results of a study which compared the effectiveness of two approaches to group problem solving, specifically: (a) non-directed group activity involving limited teacher involvement only in the form of posing problems and engaging in follow-up discussions; and (b) role-directed group activity involving direct teacher intervention through the training of social-interaction roles (recorder-reporter, checker, and leader-judge) and the organising and rotating of these roles as well as posing problems and follow-up discussion. The role directed groups appeared to: (a) perform better in maintaining task commitment and producing quality solutions; and (b) develop superior metacognitive ability. The paper concludes by proposing relationships between social-interaction roles and metacognitive processes and discussing possible ways to study these relationships.

The literature on problem solving has long espoused the efficacy of groups (e.g. Johnson & Johnson, 1989; Noddings, 1985). There is evidence that cooperative learning experiences promote higher achievement in reasoning strategies, the generation of new ideas and transfer (e.g. Johnson & Johnson, 1991) and enhance problem-solving performance and develop problem-solving skills (e.g. Assessment of Performance Unit, 1985; Clement & Konold, 1989), particularly when members of the group have been assigned social-interaction roles (Dalton & Smith, 1986).

One of the major reasons for researchers advocating groups in problem solving situations is the effect of collaboration on metacognition. Metacognitive processes (e.g. planning, monitoring, regulating, evaluating) have emerged as important components of problem solving (e.g. Costa, 1984, Reeve & Brown, 1985). Research is showing that metacognition can be developed through group activities such as reciprocal teaching (Reeve & Brown, 1985). Johnson and Johnson (1991) contended that this development is due to the interactions amongst group members resulting in improved summarising, explaining and elaborating skills. They argued that "Material being learned to be taught to collaborators is learned using higher-level strategies more frequently than material being learned for one's own use." (page 299), a position which is supported by Male (1990) who argued that the effectiveness of groups is provided by positive interdependence between group members (through setting group goals, partitioning the task amongst group members, providing only one resource to be used by the group as a whole, assigning roles and/or providing rewards for group effort and cooperation). As stated by McTighe & Clemson (1991; 108), "... the articulation of strategies and reasoning within a group helps to render the invisible process of thinking visible for participants.". In addition, as Johnson & Johnson (1991; 105) warned, unless groups are appropriately structured, they "... can be characterised by self-induced helplessness, diffusion of responsibility and social loafing, ganging up against the task, reactance, dysfunctional divisions of labour, inappropriate dependence on authority, destructive conflict and other patterns of behaviour that debilitate group performance.".

The purpose of this paper is to explore relations between roles and metacognition. It reports on a study to compare the effectiveness of social-interaction roles on group problem solving and proposes a relationship between roles and metacognitive processes.

THE STUDY

The purposes of the study were to:

(1) determine if group work constitutes an effective instructional form for developing expertise in mathematics problem solving; and

(2) compare the effectiveness of two approaches to group problem solving, specifically: (a) non-directed group activity, involving teacher intervention only in posing problems and follow-up discussions, and (b) role-directed group activity, involving direct teacher intervention through the training of social-interaction roles and the organising and rotating of these roles, as well as posing problems and follow-up discussions; on children's problem-solving performance and higher-intellectual functioning in mathematics.

Subjects

The subjects for the study consisted of thirty-six children from two year five classes in a predominantly low socio-economic status government primary school.

Instruments

The data gathering instruments for the study were group interview and observation. The tasks for the two interviews, one before the problem solving activities (the pre-interview) and one after (the post-interview), consisted of a two-dimensional dissection puzzle and an open problem concerned with determining an appropriate course of action in a social situation. The observations were taken while the students solved three open (no specific goal) and three closed problems.

Procedure

The thirty-six children were divided into twelve groups of three and these groups were randomly assigned to two treatments: role-directed and non-directed. All groups were pre-interviewed and the interviews video-taped. All groups were given five one-hour problem-solving lessons on the three open and three closed problems. The non-directed groups could solve the problems in any manner they wished as long as it was a group activity. The role-directed groups were required to solve the problems with members acting out three social-interaction roles: (a) recorder-reporter, writing down key points, preparing written material and reporting back to the group; (b) checker, checking to see if everyone in the group understands and agrees with the group's responses, evaluating the quality of what the group is doing and encouraging members to stay within their roles; and (c) leader-judge judge, overseeing the activity of the group and making decisions when the group is unable to do so (these were based on those developed by Dalton & Smith, 1986). At the end of all the problem-solving sessions, each group was post-interviewed and the interviews videotaped.

Results

The interviews showed a noticeable improvement in the role-directed group's problem solving and higher-intellectual functioning compared to that of the non-directed groups. Overall, the role-directed groups displayed improvements in the following metacognitive processes: (1) planning, brainstorming for ideas (pre-planning phase) and formulating plans; (2) acting, cooperating within the group, persisting on the problem-solving task, displaying cohesion as a group and displaying independence; (3) monitoring, displaying active listening behaviours and detecting inadequacies in solution paths; (4) regulating, opportunistically modifying plans and strategies when deficiencies are shown to exist; and (5) evaluating, reflecting on performance and verbalising the path to their thinking. The observations showed even more extreme differences. To the last session, the role-directed groups remained strongly committed to task, sustaining collaboration and planning and monitoring their work together. Typically, the role-directed groups were able to cooperatively use one set of materials, talking together and discussing options while one member recorded the findings. In contrast, by the last session, collaboration and cooperation had broken down in the non-directed groups. Children were interfering with each other, complaining and arguing, refusing to share material and being angry and aggressive. There was also large differences between the achievement of the groups, with the role-directed groups giving reasoned and complex answers to the open problems, while the non-directed groups' solutions were trivial.

DISCUSSION AND PROPOSAL

The results of the study tended to support a complex relationship between groups and performance (similar to Johnson & Johnson, 1991), i.e. that the group formation for successful improvement in problem-solving skill includes a social dynamic that encourages children to focus on their own thinking. The role-directed groups' improved performance in problem solving came with improved performance in the metacognitive skills of planning, monitoring, evaluating and regulating (similar to McTighe & Clemson, 1991).

The findings of Male (1990) and Johnson & Johnson (1991) provide a framework for understanding the success of the role-directed groups in this study, i.e. the roles provided a framework for skills in leadership, conflict management and communication and for determining who will do what and when. The study supported the contention that the recorder-reporter and checker roles enhance monitoring and evaluating, while the leader-judge (decision maker) role enhances planning and regulating as well as monitoring and evaluating.

Dalton & Smith (1986) have indicated that there are a wide variety of social-interaction roles that may prove useful in group work. It is therefore useful to analyse the metacognitive characteristics of all Dalton & Smith's social-interaction roles, because, with their possible effect on metacognition documented, a procedure for selecting the appropriate role for the needs of a group is available. A possible analysis is below. It is interesting to note the dearth of roles that relate to planning. There may be a need to consider further roles or to change roles to provide a planning focus.

ROLE	FUNCTION	METACOGNITIVE CHARACTERISTICS
Checker	Checks to determine if there is agreement	Monitors and, to a lesser extent, evaluates
Leader -Judge	Makes executive decisions if no group consensus	Involved in all aspects of metacognitive activity
Encourager	Encourages each member to participate	Monitors, regulates and evaluates
Networker	Checks information with other groups, shares info.	Super-ordinate position similar to decis. maker
Observer	Observes what group does, relates back to the group	Monitors and evaluates
Organiser	Ensures resources are available	Monitors ('housekeeper' function)
Paraphraser	Restates ideas, feelings, actions, etc.	Monitors
Recorder -Keyboarder	Writes down group's ideas, keys in info. on computer	Monitors, evaluates, summarises and acts
Reporter	Verbalises the processes /products of the group	Reflects, provides a trail of thinking
Summariser	Reviews group ideas	Monitors and evaluates
Time-keeper	Times groups' activities	Monitors and regulates
Turn-teller	Ensures everyone has turn	Monitors, regulates and evaluates

IMPLICATIONS

The implication is that teachers may be well served by concentrating their efforts on the social dynamics (e.g. assigning social-interaction roles) of the group situation, rather than directing all their efforts into intervening in a cognitive sense (e.g. providing strategic hints). Teachers will need to introduce, explain, demonstrate and clarify the roles, particularly in the acquisition stage. As students become more confident and competent in the the use of the roles, the teacher may be able to cede control. It will be important that teachers develop methods for diagnosing students' performances in group situations so that roles can be assigned which match the needs of the students. In addition to the social-interaction roles, teachers should employ a range of techniques such as providing rationales and developing micro-skills (e.g. questioning). A range of problems from open to closed would also appear to be warranted.

REFERENCES

Assessment of Performance Unit (APU). (1985). A review of monitoring in mathematics: 1978 to 1982. London: Her Majesty's Stationery Office.

Clement, J. & Konold, C. (1989). Fostering basic problem solving skills in mathematics. For the Learning of Mathematics, 9 (3), 26-29.

Costa, A.L. (1984). Mediating the metacognitive. Educational Leadership, 42 (3), 57-62.

Dalton, J. & Smith, D. (1986). Developing cooperative learning abilities. In K. Imison, L. Endean & D. Smith (Eds.). Gifted and talented children - A national concern. (Proceedings of the second national conference on gifted and talented children). Toowoomba, Qld: Darling Downs Institute Press.

Johnson, D.W. & Johson, R.T. (1991). Collaboration and cognition. In A. Costa (Ed.). Developing minds: A resource book for teaching thinking (Vol.1). Alexandria, VA: Association for Supervision and Curriculum Development.

Johnson, D.W. & Johson, R.T. (1989). Cooperative learning in mathematics education. In P.R. Trafton & A.P. Shulte (Eds.). New directions in elementary school mathematics.(1989 yearbook). Reston, VA: National Council of Teachers of Mathematics

Male, M. (1990). Cooperative learning and computers in the elementary and middle school math classroom. In N. Davidson (Ed.). Cooperative learning in mathematics: A handbook for teachers. Menlo park, Cal.: Addison-Wesley.

McTighe, J. & Clemson, R. (1991). Making connections: Toward a unifying instructional framework. In A. Costa (Ed.). Developing minds: A resource book for teaching thinking (Vol.1). Alexandria, VA: Association for Supervision and Curriculum Development.

Noddings, N, (1985). Small groups as a setting for research on mathematical problem solving. In E.A. Silver (Ed.). Teaching and learning mathematical problem solving: Multiple research perspectives. Hillsdale, NJ: Lawrence Erlbaum.

Reeve, R.A. & Brown, A.L. (1985). Metacognition reconsidered: Implications for intervention research. Technical Report No 328, Centre for the Study of Reading, University of Illinois, Urbana-Champagne, Illinois, USA

LEARNING MATHEMATICS AT UNIVERSITY LEVEL: INITIAL CONCEPTIONS OF MATHEMATICS.

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This paper reports preliminary results from an ongoing investigation to identify the conceptions of mathematics held by beginning first year university students and their orientations to their previous study of mathematics. A questionnaire was administered to approximately three hundred students during their first week at university. The questionnaire contained five open-ended questions designed to elicit students' own conceptions of mathematics and their orientations to studying it. Two of these are discussed in this paper. Phenomenographic techniques were used to analyse responses and identify qualitatively different categories of description.

In-depth interviews of a subsample of twelve students revealed details of the range of conceptions of mathematics and the related qualitative differences in approaches to learning mathematics after several weeks of university mathematics.

Analysis of responses revealed that, although a wide range of beliefs was elicited, the majority of students view mathematics as a necessary set of rules and procedures to be learned by rote that are unrelated to other aspects of their lives. The survey results also indicate a relationship between conceptions of mathematics and approaches to studying mathematics at university level. There was no evidence of gender differences in either conception of mathematics or approach to learning.

This paper will report on the evidence of relationships between conceptions of mathematics, approaches to learning and course results. These preliminary results raise questions about the impact of prior experience on approaches to learning mathematics at university level and the quality of learning outcomes.

Mathematics in school is the foundation for later learning at the university level. Research at the school level indicates that the learners' previous experiences influence the quality of their approaches to learning, attitudes to and outcomes in learning mathematics (Clarke, 1985; Clarke, Fraser and Wallbridge, in press; Crawford 1983, 1986, 1990a, 1990b; Resnick, 1987). There are indications that students' conceptions of mathematics and their orientations to study affect the quality of cognitive activity and of learning outcomes (Crawford, 1986, 1989, 1990b, 1992a, 1992b). Also, the ways in which learners interpret the context of their mathematical learning and hence the ways they relate to mathematical activities influence their mathematical thinking (Crawford 1990a; Cobb, Yackel & Wood, 1992; Lave 1988; Steffe & Cobb, 1988; Solomon 1989). Furthermore, Biggs and Collis (1982) have shown that the characteristics of learning activity as expressed by students' verbal reports about learning reflect differences in thinking in ways which are related to their learning outcomes.

There is also a growing consensus among researchers in the field of student learning in higher education (Marton, Hounsell, & Entwhistle, 1984; Ramsden, 1991) that university students learn from their experiences. It would be expected that students would enter university with a range of conceptions of mathematics and orientations to mathematics learning derived from their school experiences. There is also evidence that the quality of these experiences, from the students' perspectives, substantially relates to the ways in which they approach learning at university level and the quality of learning outcomes (Crawford, 1993; Prosser & Millar, 1989; Trigwell & Prosser, 1991a, 1991b; Volet & Lawrence, 1989).

This study has focussed on three research questions:

1. What are the nature and range of beginning university students' conceptions of mathematics and how are these distributed?
2. What are the nature and distributions of orientations to learning mathematics for beginning university students?
3. Are there relationships between students' initial conceptions of mathematics, their orientations to learning and their achievements at university level?

METHOD OF RESEARCH

A survey questionnaire on student conceptions of mathematics and mathematics learning was administered to approximately 300 first year mathematics students at the University of Sydney. The questionnaire consisted of five open ended questions that were designed to elicit students' own conceptions of mathematics and their approaches to studying it. In-depth structured interviews were also carried out with twelve selected students in order to clarify their written statements and to enhance interpretations of the questionnaire responses. The interviews were taped and transcribed.

A phenomenographic approach (Marton, 1988) was taken to analyse the students' responses. The first stage in this analysis was to identify a set of ordered categories of description to the open ended questions. The categories were hierarchical in the sense that each category in the sequence included all elements of all preceeding categories. Although the research team were influenced by knowledge of recent theories of learning and research in the field of mathematics education, every effort was made to ensure that the categories emerged as the result of a qualitative analysis of the data. This process actively involved all members of the research team in the following procedure:

1. Twenty one representative responses were selected.
2. Five researchers individually identified an initial set of categories.
3. Researchers met and compared categories and explored the relationships between them.
4. The research team agreed to a draft set of categories.
5. Each researcher took the same subset of questionnaire responses and classified them according to the draft categories.
6. The individual classifications were compared and a final set of clear statements of each category was agreed upon.
7. One researcher then classified all three hundred responses.
8. Any responses that were difficult to classify were later discussed by all five researchers.
9. Later, responses involving a mismatch between conceptions of mathematics and approach to learning mathematics were further reviewed by two researchers.

The students' responses to the questionnaire were then analysed to explore the relationships between conceptions of mathematics, approaches to learning mathematics and attainment in first year mathematics at university.

RESULTS.

The analysis of the questionaire responses yielded an ordered set of categories for each open ended question. The categories that emerged from the analysis of students' responses about the nature of mathematics (Q. 2) is shown in Table 1.

Table1: Categories of responses for Question 2.

Question 2

Think about the maths you've done so far. What do you think mathematics is?

Category[1]	Representative quote from student survey.
A. Maths is numbers, rules and formulae.	*Maths is the study of numbers, and the application of various methods of changing numbers.*
B. Maths is numbers, rules and formulae which can be applied to solve problems.	*Mathematics is the study of numbers and their applications in other subjects and the physical world.*

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[1] To economise space the statement of category B has not been repeated in the statement of the subsequent categories but its inclusion is assumed.

C. Maths is a complex logical system; a way of thinking.	*Mathematics is the study of logic. Numbers and symbols are used to study life in a systematic perspective and requires the mind to think in a logical and often precise manner.*
D. Maths is a complex logical system which can be used to solve complex problems.	*Maths is an abstract reasoning process which can be utilised to explore and solve problems.*
E. Maths is a complex logical system which can be used to solve complex problems and provides new insights which develops our understanding of the world.	*Techniques for thinking about observable, physical phenomena in a quantitative way and also for thinking more abstractly with little or no relation to the directly observable universe.*

The categories derived from students' responses to the question about the nature of mathematics (Q.2) revealed a clear shift in conception between the first two categories and later elements of the ordered set. Responses in categories A and B generally described mathematics as a fragmented collection of rules and procedures whereas responses in categories C, D and E present views of mathematics as a more cohesive theory. The original questionnaire was designed so that students had two opportunities to describe their orientations to the study of mathematics. Consequently when analysing the data questions 3 and 4a were treated as one. The categories of description and representative student quotes are given in Table 2.

Table 2: Categories of description for the responses to Questions 3/4a.

Question 3/4a
3. Think about some maths you understood really well. How did you go about studying that? (It may help you to compare how you studied this with something you feel you didn't fully understand.)
4a How do you usually go about learning some maths?

Categories[2]	Representative quote from student survey.
A. Learning by rote memorisation, with an intention to reproduce knowledge and procedures.	*I liked calculus because I could remember formulas which is how I used to study. I would rote learn all the formulas and summarise all my theoretical notes. "*
B. Learning by doing lots of examples, with an intention to reproduce knowledge and procedures. (Drill and practice)	*The way I go about studying for mathematics is by doing a lot of questions and examples. Firstly I would study the notes and learn formulas, then I put all of that to use by doing heaps of exercises. "*
C. Learning by doing lots of examples with an intention of gaining a relational understanding of the theory and concepts.	*To understand a topic well it was important to gain an understanding of the basic concepts involved, backed up by some problem solving on the topic. However, concepts which were not fully comprehended could become well understood through extra work on related questions. i.e. It is essential to do a wide range of questions on a topic to fully understand it.*

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[2] Note, each category again includes all preceeding categories.

D. Learning by doing difficult problems, with an intention of gaining a relational understanding of the entire theory, and seeing its relationship with existing knowledge.	*After listening to explanation of how a particular maths works the most essential features a repetition to develop speed (this usually consists of boring menial tasks) and an equal component of very difficult problems which require a great deal of thought to explore that area and its various properties and their consequences.*
E. Learning with the intention of gaining a relational understanding of the theory and looking for situations where the theory will apply.	*Read the relevant theory and try to get on the same "wavelength" as the person who actually discovered it. Before I attempt any problems I try to think where you can use the concept: i.e. what the concept was invented for. Then I attempt problems (on my own).*

Students' approaches to studying mathematics showed a similar shift to that discussed above. Their responses indicated a shift in intention from reproduction of knowledge and procedures, in categories A and B, towards a relational understanding of the theory and the concepts, in categories C, D and E, of increasing complexity. The Gothenburg school (Marton, 1988) describes this type of shift as being from a surface to a deep approach. Categories A and B are examples of a surface approach and categories C, D and E are examples of a deep approach

Quantitative analyses

In this section we will describe the distribution of results across categories for both the conceptions and approaches and the relationship between conceptions and approaches. As well, we will show how the conceptions and approaches relate to student performance. Table 3 shows the distribution across categories of conceptions and approaches. In this sample, there was no evidence of gender effect.

Table 3: Distribution of Prior Conceptions and Approaches

Conception/Approach	N	% of responses
Conception		
A. Maths as numbers, rules and formulae	62	26
B. Maths as numbers etc. with applications to problems	124	51
C. Maths as way of thinking	32	13
D. Maths as way of thinking for complex problem solving	18	7
E. Maths provides new insights used for understanding the world	6	3
Missing Data (no response)	52	
Total	294	
Approach		
A. Learning by rote memorisation	17	6
B. Learning by doing examples (drill and practice)	215	76
C. Learning by developing relational understanding by doing examples	30	11
D. Learning by extending understanding with difficult problems	15	5
E. Learning by extending understanding to a broader context	6	2
Missing Data (no response)	11	
Total	294	

Table 3 shows that 77% of conception statements were classified in the first two (fragmented) categories, while 23% were classified in the last three (cohesive) categories. Eighty two percent of the approach statements were classified in one of the two "surface" categories, while only 17% were classified in one of the three "deep" categories.

Table 4 shows a cross-tabulation of the relationship between prior conception and prior approach.

Table 4: Relationship between Prior Conception and Prior Approach among responses

Conceptions *Approach*

		Surface		Deep	Row totals
Fragmented		179 (91%)		17 (9%)	196
Cohesive		4 (10%)		36 (90%)	40
Column totals		183		53	236

Chi-square=126, $p<0.001$

Table 4 indicates a relationship between conception and approach. It shows that 91% of the students who had a fragmented view of high school mathematics had adopted a surface approach to the study of high school mathematics, while only 10% of those who had a cohesive conception had adopted a surface approach.

Table 5 does not indicate a relationship between both prior conceptions and prior approaches to study and achievement in the first semester. However, after a year of university study, the results indicate that students with a cohesive conception of mathematics and/or able to adopt a deep approach to study tended to achieve at a higher level.

Table 5: Relationship between Prior Conception and Prior Approach and University Achievement

Conceptions/Approaches Mathematics 1

		Semester 1 *Mean SD		Semester 2 *Mean SD
Conceptions				
Fragmented		125 28		123 21
Cohesive		134 25		134 23
T Test: T=		1.53 not significant		2.16 $p<0.05$
Approaches				
Surface		125 28		122 29
Deep		134 24		136 21
T Test: T=		1.89 not significant		2.66 $p<0.01$

* Exam result marked out of 200.

CONCLUSION

The results show a range of beginning university students' conceptions of mathematics. A large proportion of the sample of successful school leavers studied, conceive of mathematics as numbers, rules and formulae which can be applied to solve problems. In our sample, these views of mathematics are associated with surface approaches to studying mathematics. There are indications that a more cohesive conception of mathematics and/or a deep approach to studying mathematics are positively and cumulatively associated with achievement at university level. The observations on conceptions of mathematics and approaches to learning mathematics at university level raise questions about the ways in which mathematical thinking is nurtured and assessed at university level. These are issues being explored in an ongoing study.

REFERENCES

Biggs, J. and Collis, K. (1982). *Evaluating the Quality of Learning The SOLO Taxonomy (Structure of the Observed Learning Outcomes)*: New York: AcademicPress.

Clarke, D. (1985). The Impact of Secondary Schooling and Secondary Mathematics on Student Mathematical Behaviour, *Educational Studies in Mathematics*, 16 (3), pp 231–251.

Clarke, D., Fraser, S. and Wallbridge, M. (in press). *Other Consequences of a Problem Based Mathematics Curriculum.*

Cobb, P., Yackel, E. and Wood, T. (1992). The Constructivist Alternative to the Representational View of Mind in Mathematics Education, *Journal for Research in Mathematics Education* 23, pp 2–33.

Crawford, K. (1983). Past Experiences and Mathematical Problems Solving, *Proceedings of the Seventh Annual Conference of the International Group for the Psychology of Mathematics Education*, Israel.

Crawford, K. (1986). Cognitive and Social Factors in Problem Solving Behaviour, *Proceedings the Tenth Annual Conference of the International Group for the Psychology in Mathematics Education*, London.

Crawford, K. (1989). Knowing How Versus Knowing What: the need for a change in emphasis in the education of students from non-technical societies, in C. Keitel (Ed.), *Mathematics, Education and Society*, Paris, UNESCO Science and Technology Education, document series No.35

Crawford, K. (1990a). School Definitions of Work: Their Impact on the Use of Technology in Mathematics Education, *Proceedings of the Conference on the Political Dimensions of Mathematics Education*, London Institute of Education.

Crawford, K. (1990b). Language and Technology in Classroom Settings for Students from Non-technological Cultures, *For the Learning Of Mathematics* 10(1) pp 11-15

Crawford, K. (1992a). Peer Interaction and Concept Formation: The Social Context of Mathematical Development, Working Group Paper invited for the *International Congress on Mathematics Education*, Quebec City, Canada,August,.

Crawford, K. (1992b). Cultural Processes and Learning: Expectations,Actions and Outcomes, Working Group Paper invited for the *International Congress on Mathematics Education*, Quebec City, Canada, August.

Crawford, K. (in press). Cultural Processes and Learning: Expectations, Actions and Outcomes, *Proceedings of the International Congress in Mathematics Education*, Quebec City.

Lave, J. (1988). *Cognition in Practice*. London, Cambridge University Press.

Marton, F. (1988). Describing and Improving Learning, in Schmeck, R. (Ed.), *Learning Strategies and Learning Styles*, pp 53–82. New York: Plenum Press.

Marton, F., Hounsell, D. and Entwistle, N. (Eds.) (1984). *The Experience of Learning*. Edinburgh: Scottish Academic Press.

Prosser, M. and Millar, R.,(1989). The How and What of Learning Physics. *The European Education*, 14, pp 55–68.

Ramsden, P. (1991). *Learning to Teach in Higher Education*. London: Routledge.

Resnick, L. (1987). Learning In and Out of School, *Educational Researcher*, 16(9), pp 13–20.

Solomon, Y. (1989). *The Practice of Mathematics*. London: Routledge.

Steffe, L. and Cobb, P. (1988). *Construction of Arithmetical Meanings and Strategies*. New York: Springer Verlag.

Trigwell, K. and Prosser, M. (1991a). Relating Approaches to Study and the Quality of Learning Outcomes at the Course Level, British Journal of Edducational Psychology, 61, pp 265–275.

Trigwell, K. and Prosser, M. (1991b). Relating Learning Approaches, Perceptions of Context and Learning Outcomes. *Higher Education* (Special Edition on Student Learning), 22, pp 251– 266.

Volet, S. and Lawrence, J. (1989). Goals in the Adaptive Learning of University Students, *Learning and Instruction: European Research in an International Context*, by Mandl, De Corte, Bennett and Friedrich (Eds.), 2–3, pp 297–516.

STRATEGIES USED BY YEAR 9 STUDENTS TO SOLVE PROBLEMS ON SPEED

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Since 1946 there have been a number of studies reported concerning students understanding of the concept of speed (e.g., Piaget 1946, Trowbridge 1979). This paper describes one part of a study concerning Year 9 students' strategies in solving problems which involved comparing the speed of two trolleys given the starting and finishing position and the time taken for each one. It was found that students used a variety of strategies ranging from guessing to intuitive use of variables to rather sophisticated uses of ratios and a calculation. Students also revealed some misconceptions concerning the variables that determine speed. Of particular interest was the use of time or distance as a determining variable without mention of the other variable.

Piaget (1970, 1946) studied young children's concepts of speed and motion using concrete illustrations. He was able to identify stages in the development of the concept of speed for "The gradual passage from intuitive thinking, still tied to the information of the senses, towards operational thinking, which forms the basis of reasoning itself" (p. ix). Trowbridge (1979) extended Piaget's ideas and questions in a study of first year University students. He was able to reveal "some of the ways in which college students in introductory physics courses think about velocity" (Trowbridge and McDermott 1980, p. 1028). The major misconception that these students had "was an inability to discriminate between position and velocity" (p. 1028).

This paper reports on a small component of a larger study involving secondary school students' concepts and responses to problems involving speed. The results are confined to the responses of students in an unstreamed Year 9 class to a set of questions called Dual Focus Closed Comparison Questions. Each question consisted of two diagrams representing the starting and finishing points of a trolley and the time taken for the trolley to get from its starting point to its finishing point. The students were asked to identify which trolley had the greatest speed and give an explanation of how they obtained their answer.

Originally ten questions were developed covering the ways that the variables: time, distance and starting points, could be arranged with the same and different speeds. In the final test, of which the Dual Focus Closed Comparison Question formed only a part, only five of these questions were used. These five questions are given in the appendix. Table 1 reports a summary of how the variables differ in each of the questions. Note that the two speeds in each question are the focus of the comparison.

Table 1
Variables in Questions

	Variables			
Question	**start position**	**time**	**distance**	**speed**
1	same	same	same	same
2	same	differ	differ	same
3	same	same	differ	differ
4	differ	differ	same	differ
5	differ	differ	differ	differ

At the time the test was administered 20 students were present: eleven girls and nine boys (average age 14 years, 6 months). The test was given during normal class periods. On completion of the preliminary analysis of the tests, interviews were conducted with five students. Both the test and the interviews were given without prior warning to the students. In the analysis that follows, the focus of attention is on the quality of each response as well as on the strategies of solving the problem implied by the responses.

RESULTS

Table 2 details a summary of the quality of responses that were given by the students tested. The responses were categorised into Full, Partial and Inappropriate explanations. A Full response occurred when a student referred to all variables that were required to obtain an answer. A partial response occurred when only some of the variables were referred to in the explanation. An inappropriate answer occurred when the response referred to some aspect of the question which did not reflect a correct approach to the problem. An answer that was correct but for which the explanation was inappropriate was considered to be an incorrect response.

Table 2
Type of Quality Response

Type of Response		Questions 1	2	3	4	5
correct responses						
- Full explanation	CF	12	5	9	7	7
- Partial explanation	CP	5	5	4	2	5
- no reason, or guess	CU	3	2	2	1	1
total correct		20	12	15	10	13
Correct response						
Inappropriate reason	CR	0	2	3	0	3
Incorrect responses						
- Full explanation	IF	0	0	0	1	1
- Partial explanation	IP	0	2	2	4	1
- Inappropriate reason	IR	0	4	0	3	1
- no reason or not done	IU	0	0	0	2	1
total incorrect		0	8	5	10	7

While a number of interesting features are revealed by this analysis, two warrant comment. First, Question 1 was included to test the students' ability to comprehend the diagram when all variables were identical. All students were able to do this. Second, more students were correct with Question 5 than Question 4. This was a surprise as in Question 5 all variables were different. The contingency table below shows the relationship of the correct responses to Questions 4 and 5.

		Q. 5	
		C	I
Q.4	C	10	0
	I	3	7

C = correct
I = incorrect

From this table it can be seen that all students who had Question 4 correct also had Question 5 correct. The three students who had Question 4 incorrect and had Question 5 correct, gave only partial answers to both questions which in the case of Question 5 reflected a lack of understanding of the problem. During interview two students said that "(trolley) A got further" and one student said that "(trolley) B was behind by 1 sec".

STRATEGIES

A summary of the different strategies used by students to solve the questions is given in Table 3.

Table 3
Frequency of Strategies

Strategy	Frequency				
	1	2	3	4	5
1. Calculation	0	1	0	0	3
2. Ratio	0	4	1	3	1
3. Informal (time and distance)	11	6	10	6	4
4. Time only	2	4	1	3	1
5. Distance only	0	2	5	5	5
6. One in front	0	0	1	0	3
7. Faster/speed	3	1	0	0	0
8. Guess	1	1	1	0	1
9. Unknown	3	1	1	3	2

Below are some of the responses of the students using the different strategies in the speed test with the code for quality of response. Also included are some extracts taken from the interviews. (The Colon : separates the answer from the explanation)

1. **Calculation** Students actually calculated the speed of both trolleys using distance ÷ time.

Student 920 Question 5 (CF)

[Trolley] A: $A = \frac{5}{4} = 1.25$ secs units per sec.' $B = \frac{3}{3}$ secs = 1 unit per sec so [trolley] A went faster.

It was interesting to note that the student did not quote a formula but merely divided distance by time to be able to compare the two ratios. In general it was only when the better students got to Question 5 that they reverted to a calculation approach. On analysing their previous responses it was found that all of these students had used the ratio approach at least once in other questions. They may not have needed to find the quotient of these ratios because at least one of the components of the ratio was the same. But with Question 5 all components of the ratio were different and one easy way to compare the speeds is to find the quotients of the ratios.

2. **Ratio** Students compared the ratio of distance and time for each trolley without evaluating the quotient.

Student 906 Question 4 (CF)

[Trolley] B: Because [trolley] A took 5 secs to go 3 units and it took [trolley] B 3 secs to go 3 units.

Student 915 Question 2 (CF)

Same [speed]: Because they are same speed by time and cm (equivalent).

This student could give a response that was not tied to the actual values of the problem, i.e., he was able to focus on the variables themselves.

Interview:

S They are going the same speed at 1 sec a unit.
I How did you get 1 sec a unit?
S Um well in [trolley] A, 3 divided by 3 equals 1 that is 1 unit per sec and in [trolley] B, 5 divided by 5 gives 1, 1 unit per sec.

Note that in the test and in the initial part of the interview this student used the ratio 'time : distance' but in the final part of the interview he reverted to 'distance : time" and calculated the quotient to find the speed, which indicates that, in this case, speed is neither a learned formula nor a triggered response.

3. **Informal** (Time and distance) Students used both time and distance in their response.

Student 903 Question 3 (CF)

[Trolley] B: it reaches the farthest [sic] in only 3 secs.

Interview

S um··[trolley] B because it reached 5 in 3 secs and [trolley] A only reached 3 in 3 secs and 5 is going the fastest.
I Why? [can the student generalise and talk about the variables and not only the particular values?]
S Because it only took 3 secs to go to 5.
I But [trolley] A took 3 secs to get to 3.
S ··well I don't know um um the secs are shorter to get to a lower number.
I Are you saying it went a bigger distance?
S Yea in fewer secs.
I Righto? But the seconds are the same.
S Yea but ·· but in 3 secs [trolley] A only stopped at 3.
I Yes so [trolley] A stopped at 3.
S And [trolley] B took 3 secs and it got to 5.
I So [trolley] B had greater speed?
S Yes.

This student could not make the jump to using variables and kept on using the values associated with the variables.

Student 908 Question 4 (CF)

[Trolley] B: [trolley] A started at 2 [trolley] B started at 0 and [trolley] B got the same distance in less time.

This student started using the values of the variables in his explanation but was able to finish by using the variable names correctly.

Student 915 Question 5 (CF)

[Trolley] A greater: Quicker time for distance

This student is fairly typical of those who used this type of strategy for Question 5. It would appear that the strategy these students are using is: Pull trolley B back one unit so that it starts at the same place as A. It then reaches 3 units in 3 secs. So A should get to 4 units in 4 secs but it went to 5 units so it must be going faster, or it should get to 5 units in 5 secs but it only took 4 secs, i.e., "it had a quicker [shorter] time for [its longer] distance". This strategy of referring to both distance and time was the most common strategy that was used in the responses.

4. **Time only** Students used only time in their explanation.

Student 913 Question 2 (IR)

[Trolley] B: Because its taking 3 sec for [trolley]A.

This student has focused on the trolley that has the shortest time but has ignored the fact that Trolley B has also gone a larger distance.

Student 914 Question 2 (CP)

Same [speed]: Train A has only been on for 3 seconds but B has been on for five.

In contrast to student 913 this student has only referred to time in his response but he seems to be aware that the larger distance does make a difference. The students who referred to 'time only' in Question 2 were not consistent in using this strategy across questions. They also used 'distance only' and 'distance and time' strategies.

5. **Distance only** Students used only distance in their explanation.

Student 904 Question 2 (IP)

[Trolley] B: cause [trolley] B got further than [trolley] A.

Students who used this strategy seem to be focusing on the actual pictures that are given and seeing which trolley has gone the greater distance. It is similar to the next strategy where the student looks for the trolley that is in front.

6. **One in Front** Students used the term "in front" to explain the answer.

Student 917 Question 5 (CP)

[Trolley] A: [trolley] A is one second in front.

This answer uses a similar reasoning to that presented by student 915 above. Students who identify the object in front as having the greater speed are using visual information without processing the other information that is given.

7. **Faster** Students used the term "fast" or "faster".

Student 901 Question 2 (IR)

[Trolley] A: Because it is faster than [trolley] B.

The term "faster" might mean 'speed' but it is more likely to refer to 'time being shorter'. Other students often use 'speed' to refer to the object with the shortest time.

DISCUSSION

Students who were able to give a full explanation used the strategies of ratio, calculation and distance-time comparisons. In the interviews these students were generally consistent with the approach they took in the written test. Some students who gave a partial explanation in the test could give a full explanation in the interview with a little prompting. Most other students were found to be inconsistent with their approach to questions, trying to close quickly on a solution without giving proper regard to all the variables.

The most popular strategies involved the use of both the variables, distance and time. However at least six students focused only on one variable, either distance or time. They saw this as the determining variable and they were not consistent with the use of this variable across problems. They may have seen that one variable was held constant and then focused on the variable that was changing. An implication of this is the importance for the teacher to draw students' attention to all variables and discuss which ones are constant and which ones vary and how they interact with each other.

Student performance decreased with the increase of the number of variables that differ in the question. For those students who were more ratio-orientated, the final question "forced" them to do a calculation which was identical to the formula used to find speed. For the other students, the more the variables altered the more they had difficulty in identifying the speed and explaining it in a clear way.

One other aspect of the students' responses was the observed confusion over the meaning of the terms 'speed', 'fast(er)', 'quick(er)', 'shorter', 'time'. Often these words were used synonymously. In interpreting their responses this confounding feature had to be taken into account.

CONCLUSION

It is interesting to note that most students attempted these problems in an intuitive way. There was not a standard learned method of attack and most did not use mathematics in an algorithmic sense. Perhaps the style of presentation of the questions contributed to this. The results showed that students in Year 9 are not limited to one method of attack. Even individual students exhibit a range of strategies across questions. There was even marked differences in the variables that students chose to focus on in an attempt to solve problems. Students who could respond by referring to the variables and how they were related gave a higher quality response than those who attempted to explain by using values only. Some students used a mixture of the two which suggests a transition phase from 'value to variable thinking'.

Finally, the style and content of the questions proved to be most valuable in revealing substantial variability in student strategies. Further research is currently underway to capitalise on this work. In particular, the focus is on identifying the triggers that elicit different strategies and, more importantly, what features encourage an intuitive rather than algorithmic response.

The authors acknowledge the help given by the teachers and students of Mabel Park State High School in Logan City for taking part in the tests and interviews.

REFERENCES

Piaget, J. (1970). *The Child's Conception of Movement and Speed.* New York: Basic Books. (Originally published in French by Presses Universitaires de France, Paris 1946)

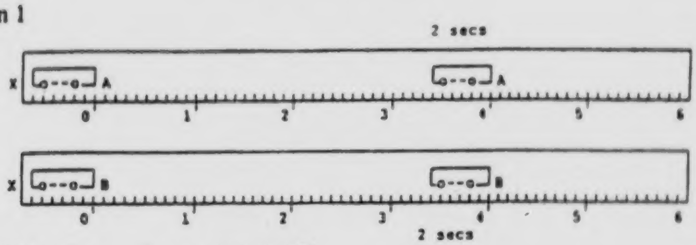
Trowbridge, D.E. (1979). *An investigation of understanding of kinematical concepts among introductory physics students.* Unpublished Ph.D Thesis, University of Washington.

Trowbridge, D.E., & McDermott, L.C. (1980). Investigation of student understanding of the concept of Velocity in one dimension. *American Journal of Physics,* **48,** p.1020-1028.

Appendix - Five Questions

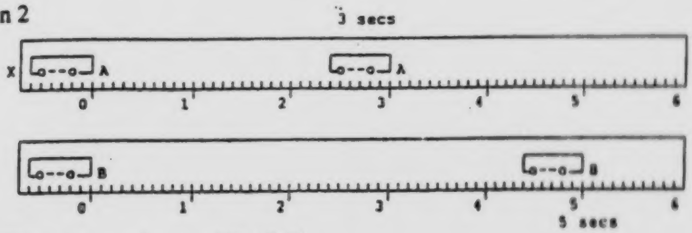
You must decide EITHER which cart is going the GREATER SPEED
OR if they are going the SAME SPEED.
Then explain your choice of answer.

Question 1


2 secs
X
A
A
0
1
2
3
4
5
6
X
B
B
0
1
2
3
4
5
6
2 secs

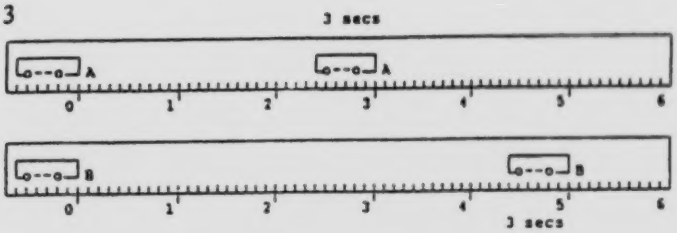
Answer: A B same speed

Question 2


3 secs
X
A
A
0
1
2
3
4
5
6
B
B
0
1
2
3
4
5
6
5 secs

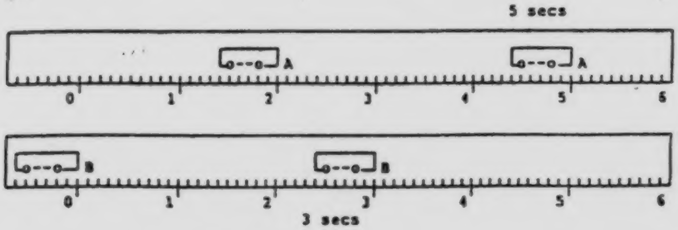
Answer: A B same speed

Question 3


3 secs
A
A
0
1
2
3
4
5
6
B
B
0
1
2
3
4
5
6
3 secs

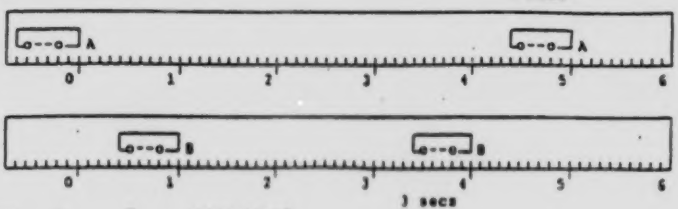
Answer: A B same speed

Question 4


5 secs
A
A
0
1
2
3
4
5
6
B
B
0
1
2
3
4
5
6
3 secs

Answer: A B same speed

Question 5


4 secs
A
A
0
1
2
3
4
5
6
B
B
0
1
2
3
4
5
6
3 secs

Answer: A B same speed

VISUAL IMAGERY AND METACOGNITION IN PROBLEM SOLVING

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This paper describes a research project which brings together current interests of the co-investigators. Anderson has been investigating the metacognitive strategies used by junior secondary students, to solve non-routine problems in mathematics. Dawe has been studying the use of visualisation by teacher trainees, in the comprehension and solution of higher order non-routine problems in mathematics. The project examines the introspections of teachers and students as they solve non-routine problems, and the implications for cooperative classroom learning. The increasing use of computer graphics, and graphics calculators in schools and tertiary institutions, has re-awakened interest in the specific role of visualisation in the problem solving process.We are particularly interested in the interaction of visual imagery with metacognitive strategies in non-routine problem solving. The data and its analysis will provide an indepth qualitative case study, of the role of verbal and non-verbal mental activity in comprehending and solving non-routine problems.

Despite the significant progress made through research and curriculum innovation in Australian schools, there are still many teachers who lack confidence in addressing the needs of their students with respect to problem solving ability and mathematical inquiry generally. They have been used to teacher centred approaches which rely heavily on textbooks. Students are still expected to sit and listen, follow demonstrated procedures, and practice routines and skills for reproduction in examination papers. The revolutionary changes taking place, not only with respect to the teaching of mathematics but in the wider context of school reform, has shaken the confidence of many teachers. Previously tried and trusted methods are threatened as the purposes of schooling and preparation for the world of work are redefined. The problem has been exacerbated by higher retention rates in schools, and demands for all students entering the workforce to be competent in clearly identifed areas.

Research studies over the last decade have pointed to metacognitive factors which have a profound influence on the confidence of both students and teachers to solve problems, including beliefs about mathematics as a discipline and how it should be taught and learned (Schoenfeld,1992). Australian researchers have made a very significant contribution to the field - see for example the review by Putt and Isaacs (1992). This study builds particularly on their work. It addresses the problem solving strategies of two groups of students at the University of Sydney, completing their training as teachers of mathematics in secondary schools. The study is intended to be exploratory, and serve as a pilot project for a larger and more rigorous study in 1994-95 which will involve teachers (stage2) and their pupils (stage3). The research has its origins within the broad framework of how teachers think, plan and conceive of their classroom practice. The notion of the reflective practitioner is a particular hallmark of significant educational research and teaching practice (Smith and Lovat, 1990). Such work has made an important contribution to the development of practical strategies to enable teachers to improve their work. In the light of the fact that the quality of teaching is a major educational issue today, critical reflection upon practice is an excellent mechanism through which quality can be enhanced.

In the field of mathematics education, there has been some research which has attempted to address the critical reflection of mathematics teachers on their work. Jaworski (1991) observed and interviewed six secondary teachers who espoused a constructivist philosophy, in an effort to discover the issues the teachers faced as they attempted to put this philosophy into practice. She concluded (p214) that " the teacher working from a constuctivist philosophy will be a reflective practitioner." Constructivists believe that learning is personal and that knowledge is personally constructed within the social context of the classroom (Cobb, 1986). This view of learning contrasts with the more traditional transmission mode of learning whereby the teacher possesses the knowledge which is presented to the students. Learning is assessed by testing to see if particular objectives have been met which means that the products of learning are valued rather than the process of how children learn (Rowland, 1984). If teachers are to become reflective practitioners, traditional views of teaching and learning will need to be challenged.

This project is designed to investigate the introspections of teachers as they solve non-routine problems themselves, and are encouraged to reflect on the implications for their classroom practice. Problem solving most closely resembles the way that mathematics is used in everyday life, and has been identifed as a key competency expected of all young people in preparation for work (Finn,1991). Further, the National Statement on Mathematics for Australian Schools (Australian Education Council,1990) sets goals for all Australian children which encompass problem solving strategies within the more general area of mathematical inquiry. This has become increasingly important for mathematics teachers as school retention rates continue to rise. There is now an urgent need to provide guidance for teachers in problem solving activities for the much wider ability range of the students in their care. Learning to think mathematically must become a focus of research and investigation if we are to meet the goals of these documents.

Within the very large body of problem solving research which has been carried out in the last decade, many studies clearly indicate that teachers need to be aware of their own thinking processes and the strategies they use in solving problems, before they can successfully help students to solve problems too (Schoenfeld, 1992). Problem solving requires the ability to control and organise thinking. This has been referred to as metacognition. The metacognitive strategies which students use when they solve problems can enhance their problem solving performance (Siemon, 1986). It is therefore of great interest to study the strategies students and their teachers use in solving problems. Hart (1991) for example organised problem solving workshops for middle school teachers aimed at improving problem solving abilities by focusing on metacognitive and constructive processes. Schoenfeld (1987) lists three related but distinct categories of intellectual behaviour which have been the focus of metacognitive research. We have specifically applied them to teachers for the purpose of this study.

1. A teachers knowledge of their own thought processes; how
2. How well a teacher controls or self-regulates their problem solving actions
3. The beliefs and intuitions of the teacher; that is, the ideas about mathematics which a teacher brings to their work and how such ideas shape the way the teacher does mathematics.

The introspections of teachers as they go about solving non-routine problems is an important goal of the study. By coming to reflect critically on their own thought processes, how they can be monitored and controlled, their beliefs about mathematics, and what it means to come to know mathematics, it will be a short step to critical reflection on their classroom practice. In particular the role of visual imagery in comprehending a problem and its subsequent solution will be of great interest.

THEORETICAL CONSIDERATIONS

There are two theories of cognition which particularly lend themselves to the design and the interpretation of the results of the research to be carried out.

Schema theory

Research has found that students who have schemata for meaningful problem types, evidenced by the ability to classify problems on the basis of their semantic structures, are better problem solvers than students who do not have knowledge of problem types (Goldin,1992). Schemata are theorised to be abstract knowledge structures for representing generic concepts stored in memory, although there is no agreement in the literature as to how a schema should be defined. In the field of mathematics education Sweller (1989), for example, defines a schema as a cognitive construct that permits problem solvers to classify problems into particular categories which require particular moves for solution. As a result of many studies which have been driven by this theoretical perspective, there is a widespead view that more explicit instruction in representation skills for comprehending the semantic structure of word problems would be beneficial. Lewis (1989), for example, found that training in the use of diagrams to represent the semantic structure in a word problem improved problem solving performance. Further, the training resulted in transfer of new representation skills to related, but more complex, problem solving tasks.

Schema theory has an honoured place in research in mathematics education. Richard Skemp, for example, has developed his schematic theory of mathematics learning over the last 30 years (for an up to date summary of his work see Skemp, 1992). Many other writers in the field of the psychology of mathematics, have made excellent theoretical contributions to our understanding of mental representations of mathematical ideas, and how mathematical knowledge systems are developed. Goldin (1992) has recently attempted to pull the diverse theoretical contributions together into a unified whole. Mental imagery plays a very important role in these theories, and the literature abounds with studies of "image schemata" (including visualisation) of mathematical concepts. Further, in contrast to reading research, schematic approaches to mathematics learning have often incorporated affective representations into their models.

In discussing the impact of schema theory on research in mathematical problem solving, one can hardly pass the work of Sweller and his colleagues. Putt and Isaacs (1992) summarise the contribution these researchers have made to our understanding of the problem solving process and the implications for classroom practice. According to Sweller (1989) the two keys to skilled problem solving performance are schema acquisition and rule automation, the latter being a rule that can be used without conscious processing. This has sharpened the constructivist debate because, as Putt and Isaacs (1992, p223) point out

> The work of Sweller and his colleagues suggests that, at the secondary level, teachers should be focusing on domain specific strategies which will enhance the development of multiple schemata to meet differing problematic situations of school mathematics. Such an approach to problem solving will be much easier to pass on to teachers of mathematics operating in the conventional role of imparters of knowledge and skills, as it requires only small changes to their current practice. The Polya model on the other hand requires radical changes in the belief systems of both teachers and students, as well as substantial role changes on the part of the teacher.

Dual Coding Theory

Like schema theory, dual coding theory is not specifically a theory of how mathematics is learned, but a theory of cognition. The following summary has been taken mainly from Sadoski, Paivio, and Goetz (1991), but a full description of the evolution of the theory and its empirical record, are extensively documented in Paivio (1986,1991). Dual coding theory holds that cognition consists of two separate mental sub-systems, one specialised for the representation and processing of information concerning nonverbal objects and events, and the other specialised for dealing with language. The former is often referred to as the imagery system because it functions include the generation of mental images (visual, auditory, haptic and affective). The language specialised subsystem is referred to as the verbal system. These systems are separate but interconnected, so that they can function independently, in parallel, or in an integrated manner.

The two systems have different organizational and processing characteristics. Information in the verbal system is organised in a way that favours sequential, syntactic processing. Nonverbal information, especially in the visual mode, is organised more in the form of holistic nested sets, with information available for processing in a synchronous or parallel manner. Interconnections between the systems allow for great variety in cognitive activity. Language can evoke imagery, and imagery can evoke language. Such cross system activation is called referential processing. However the connections are not one to one but one to many. Hence, language could referentially evoke numerous images or none at all. Mental images could evoke much language or none at all.

Within system processing is said to be associative. This refers to spreading activation among verbal representations or among images. Thus words or phrases are associated, through experience or learning, with other words or phrases. Likewise, mental images could evoke other (associated) images. A mental image of a tyre for example, might evoke an image of a car and even a driving experience. Paivio (1991:379) points out that individuals differ in the degree to which they have developed representations corresponding to particular words and objects, interconnections between them, and processing skills involved in making use of the available structures and pathways.

Processing in dual coding theory includes external and internal variables. External contexts, situational constraints, instructions and so on, interact with a person's two existing symbol systems, as determined by prior experience and individual differences. Processing may be conscious or unconscious, and often involves the transformation and recoding of representations. However the theory does not include any separate abstracted structure such as a schema or processing under the guidance of a schema. This, it is claimed, enables the theory to better account for the great detail found in the memory for events, as well as the flexible construction and reconstruction of events in memory.

The application of dual coding theory to problem solving

According to dual coding theory, problem solving performance is mediated by the joint activity of the individual's verbal and imagery systems. The relative contribution of each system will depend on the characteristics of the task and the cognitive abilities and habits of the problem solver. The theory predicts that the more concrete and nonverbal the task, the greater the contribution of the imagery system. The more abstract and verbal the task, the greater the contribution of the verbal system. The demands of the task and individual differences variables will determine the degree to which the various dual coding processes (described above) are brought into play. The imagery system contributes richness of content and flexibility in the processing of that content, so that diverse bits of information can be quickly retrieved, compared, evaluated, transformed and so on. Paivio (1986:201) suggests that these attributes may underlie the intuitive leaps of imagination that often characterise creative thinking. He argues that language and the verbal system on the other hand, provide precise means (conceptual pegs) for retrieving images from memory and guiding the processing of retrieved information. Its capacity for sequential organisation in particular suggests that the verbal system contributes logical order to ideation that would not be possible on the basis of imagery alone.

> That logical potential implies, too, that verbal processes may predominate in the later stages of the task sequences, eg, the so called verification stage of discovery, whereas imagery may predominate at earlier stages, at least in some tasks. Paivio (1986: 202)

Kaufmann's (1980) theory of problem solving has much in common with the dual coding theory approach, although he disagrees with some specific points. He emphasises similar functional distinctions between imagery and linguistic representations. He agrees that visual imagery is crucial during the discovery phase of problem solving. In his general theory, the fuctional usefulness of the two systems is related to the novelty of the task. Thus, he sees linguistic representations as more appropriate when the task is familiar, whereas imagery becomes increasingly appropriate and adaptive as the novelty of the task increases.

In the particular case of informing research into the role of visual imagery in problem solving in mathematics, there is much to be learnt from the above analysis. Individuals differ in their preferred strategies and abilities, so that some rely on imagery to solve certain types of problems, while others use a linguistic strategy. This is in accord with dual processing theory. However, there is good evidence to suggest that, for most people, visualisation may play an important role in comprehending a problem, even if they prefer a linguistic strategy in arriving at the solution. Kaufmann's novelty hypothesis would be interesting to test in the context of a wide range of mathematical tasks.

AIMS OF THE PROJECT

1. To investigate the metacognitive strategies used by two groups of student teachers differing in training and mathematical ability, in solving non-routine mathematical problems
   (i) Diploma of Education students
   (ii) Bachelor of Education students in their final year
2. To study the effect of individual learning style on the metacognitive strategies used.
3. To investigate the specific use of visual imagery in comprehension and analysis in the problem solving

process.
4. To see if there is any evidence to suggest that the students' beliefs about mathematics, and in particular beliefs about problem solving, effect problem solving performance.
5. To see if there is any evidence to suggest that emotional factors effect the metacognitive strategies employed.
6. To generate hypotheses for further research.

THE RESEARCH DESIGN

Methodology

The purpose of the research is to study the thinking processes of two small groups of final year mathematics students as they go about solving non-routine problems. The data and its analysis will then provide an indepth qualitative case study, of the role of verbal and non-verbal mental activity in comprehending and solving such problems. It is intended to be a preliminary pilot study, to inform the planning and design of a study of teachers (and finally school students) as they solve non-routine problems.

The study lends itself to an ethnographic research design, and generally follows the methodology adopted by the co-investigator in her successful study of the metacognitive strategies adopted by her year 8 students in solving non-routine problems (Anderson, 1993). In particular, the protocol developed by Schoenfeld (1985) for use with undergraduate students, will be used to address aims 1 and 3 of the study. For Aim 2 we will use the learning style inventory developed by Kirby et al (1988) and the instrument used by Brumby (1982) - see sample below- while Aim 4 requires a special instrument adapted from Anderson (1993). Aims1-5 will be addressed from observation, interviews and student reporting. The problem solving sessions will be videotaped, and students will be asked to participate in the analysis with the investigators by reviewing the recordings.

It is not possible to fully justify the methods we have decided to use here. After a critical review of the methods other researchers have used, and in particular the results of Anderson's work, we have settled on a combination of methods to gather the data in an effort to overcome weaknesses and to strengthen the findings using triangulation. These include non participant observation of the students (working alone, in pairs and in a group of four), written student recordings, audio and video recordings, student responses from video replays, student interviews, and researcher's field notes. It is hoped that a wide variety of methods will inform the methodological decisions for the next stage of the research with teachers. We have also decided to try out a considerable range of problems which include the analysis of mathematical images generated by computer.

The sample

We plan to include two contrasting groups of students at the University of Sydney in the pilot project: the 1993 intake of Dip Ed(sec) mathematics students, and the final year students in the integrated BEd (sec) mathematics program. The two groups differ markedly on a wide range of variables including mathematical ability, work experience, knowledge of how children learn mathematics and practicum experience. However they are roughly the same age. Having reached the end point of their training they will be appointed as four year trained teachers to schools in 1994. It is likely that the very different training the two groups have received, together with differences in academic ability, will produce some interesting contrasts in the data. It is expected that there will be implications for the inservice needs of particular teachers in the workforce, as the competencies identified by the Finn committee are implemented in schools.

The Dip Ed group comprises 28 students, 16 women and 12 men. The undergraduate group completing the integrated degree comprises 33 students, 18 women and 15 men. It is intended to administer the Visual and Verbal Learning Style Inventory (Kirby, Moore and Schofield,1988) to each group to identify those who prefer visual to verbal modes of processing information. This instrument was designed for adult students and is particularly appropriate for the theoretical basis of Dual Coding theory. Although this will provide helpful initial information, it is inadequate for the selection of the sample. We will also administer the instrument developed by Brumby(1982), who investigated differences in cognitive style for qualitative problem solving in biology. She identified two distinct learning styles which have been described as analytic or serialist, and global, holistic or intuitive. The essential characteristic of the analytic style appears to be that a given problem is immediately broken up into its component parts. These are studied step by step as discrete entities in isolation from each other and their surroundings. The second, holistic, style seeks an overall view of a problem. The various parts are integrated and related and seen in the context of their surroundings. Overall 42% of Brumby's sample maintained an analytic style, 8% used only the holistic style and 50% used a combination of both. Brumby described the latter group as "versatile learners".

We will then select a sample of 4 "visual" and 4 "verbal" and 4 "versatile" students from each group to take part in the study. The distinction between the visual/ verbal styles is seen in terms of the type of processing described in the dual processing model, and by such descriptors as simultaneous versus successive, or holistic versus analytic. The versatile group are more likely to use both, to switch from visual to verbal mode depending on the nature of the problem. The sample does not need to be random and we will choose a balance of men and women, taking into account academic background, to participate in the study.

DISCUSSION

With respect to problem solving, strategies which encourage the development of holistic thought processes, linking them to sequential deductive thinking, are likely to result in increased mathematical performance. Versatile learners able to think in holistic and analytic modes, and use them in harmony, are more likely to be successful, particularly at higher levels of mathematics. Whether visual imagery per se plays an additional role in the actual method of solution seems to be a personal matter. We will be interested in getting further data on this interesting question. What does seem to be important is the role of visualisation in comprehension. In parallel to the comprehension of narrative text, reading a mathematical task with understanding may require an automatic imaging, in which parts of the problem are visualised and brought together to form a whole (gestalt) of information (Dawe, 1993).

Previous research has also shown that students need to be aware of and in control of their thinking, ready to switch to a new line of thought when it becomes clear that the present one is going nowhere. This requires sitting back, asking themselves how they are going, and getting a holistic view. This "internal talk" is characteristic of self monitoring and self regulation. The ability to switch one's viewpoint from a local analytical one to a holistic one, in order to be able to place the details as part of a structured whole, is crucial. Such metacognitive abilty is characteristic of good problem solvers.

What then should teachers be doing about these matters in the classroom? How do teachers enable students to develop their powers of mental imagery? How do teachers identify, and monitor, the metacognitive strategies their students use in problem solving? What about the role of affective variables? If our goal is to encourage the development of all these internal systems of representation, should we pay equal attention to each? These and other issues will be discussed in the presentation.

REFERENCES

Anderson, J. (1993) An investigation of the metacognitive strategies which students use while solving non-routine problems , unpublished MEd (hons) essay, University of Sydney.

Australian Education Council (1990) A National Statement on Mathematics or Australian Schools the Curriculum Corporation, Victoria

Brumby, M. N. (1982) Consistent differences in cognitive styles shown for qalitative biological problem solving. British Journal of Educational Psychology 52, 244-257

Cobb, P. (1986) Contexts, goals, beliefs and learning mathematics. For the Learning of Mathematics, 6(2), 2-9.

Dawe (1993) Visual imagery and communication in the mathematics classroom, in Communicating Mathematics: Perspectives from Research and Classroom Practice , W. Stephens, D.J. Clarke, J. Izzard and A. Waywood (eds), ACER, Melbourne (in press)

Finn, T. B. (Chair,1991) Young People's Participation in Post - Compulsory Education and Training Report of the Australian Education Council Review Committee, Canberra; AGPS

Goldin, G.A. (1992) On developing a unified model for the psychology of mathematical learning and problem solving Proceedings of the Sixteenth Conference of the International group for the Psychology of Mathematics Education vol3, 235-261, University of New Hampshire.

Hart, L.C. (1991) Monitoring change in metacognition. In Furinghetti, F. (ed) Proceedings of the 15th Psychology of Mathematics Education Conference, Italy, Vol 11, 141-148.

Jaworski, B. (1991) Some implications of a consructivist philosophy for the teacher of mathematics. In Furinghetti, F. (ed) Proceedings of the 15th Psychology of Mathematics Education Conference, Italy, Vol 11, 213-220.

Kaufmann, F. (1980) Imagery, language and cognition NewYork, Columbia University press

Kirby, J.R., Moore, P.J. and Schofield N.J. (1988) Verbal and Visual Learning Styles Contemporary Educational Psychology 13, 169-184

Lewis, A.B. (1989) Training students to represent arithmetic word problems Journal of Educational Psychology 81, 4, 521-531

Paivio, A. (1986) Mental Representations: A Dual Coding Approach New York, Oxford University Press

Paivio, A. (1991) Images in the Mind: the evolution of a theory New York, Harvester Wheatsheaf

Putt, I.J. and Isaacs, I. (1992) Research in Mathematical Problem Solving B. Atweh and J. Watson (eds) MERGA, 208-230

Rowland, S. (1984) The Enquiring Classroom. London : Falmer Press.

Sadoski, M., Paivio, A. and Goetz, E. (1991) A critique of schema theory in reading and a dual coding alternative Reading Research Quarterly XXVI,4, 463-484

Schoenfeld, A.H. (1985) Mathematical Problem Solving Orlando: Academic press

Schoenfeld, A.H. (1987) "What's all the fuss about metacognition?" in Cognitive Science and Mathematics Education Hillsdale, N.J., Lawrence Erlbaum Associates, 189-215

Schoenfeld, A.H. (1992) "Learning to think mathematically: Problem solving, metacognition and sense making in mathematics". In Grouws, D.A. (Ed) Handbook of Research on Mathematics Teaching and Learning New York, Macmillan, 334-370

Siemon, D. (1986) "Problem Solving- a challenge for change or a passing fad?" in Swinson, K. (Ed) Mathematics Teaching: Challenge for Change Brisbane, Australian Association of Mathematics Teachers, 31-37

Skemp, R.R. (1992) Bringing theory into the classroom: the school as a learning environment for teachers in B. Southwell, B. Perry and K. Owens (eds) Space- the First and Final Frontier Proceedings of the Fifteenth Annual Conference of the Mathematics Education Research Group of Australasia, University of Western Sydney.

Smith, D. and Lovat, T. J. (1990) Curriculum: Action on Reflection 2nd edition, Social Science Press

Sweller, J. (1989) Cognitive technology: some procedures for facilitating learning and problem solving in mathematics and science. Journal of Experimental Psychology, 81,4, 457-466

Useful references which address the role of technology in developing visualisation in advanced mathematics

Boers, M. and Jones, P. (1992) The graphics calculator in tertiary mathematics in B. Southwell, B. Perry and K. Owens (eds) Space- the First and Final Frontier Proceedings of the Fifteenth Annual Conference of the Mathematics Education Research Group of Australasia, University of Western Sydney.

Ruthven, K. (1990) The influence of graphic calculator use on translation from graphic to symbolic form. Educational Studies in Mathematics 21, 5, 431-450

Tall, D. and Thomas, M. (1991) Encouraging versatile thinking in algebra using the computer. Educational Studies in Mathematics 22, 2 (April) 125-147

West, T.G. (1991) In the Mind's Eye: visual thinkers, gifted people with learning difficulties, computer images and the ironies of creativity, New York, Prometheus Books

Zimmermann, W. and S. Cunningham (1991) eds, Visualisation in teaching and Learning Mathematics The Mathematical Association of America

WHAT DO CHILDREN BELIEVE ABOUT CALCULATORS?

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This study looks at two groups of children – those for whom the calculator is a part of their everyday mathematics at school and those for whom it is not. The research methodology adopted was that used in the innovative Victorian science study (1990). Children at grade three level were asked to complete statements about calculators, sometimes by writing and sometimes by drawing. The analysis categorized the children's responses into mutually exclusive types, which were then assigned integer 'level' labels. These categories were then tested for cohesion using an IRT partial credit analysis. The last stage of the analysis was to construct descriptors for each of the categories, thus establishing a developmental continuum for beliefs about calculators. While the number of children was only a few hundred, it is clear that further investigations of children's beliefs in this area would contribute significant information for those implementing a 'calculator aware' mathematics curriculum in their school.

The calculator is seen by educators as an instructional aid as well as a computational tool (AAMT, 1988). But what do the users, the children, think? Research into so-called 'misconceptions' would indicate that children do indeed hold views about the content and processes they encounter in their schooling (see for example, Confrey (1990) for a review of this research). There appears to be no research however, investigating children's ideas or opinions regarding either the use of calculators or exactly how a calculator functions.

Surely most, if not all, children are exposed to calculators in different situations in their classroom mathematics, or in the 'real' world? Such exposure will be the foundation of children's beliefs about what calculators are and what they are for. The present study is a preliminary attempt to collect such data and perform such analyses as will help in determining the nature of children's beliefs about calculators. As Ausubel said '[T]he most important single factor influencing learning is what the learner already knows' (Ausubel, 1968: vi).

The evidence of research into learner beliefs (prior to teaching in science) shows that they are indeed critical to the outcomes of instruction and have been well documented (Adams, Doig and Rosier, 1991). In mathematics learning however the role of affective variables has not received the same attention. In his review of research into affective variables M^{c}Leod (1992) categorizes this research into the following categories: beliefs about mathematics; beliefs about self; beliefs about mathematics teaching and beliefs about the social context. However none of the reported studies focus on learners' beliefs about calculators or their role in mathematics learning.

RESEARCH QUESTIONS

The investigation reported here forms a small part of a larger study involving number and calculators which is an adjunct to CAN-like projects being conducted by Deakin University (Groves, Cheeseman, Allan and Williams, 1992). While this project has sought to answer many questions, the specific foci selected for this report are a description of children's beliefs about calculators. In order to achieve this, the following questions were posed:

1 **Question 1 What can a calculator do?** The purpose of this question is to explore the range of uses that children believe exist for a calculator. While most adults would think that a calculator is for computation, children's experiences of mathematics at this age level are still dominated by counting, basic addition and subtraction, fractions and to some degree exploration of the number system.

2 **Question 2 Does a calculator always give the correct answer.** The purpose of this question is to explore the degree to which children trust their calculators. While adults are ready to believe in the machine's infallibility children, who are apt to press the wrong key more often, may be less willing to put their trust in one.

3 **Question 3 How can you check your work?** The purpose of this question is to explore the range of strategies that children use for checking their work. Do they rely upon another 'go' with the calculator, or do they have alternative methods? Many critics of children's use of calculators argue that the children become dependent upon them, but is this so?

4 **Question 4 Can a calculator teach you?** The purpose of this question is to explore whether children believe that they can learn from a calculator. While it is true that there are guides available to assist teachers to teach *with* the calculator (Open University, 1982), whether children think that they learn *from* a calculator is unclear. Perhaps the children in this study can shed some light on the matter.

5 **Question 5 The inside workings of a calculator.** The purpose of this question is to explore what children believe happens inside a calculator when a button is pressed. Are there little people inside? Or is it simply a collection of wires and batteries? Children's drawings are always revealing of their inner ideas, and this question, while not implying that children should be taught binary arithmetic or electronics, should enable us to gauge to what extent there may be a 'black box' view of calculators emerging.

METHODOLOGY AND INSTRUMENTATION

To gather information on children's beliefs at year three (approximately nine years of age) would usually involve one-to-one interviews. However to gather sufficient data to be able to make justifiable inferences makes interviewing not feasible. Fortunately there has been developed recently techniques for gathering and analyzing such data using interview-like written formats and modern statistical tools. To date these have been used in science and social science only but there was no reason to doubt that the technique would apply equally well to mathematics. For a full description of these formats and their application to science see Adams, Doig and Rosier (1991) and for partially similar methods in mathematics Streefland and van den Heuvel-Panhuizen (1992) and Tirosh and Stavy (1992). The particular format selected for this investigation was that of a short story entitled 'What happened last night'. In this story an alien visitor asks questions of the child (reader) who responds by completing 'gaps' left in the text. In all administrations the entire story was read to the children by the author then the children read and completed the story in their own time, approximately thirty minutes.

SUBJECTS

The subjects were from four Melbourne (Victoria) schools. Two schools were where the children had had complete access to calculators and two non-calculator schools matched on socio-economic variables. The number of subjects in each school is presented in Table 1.

Table 1: Number of year three subjects in calculator and non-calculator schools.

Calculator schools	Non-calculator schools
Total subjects = 105	Total subjects = 94
Total subjects = 199	

ANALYSIS

As outlined earlier, the methodology used was that of written responses to leading questions. This meant that two hundred scripts each of five responses had to be prepared for analysis. The procedure used was that pioneered by Adams and Doig (Adams et al, 1992) in their study of science beliefs. First all responses to a particular question are read to give an overall 'feel' for the range of responses. Theoretically each response is unique, but in practice responses tend to 'group' themselves in a qualitative sense. Thus after the initial reading, it is possible to describe tentative qualitative categories. All responses are then placed into one of these mutually exclusive categories. If necessary this process is repeated until all responses can be accommodated. Each category is now given an integer 'level' label, which describes its ranking from being the most to the least sophisticated response. The level labels cannot be equated across questions, and in some cases two qualitative responses have been assigned to the same level. The analysis of the labelled data was via the Quest® interactive analysis program (Adams and Khoo, 1992). The analysis was of two forms, a simple frequencies analysis and a Rasch partial credit analysis (Wright and Masters, 1982). The application of this model enables the construction of a developmental continuum for the set of questions as a group. This is entitled the 'Beliefs about calculators' continuum.

RESULTS

Below are the response percentages by category for the children, in both calculator and non-calculator schools. For each question the highest value label indicates the response considered to be the most sophisticated.

Table 2: **Question 1** **What can a calculator do?**

LABEL	%	DESCRIPTION
3	34.7	+/–/∗/÷
2	40.2	sums/get answers
1	22.1	maths/make life easier
0	3.0	uninterpretable

The purpose of this question was to explore the range of uses that children have for a calculator. By year three most of the numerical experiences of Victorian children have been to do with counting, with some work on addition and subtraction, mainly non-algorithmic, and fractions. It is no surprise then that the overwhelming majority of children believe that either 'sums' or calculating (addition was the operation most frequently mentioned) are the major uses for calculators.

TABLE 3: **Question 2** **Does a calculator always give the correct answer**

LABEL	%	DESCRIPTION
3	62.8	no
2	18.1	sometimes
1	10.1	yes
0	9.0	uninterpretable

The purpose of this question was to explore the degree to which children trust their calculator to give the correct answer. While the tendency of adults is to consider the 'yes' option as being more correct, those students who said 'sometimes' indicated that there was the possibility of human error. The high percentage of children responding 'no' would appear to indicate that incorrect keying is a problem with young children, and also vindicates the 'sometimes' responses.

TABLE 4: **Question 3** **How can you check your work?**

LABEL	%	DESCRIPTION
2	39.2	mental calculation/paper&pencil /concrete materials
1	54.3	do it again/ask someone
0	6.5	uninterpretable

The purpose of this question was to explore the range of strategies that children use for checking their work. The most surprising responses were those which simply suggested asking someone (usually mum) rather than any attempt to try again. Those suggesting a further attempt (do it again) appear to believe that a keying error had occurred. On the other hand, a large group suggested that they had alternative strategies at their disposal.

TABLE 5: Question 4 Can a calculator teach you?

LABEL	%	DESCRIPTION
3	11.1	no
2	12.1	sometimes
1	50.8	yes
0	26.1	uninterpretable

The purpose of this question is to explore whether children believe that they can learn from a calculator. While one might be tempted to say 'yes, of course' a number of children who said 'no' did so with the explanation that the calculator gave the answer, but the person had to know 'which button to push'. The 'yes' group usually suggested that one needed only to memorize the calculator's answers to questions to learn.

TABLE 6: Question 5 The inside workings of a calculator.

LABEL	%	DESCRIPTION
4	3.5	circuit complete (with 'brain')
3	6.5	button/wires/ screen/batteries
2	50.8	circuit (generalised)
1	3.0	animism/ mechanical
0	36.2	uninterpretable

The purpose of this question was to explore what children believe happens inside a calculator. Children's ideas about what happens encompasses a wide range. Mechanical or animistic notions are rare, but not unexpected for children of this age. Most children knew, or felt that, there was some sort of circuit, although these were not expressed in standard form by any means. The inclusion of a 'brain' in a response was taken as indicating the highest level of response, and the rarity of this response indicates that this was a very difficult question.

CONCLUSIONS

The questions and their responses reported here, while only a subset of a larger study, do indicate that there are reasons to believe that there is a widespread set of beliefs being developed by children. The hidden-curriculum involved when children use tools such as the calculator must in some way interact with the taught curriculum and the effects of such interactions can only be interpreted if the underlying ideas of the children in question are known and taken into account. The large number of responses suggesting that calculators are only for 'sums' is certainly an example of the curriculum influencing opinion, while the implied high incidence of keying errors appears to be working against children's routine use of calculators. On a more positive note is the finding that so many children have developed a range of alternative computational methods, despite the availability of calculators, a finding generally not predicted.

In regard to teaching or learning with, or from, a calculator, it is a sad fact that these children equate teaching with 'telling the answer' and learning with 'memorizing the answer'. No response indicated that a calculator could be used for exploring numbers, patterns or operations, despite teachers, in the calculator project schools especially, using calculators for such activities. The only explanation that springs to mind is that by grade three the curriculum has turned from numbers *per se* to operations with them and exploration has been 'left behind'.

Is the calculator a 'black box'? Nearly every child who gave an interpretable response indicated that there was some form of circuitry inside the case. Most gave wires, batteries, buttons, and display panels as part of their circuits (all of which are reasonably visible) but a few indicated that there was a 'brain' inside to do the work. The exact nature of the workings was not specified; should we be concerned? Is there a place for understanding the tools we use as well as how to use them?

From a research perspective, the high number of responses that were uninterpretable indicates that better questions need to be formulated and follow-up interviews conducted. While some findings are interesting in themselves, others do have curriculum implications and need to be further investigated. This is particularly true of fractions, which were not mentioned in any response at all. Decimal fractions are an area ready to be explored with calculators, but there was no evidence of children's acquaintance with these at all, which, considering the wide use of this form of fraction in the real world seems inexplicable.

(The second form of analysis, providing a 'Beliefs about calculators' continuum, is not included here due to space restrictions, but will be presented at MERGA itself).

REFERENCES

Adams, R., Doig, B. and Rosier, M. (1991). *Science learning in Victorian schools 1990.* Hawthorn: The Australian Council for Educational Research.

Adams, R. J. and Khoo, S-T. (1992). *Quest.* Hawthorn: The Australian Council for Educational Research.

Australian Association of Mathematics Teachers (1988). *A National Statement on the use of Calculators for Mathematics in Australian Schools.* Canberra: Curriculum Development Centre.

Ausubel, D. (1968). *Educational psychology: a cognitive view.* New York: Holt, Rinehart and Winston.

Confrey, J. (1990). A review of the research on student conceptions in mathematics, science and programming. In Cazden, C. (ed.). *Review of research in education, volume 16.* Washington: AERA. p 3-56.

Groves, S., Cheeseman, J., Allan, A. and Williams, M. (1992). Calculators in primary mathematics – the calculator projects reach grade 3. In Horne, M. and Supple, M. (eds) (1992). *Mathematics: meeting the challenge.* Melbourne: The Mathematical Association of Victoria. p 379-385.

M^c^Leod, D. B. (1992). Research on affect in mathematics education: a reconceptualization. In Grouws, D. A. (ed) (1992). *Handbook of research on mathematics teaching and learning.* New York: Macmillan Publishing. p 575-596.

Open University. (1982). *Calculators in the primary school.* Milton Keynes: Open University Press.

Streefland, L. and van den Heuvel-Panhuizen, M. (1992). Evoking pupils' informal knowledge on percents. In Geeslin, W. and Graham, K. (eds) (1992). *Proceedings of the sixteenth PME conference, volume iii.* Durham: IGPME. p 51-57.

Tirosh, D. and Stavy, R. (1992). Overgeneralization between science and mathematics: the case of successive division problems. In Geeslin, W. and Graham, K. (eds) (1992). *Proceedings of the sixteenth PME conference, volume iii.* Durham: IGPME. p 7-81.

Wright, B. D. and Masters, G. N. (1982). *Rating scale analysis.* Chicago: MESA Press.

CHILDREN'S STRATEGIES AND REASONING PROCESSES IN SOLVING NOVEL COMBINATORIAL AND DEDUCTIVE PROBLEMS

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This paper reports on a current study investigating 9 to 12 year-olds' strategies and reasoning processes in solving novel combinatorial and deductive problems. Equal numbers of children were selected from low, average, and high achievers in school mathematics and were individually administered five sets of problems, namely, two sets of combinatorial problems (2-dimensional and 3-dimensional) and three sets of deductive reasoning problems. Both problem types were presented in "hands-on" and written formats. The written problems were isomorphic to the hands-on examples. The nature of children's strategies and reasoning processes in solving these problems is addressed. Of particular interest are the differences in the responses of children classified as low and high achievers in school mathematics.

Research on children's mathematical problem solving has focussed largely on their ability to solve routine problems which call for a standard method of solution and which offer little opportunity for observing the processes of learning (e.g., Baranes, Perry, & Stigler, 1989; Riley & Greeno, 1988). While the mathematics education community has espoused the importance of problem solving during the past two decades (e.g., National Council of Teachers of Mathematics, 1989), comparatively little research has addressed novel mathematical problem solving where children do not have an efficient solution procedure and must develop their own strategies for goal attainment. Investigations into children's reasoning processes in solving novel problems can provide valuable data on their potential for mathematical learning -- information which might otherwise go undetected in routine mathematical activities.

Current theories on children's learning and development strongly support the call for novel problem solving in the mathematics curriculum. Several studies have portrayed children as self-directed learners whose problem-solving efforts are likened to those of a scientist, creating theories-in-action which they challenge, modify, and extend on their own (Brown & Palinscar, 1989; Burton, 1992; Carey, 1985; Gelman & Brown, 1986; Karmiloff-Smith, 1984). Studies of very young learners show that they possess some powerful problem-solving skills. For example, they are able to direct their attention to tasks that interest them, they independently monitor their progress towards goal attainment and modify their actions accordingly, and they persist at a problem while moving through increasingly sophisticated levels of solution procedures (Andreassen, Kelly, & Waters, 1991; Brown & Reeve, 1987; Deloache, Sugarman, & Brown, 1985; Gelman & Greeno, 1989; Klahr, 1985). If young children display these skills, it could be assumed that older children will demonstrate similar abilities in novel situations provided they are motivated to do so.

Previous studies (e.g., English,1988, 1991a, 1993) had shown that primary school children can generate their own strategies for solving novel, two-dimensional ($X \times Y$) and three-dimensional ($X \times Y \times Z$) combinatorial problems set within a meaningful context (dressing toy bears in different combinations of clothing items and tennis rackets). The present study further explored this ability by extending the hands-on problems to include written isomorphic examples. Of interest here was whether children could recognise the similarity in problem structures and could apply analogical reasoning (Halford, 1992) in solving the isomorphic examples.

Novel problems involving deductive reasoning were also included in this study, given that previous work (e.g. English, in press) had highlighted children's ability to reason deductively in solving logical and illogical syllogisms. Furthermore, since the combinatorial and the deductive domains both represent novel situations for children and thus demand considerable use of general reasoning processes, it was considered worthwhile to compare children's performance across the two areas.

In addition to their novelty feature, combinatorics and deduction were chosen for their mathematical and developmental significance. The combinatorial domain, involving the selection and arrangement of objects in a finite set, comprises a rich structure of significant mathematical principles which underlie several areas of the mathematics curriculum, including counting, computation, and probability. The domain has also featured prominently in theories of cognitive development. The establishment of a combinatorial system plays a significant role in Piaget's theory of cognitive growth, heralding the onset of formal thought (e.g., Inhelder & Piaget, 1958; Piaget, 1957; Piaget & Inhelder, 1975).

Deductive reasoning problems were included because of their importance in the development of mathematical thinking. There is the assumption that because mathematics is a logically developed discipline, the logic will be absorbed by the students as they pursue a study of the formal content areas (Burton, 1984). However this is rarely the case. A facility to think mathematically is not a natural consequence of acquiring mathematical content. This facility relies on the development of a number of mathematical processes or operations which are applied to a range of content areas (NCTM, 1989). The ability to detect relationships, make hypotheses and generalisations, and test conjectures are major components of mathematical thinking and problem solving.

METHOD

Subjects

Three hundred and thirty-six children in grades 4 through 7 have been involved in the study. Four age groups are represented here (84 children per group): 9 yrs 0 mths to 9 yrs 6 mths, 10 yrs 0 mths to 10 yrs 6 mths, 11 yrs 0 mths to 11 yrs 6 mths, 12 yrs 0 mths to 12 yrs 6 mths. The children were selected from four state schools and three non-state schools in low to middle socio-economic suburbs of Brisbane. Equal numbers of children were selected from the low, average, and high achievement levels in mathematics (as defined by the class teacher).

Instruments

Five sets of problems (three problems per set) were developed for the study, as follows:
Set 1: one 2-D (X x Y) and two 3-D (X x Y x Z) combinatorial problems (hands-on)
Set 2: three written combinatorial problems, isomorphic to the first set
Set 3: three hands-on deductive reasoning problems (non-numerical)
Set 4: three written deductive reasoning problems, isomorphic to Set 3
(non-numerical)
Set 5: three written, numerical deductive reasoning problems.

Set 1 required children to dress toy bears in all possible combinations of coloured tops and pants (2-D examples) or coloured tops, pants, and tennis rackets (3-D examples). The bears were made of thin wood and were placed on a stand so that the children could clearly see their completed combinations. The number of possible combinations ranged from 6 to 12. The nature of the written combinatorial examples (Set 2) is illustrated in Figure 1.

The Select-A-Card company plans to make new boxes of greeting cards.
In each box there will be greeting cards that are:
either GREEN or YELLOW, and have
either CHRISTMAS greetings or BIRTHDAY greetings or EASTER
greetings, and have either GOLD LETTERING or SILVER LETTERING.
How many different greeting cards will there be in each box?

Figure 1. Example of a written three-dimensional combinatorial problem

Sets 3 and 4 required children to work through a series of clues to solve problems of arrangement or association. For example, in one of the hands-on problems, children were provided with regular playing cards and given a series of clues as to how the cards were to be arranged. An example of a written deductive problem involving associations appears in Figure 2.

Four famous sports people entered a television studio. One was a tennis player, one was a swimmer, one was a golfer, and the other was a chess player. Use the clues to find out who played what sport.

- *Mr Bowler is not good at chess.*
- *Both Mr Big's and Ms Ace's sports involve a ball.*
- *Ms Fish can't swim at all.*
- *Neither Ms Ace nor Ms Fish play tennis.*

Figure 2. Example of a written deductive reasoning problem (non-numerical)

The final set of problems was designed to assess whether the inclusion of number interferes with children's deductive reasoning processes. A sample problem from this set appears in Figure 3.

Sue bought two items from the tuckshop. Use the clues to find out which two items she bought.

icecream.........85c *bun........45c* *drink........92c*
chocolate..........50c *sandwich...90c* *apple.......20c*
CLUES:

- *Sue only had $1.50 to spend.* * *She does not like sandwiches.*
- *One item cost less than 50c.* * *If Sue buys an icecream, she always buys a drink as well.* * *If Sue buys a bun, she always buys a chocolate as well.*
- *After buying her two items,Sue had more than 50 cents change.*

Figure 3. Example of a written numerical deductive reasoning problem

Procedure

The children were presented the problems on a one-to-one basis in two or three sessions on consecutive days. The order in which the children were presented the problems was counterbalanced for Sets 1 to 4. The children were expected to complete the problems without assistance from the research assistant. To avoid any reading difficulties for the child, the research assistant read aloud each of the written problems which were presented on an activity sheet. After the two sets of combinatorial problems had been completed, the research assistant asked the child if solving one set of problems helped him/her to solve the other set. The child was also asked to identify ways in which the two sets of problems were similar. This procedure was repeated with the deductive reasoning problems, sets 3 and 4. The responses of all children were videotaped for subsequent analysis.

Data analysis

The children's responses on the combinatorial problems were analysed in terms of their strategies, their written procedures, goal attainment, and their detection of similarities between problem sets (structural or surface level).

The children's responses on the deductive reasoning problems were analysed in terms of their:

1. overall plan of attack, such as drawing an appropriate diagram;
2. specific problem-solving and reasoning processes, including, seeing connections and relationships among items of information, use of means-ends analysis, consideration of alternatives; 3. application of mathematical knowledge and skills; 4. use of appropriate repairs;
5. checking and monitoring processes;
6. detection of similarities between problem sets.

Results

As this study is still in progress and the data analysis incomplete, only a selection of findings can be reported here. Children's strategies for solving the combinatorial problems had been identified previously (e.g. English, 1992; 1993) and were confirmed in the present study; they are revisited briefly here.

Children's combinatorial strategies

The interesting feature of the strategies children used in solving the combinatorial problems was their diverse nature, reflecting varying levels of domain knowledge. Their strategies for solving the two-dimensional problems ranged from inefficient, trial-and-error procedures, to a uniform cyclic pattern in item selection (e.g. red top, blue top, green top, red top, blue top, green top.....), and finally, to a sophisticated odometer strategy which is the most efficient for generating all possible combinations. Here, one item is held "constant" while the other is systematically varied (e.g. red top/blue pants, red top/green pants, red top/yellow pants.....). This process is repeated until all constant items have been used. Children's three-dimensional strategies also reflected a hierarchy of five, increasingly complex, procedures with the most advanced involving the simultaneous use of two constant items. A detailed analysis of these strategies can be found in English (1993).

At this stage of the analysis, it appears that children classified as low achievers in mathematics gained from completing the hands-on combinatorial problems (set 1) prior to the written examples (set 2). The order of problem presentation did not appear to affect the responses of the average and high achievers. Children from each achievement level were able to transfer their combinatorial strategies from one problem set to another, irrespective of the order in which these were presented. When solving the written combinatorial problems, children mainly employed a listing procedure. Some children however, wrote each of the possible combinations in set form, with the names of the items encircled. The use of a tree diagram or array format was rare, indicating that such procedures need to be specifically taught. When asked if they could see any similarities between the hands-on and written combinatorial problems, many children were unable to recognise the common underlying structural similarities. They tended to focus on the surface features, this being a frequently observed feature of novice problem solving (Biggs & Moore, 1993).

Children's deductive reasoning processes

A particularly interesting finding to date pertains to the deductive reasoning processes of children classified as low achievers in mathematics. For example, these lower achievers appeared to quickly see relationships and connections between items of information, using these to streamline the solution process. The self-monitoring processes of many of the lower achievers also seemed comparable, if not better, than those of the other children. The response of 10 year-old Gabrielle below, classified as a low achiever in mathematics, serves as an example here. She solved the problem of Figure 2 as follows:

> *Gabrielle listened to the clues given and listed down the names of the four people. Her first step was to write the word, "golf" beside the name, "Ms Ace." Gabrielle explained that she did this, "Because Ms Ace doesn't like tennis and she uses something with a ball." Gabrielle then claimed that Ms Fish plays chess, "because she can't play golf and she can't swim and she can't play tennis." This was followed by the matching of Mr Big with tennis ("because he uses a ball") and Mr Bowler with swimming.*

It was interesting to observe the responses of children like Gabrielle and compare them with those of her high achieving counterparts. Several of the latter appeared quite inefficient in solving these deductive problems (in some cases, transcripts of their responses on just one example took several pages). They appeared unable to work logically through the clues, failed to make connections, moved repeatedly from one clue to another, and made numerous repairs during their solution attempts.

The order in which children were presented the deductive reasoning problems had a bearing on their performance. Irrespective of their achievement level, children who were administered the hands-on deductive problems prior to the corresponding written examples appeared more competent in solving the written examples than those children who received the problems in the reverse order.

Children's procedures in solving the written examples included the use of diagrams and lists to display the arrangements or associations of items. Some children worked the written examples mentally and recorded only their answer. This suggests that they were using quite sophisticated reasoning processes, given the nature and amount of information to be processed. With the exception of one child, the children did not use a matrix to solve the problems. Given the obvious benefits of such representational formats (Novick, 1990; Polich & Schwartz, 1974), it is important that children be exposed to these after prior experiences with more informal methods.

When recognising similarities between the problem sets, children in each of the achievement levels performed better here than on the combinatorial problems. They appeared better able to detect the underlying structural similarities between the deductive problems. The final set of deductive problems which draws upon children's number sense, appeared more challenging even though these problems have a similar structure to the previous two sets. Weaknesses in children's knowledge of numeration and understanding of terminology such as, "3 times as old as", hampered their progress here.

CONCLUDING POINTS

Given that this study is still in progress, it is difficult to draw major conclusions. Nevertheless, the study does highlight children's ability to reason logically in solving novel problems and, in the process, independently develop more sophisticated procedures. Of significance is the mathematical potential of children classified as low achievers in school mathematics, this potential being particularly evident in their responses on the deductive reasoning problems. Their performance here highlights the importance of broadening the mathematics curriculum to include a range of novel problems; such problems should provide all children with the opportunity to display their reasoning skills, generate their own solution procedures, discover new ideas and principles, and in essence, to take control of their learning (English, 1991b).

REFERENCES

Andreassen, C., Kelly, H., & Waters, H. S. (1991). The development of planning skills in children two to five. Paper presented at the annual meeting of the Society for Research in Child Development (Seattle, Washington, April).

Baranes, R., Perry, M., & Stigler, J. W. (1989). Activation of real-world knowledge in the solution of word problems. Cognition and Instruction, 6 (4), 287-318.

Biggs, J., & Moore, P. (1993). The process of learning. New York: Prentice Hall.

Brown, A. L., & Palincsar, A. S. (1989). Guided, cooperative learning and individual knowledge acquisition. In L. B. Resnick, (Ed.), Knowing, learning, and instruction. Essays in honor of Robert Glaser (pp. 393-451). Hillsdale, New Jersey: Lawrence Erlbaum Associates.

Brown, A. L., & Reeve, R.A. (1987). Bandwidths of competence: The role of supportive contexts in learning and development. In L. S. Liben (Ed.), Development and learning: Conflict or congruence? (pp. 173-223). Hillsdale, New Jersey: Lawrence Erlbaum.

Burton, L. (1992). Do young children think mathematically? Early Childhood Care and Development Journal, Special Edition on Young Children and Mathematics, 82, 57-63.

Burton, L. (1984). Mathematical thinking: The struggle for meaning. Journal for Research in Mathematics Education, 15 (1), 35-49.

Carey, S. (1985). Are children fundamentally different kinds of thinkers and learners than adults? In S. F. Chipman, J. W. Segal, & R. Glaser (Eds.), Thinking and learning skills, vol.2: Current research and open questions (pp. 485-517). Hillsdale, New Jersey: Lawrence Erlbaum.

DeLoache, J. S., Sugarman, S., & Brown, A. L. (1985). The development of error correction strategies in young children's manipulative play. Child Development, 56, 928-939.

English, L. D. (in press). Evidence for deductive reasoning: Implicit versus explicit recognition of syllogistic structure. British Journal of Developmental Psychology.

English, L. (1993). Children's strategies in solving two- and three-dimensional combinatorial problems. Journal for Research in Mathematics Education, May.

English, L. (1992). Children's use of domain-specific knowledge and domain-general strategies in novel problem solving. British Journal of Educational Psychology, 62, 31-45.

English, L. (1991a). Young children's combinatoric strategies. Educational Studies in Mathematics, 22, 451-474.

English, L. (1991 b). Young children as independent learners. In G. Evans (Ed.), Learning and teaching cognitive skills, ACER Monograph Series on Cognitive Processes and Education (pp. 70-86). Melbourne: Australian Council for Educational Research.

English, L. (1988). Young children's competence in solving novel combinatorial problems. Unpublished PhD thesis. University of Queensland.

Gelman, R., & Brown, A. L. (1986). Changing views of cognitive competence in the young. In N. Smelser & D. Gerstein (Eds.), Discoveries and trends in behavioral and social sciences (pp. 175-207). Commission on Behavioral and Social Sciences and Education, Washington, DC: NRC Press.

Gelman, R., & Greeno, J. G. (1989). On the nature of competence: Principles for understanding in a domain. In L. B. Resnick, (Ed.), Knowing, learning, and instruction. Essays in honor of Robert Glaser (pp. 125-186). Hillsdale, New Jersey: Lawrence Erlbaum Associates.

Halford, G. S. (1992). Analogical reasoning and conceptual complexity in cognitive development. Human Development, 35 (4), 193-217.

Inhelder, B., & Piaget , J. (1958). The growth of logical thinking: From childhood to adolescence. London: Routledge and Kegan Paul.

Karmiloff-Smith, A. (1984). Children's problem solving. In M. Lamb, A. L. Brown, & B. Rogoff (Eds.), Advances in educational psychology. (vol.3, pp. 39-90). Hillsdale, New Jersey: Lawrence Erlbaum.

Klahr, D. (1985). Solving problems with ambiguous subgoal ordering: Preschoolers' performance. Child Development. 56, 940-952.

National Council of Teachers of Mathematics (1989). Curriculum and evaluation standards for school mathematics. Reston, Virginia: National Council of Teachers of Mathematics.

Novick, L. R. (1990). Representational transfer in problem solving. Psychological Science. 1, (2), 128-1332.

Piaget, J. (1957). Logic and psychology. New York: Basic Books.

Piaget, J., & Inhelder , B. (1975). The origin of the idea of chance in children. New York: W. W. Norton & Company.

Polich, J. M., & Schwartz, S. H. (1974). The effect of problem size on representation in deductive problem solving. Memory & Cognition. 2, 683-686.

Riley, M. S., & Greeno, J. G. (1988). Developmental analysis of understanding language about quantities and of solving problems. Cognition and Instruction. 5 (1), 49-101.

DEVELOPMENT OF THE NUMBER CONCEPT \\ THROUGH NUMBER WORDS IN LANGUAGE

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Our purpose is to exploit linguistic evidence derived from theuse of number words in languages in order to formulate conjectures of how the number concept emerged in human thinking. Comparison of structures in naming numbers gives us a hint about what thoughts are at the background in the evolution toward the mathematicalconcepts. Two distinct procedures seem to activate the process: the firstconsists in developing an awareness of quantity and leads into the cardinalnumber concept, the second relies on order and introduces the ordinal numbers.Four succesive phases may be distinguished in the development of a numbersystem. Phase I is related to the recognition of a number of objects, involvingawareness of the concept without exteriorisation, without using namesfor it. Phase II comprises the introduction of number words with classifiers: thenumber concept is closely associated with objects of a certain kind.In phase III the development of a number system occurs through associationwith a standard: the human body. Eventually, in phase IV abstraction yieldsthe creation of an abstract number concept. A word for a concrete object whichis part of the vernacular is totally freed from any concrete connotation and becomes an abstract measure.

Anthropologists consider counting as a cultural universal; useless to say that mathematicians and mathematics educators are more than willing to support that opinion. Psychologists often consider numbers as archetypes (Jung). Linguistic evidence derived from the occurence, form and usage of number words in different languages allows for a comparison of structures in naming numbers and for the formulation of conjectures of how the number concept emerged in humanthinking and of what kind of thoughtstructures are at the background in devising number systems. In particular, what we know about the transition from concrete numbers to themathematical concept may be very important for the early teaching of arithmetic, as the construction of a mathematical concept in the individualmind seems to follow procedures which are closely related to the ones whichhave been in action during its historical development.

Of much importance for the development of the counting ability is the introduction of words which may be considered to be rudimentary forms of symbols for numbers, not yet endowed with all of the abstract power which our present symbols 1, 2, 3, ... have. As counting becomes very soon a symbolic process, lack of words/symbols may hamper the smooth development of the counting faculty. There is a striking similarity with another abstraction faculty of the mind: the recognition and naming of colours. Although the human eye has the ability to distinguish among 10 or 11 basic colours, many primitive languages didn't develop words for all of these, f.ex. using the same word for blue end green, and are unable to give an accurate description of the full sprectrum of the rainbow ([5] p.10, [13] p.33).

ASPECTS OF THE OCCURRENCE OF NUMBER

The intricate cultural development of an abstract concept such as number may have passed through some distinct phases, which however didn't have to be strictly successive, as parts of some of them may have evolved inparallel. It seems to us that an overall scheme for organizing the evolution of the number concept has to take into account at least four phases.

1. **PHASE I:** Recognition of a number of objects; awareness of the number concept without verbal exteriorisation, i.e. without using names for it. Groupings of objects are perceived as permanent, discrete and distinguishable items, such that the difference with another grouping is part of the conscious mind of the observer. But *number words* are not yet introduced in the language, if language has already come to exist at all ! It is likely that the counting process preceded the use of fully developed number words; study of the

content of the oldest Sumerian tablets reveals the fact that tablets containing arithmetical operations (an elementary form of bookkeeping) are one or two centuries older than the very first literary texts ([6]). The same order has been found in the development of writing in the oldest tablets of Mycen\ae an Greece, about 1500 BC ([2]).

2. PHASE II: The use of collective nouns and of classifiers. A specific term is used to signify ``a pair" or ``a ten of ". But neither the numbers 2 and 10, nor the name of the object are mentioned. Words are used to name certain collectivities of objects of a rather particular kind, often animals. The collectivity may be small (2, 3) or big (no precise number: herd, flock, crowd,...). Other examples are *a dozen, a gross*. Classifiers control the introduction of different classes: different number concepts are associated with objects of a given class. Number words exist but refer to specific objects with a common characteristic, considered as members of a well defined class, see [4]. Remnants of this phase subsist till the present day in many modern languages.
3. PHASE III: Development of a number system. The material collected by anthropologists reveals that this occurred almost universally through association with a standard, the human body. Counting of a grouping of objects is done by establishing a correspondence with the speaker him/herself, counting on fingers, toes and the whole of the body.
4. PHASE IV: Development and communication of an abstract number concept using specific terms which are integrated into the vernacular, followed by the creation of a fully developed number system.

THE EARLY BEGINNING OF COUNTING.

Archeological findings have led to the conjecture that some form of counting had already developed in palaeolithical times (20,000-10,000 BC). Scratches on bones or stones have been found which seem systematicallyorganized in groupings, separated by gaps ([10]). The groups have 29 or 30 marks, with occurrence of subgroups of 7 or 14. It is acceptable to relate those scratches to the phases of the moon, hence that could be a primitive (lunar) calendar and implies the ability of some elementary form of counting. If there were already words for numbers at that time is a question which must remain unanswered. But it is impossible to imagine an endeavour to keep track of the passing of days and nights without at least a vague concept of number. Counting may very well have started with unnamed numbers.

Those facts bring us to the question if there is an instinctive component in counting. In [12] p.6, D.E.Smith mentions ``it is said that a shepherd may recognize that one of his sheep is missing without being able to count his flock, and even his dog could do the same. There is in such circumstances no need for counting up to large numbers". The same author refers also to the example of a deaf-and-dumb boy who acquired a knowledge of numbers from observing his fingers, even before he was taught to count, which shows us that the idea of number did not have to await the development of spoken language, and primitive man may have appreciated ``3" without having a name for numbers beyond two.

A CONJECTURE.

In the development of number two elementary but distinct factors seem to be involved, and there is no need to use words for numbers in the cultural background which supports this evolution.

1. The first factor is related to the act of ``seeing" numbers, without naming them. It seems to work only with small numbers and allows for the visual distinction between, say, a group of three and another group of five objects. The limit has been fixed experimentally around six or seven, and as such there exists already a boundary to counting. This process may be called ``the use of unnamed numbers", or alternatively ``instinctive number use" and leads into the cardinal number concept of present day mathematics.
2. The second factor relies strongly on memory and is based on remembering succession of events, f.ex.\ the succession of a number of specific days (the idea of a calendar). If the conjecture of the existence of a lunar calendar in prehistoric times is correct, we may infer that the second factor was predominant in the early development of number. The limit to the registration of succession/order seems to be much larger than the preceding one: primitive peoples remember very long lists, f.ex. of a succession of kings. An example is the list of

Mwami's of Burundi and Rwanda, starting from the 16th century and forwarded by oral tradition ([7]). Clearly, this factor is related to an idea of`ordering" and preludes the ordinal number concept.

In some modern languages both aspects have been preserved in some form: there is still a difference in words used to name an absolute quantity and the corresponding words to designate a succession. This is the case in modern Japanese with measures of time: the names for hours, weeks, months (which require the use of a classifier). For hours, considered as a moment of the day, the classifier ``-ji" is used: sanji is 3h, but for a time-period the classifier ``-jikan" is used: sanjikan is a period of three hours (an absolute quantity). For weeks (shu) the classifier shukan is used; goshukan is a period of five weeks. But there seems to be no other classifier for a succession of weeks (neither in I.E.languages). For the succession of months in a year the classifier ``-gatsu" is used: nigatsu is the second month of the year (february). But for a time period (an absolute amount of time) the classifier ``-kagetsu" is used: nikagetsu is a period of two months.

THE USE OF NOUNS FOR A COLLECTIVITY; CLASSIFIERS

Some languages developed particular terms to express a certain number (often five, ten or twenty) of objects; using the term there is no need to express the number nor the name of the object. This is the case in Polynesia (Fidji, Florida Island), see [3]. Modern languages have conserved some relicts of this way of expressing numbers; some of these are still in use such as the terms *couple, pair, dozen*, but others have an archaic status, as a . More examples are: brace, heap, crowd, school (of fish), pack (of hounds), flock (of sheep). The existence of a dual in some, extinct as well as presently spoken, languages as a distinct form between singular and plural relies probably on the same background concept of using a special word for a set of two objects.

A possible broadening of the idea of relating specific number terms to the nature of the objects counted may have led to the use of *classifiers*: different kinds of objects are counted using different terms, mostly suffixes (see examples in [4]). It is likely that primitive man could count only by pointing to the objects counted, one by one, just as little children do. In the basic motivation for counting the object is all-important and the number may be seen as only the answer to a question raised in relation with the objects (adjectival use of numerals). There is no separation of the number word from the class of objects counted; it is not at all clear that, once primitive man had counted one collection of objects, he could immediately count another collection of the same number of quite different objects. Prelogical mentality has no abstract concepts at command; it does not distinctly separate the number from the objects numbered [9]. This process resulted eventually in linking the number concept very intimately to the counted object, using *different* words for the same number of objects of a *different* kind. This procedures is already present f.ex. in the early Sumerian system [6]. Linguistically the conceptual linking of objects (of a kind) to numbers resulted in the use of classifiers, used in many primitive languages, f.ex. in several Maya languages, but also in a modern language as Japanese. An extreme case is Tzeltal (Mexico), as reported in [1], which has 528 different classifiers.

DEVELOPMENT OF A NUMBER WORD SYSTEM

The way primitive number systems have developed is basically such that new words were in general not introduced until there was a need for them and there was no attempt at, and no idea of, a continuing sequence but only a pragmatic introduction of new terms as the need arose. Two typical examples are the introduction of words for large numbers in Chinese under the influence of Buddhism, which on religious grounds cultivates a predilection for such numbers ([11], p.451) and the borrowing of terms for tenths, hundreds or thousands in many primitive languages during the 19th century due to the developing contact with western cultures. The salient features of the particular primitive systems may be summarized as follows ([5], p.9):

1. New terms are added as the need arises; one may consider this ability as based on an instinctive number use.
2. There is no obvious stopping place; once the idea of counting has emerged then the idea of going on counting is not far away. There is clearly (from our modern viewpoint) the possibility of continuing the sequence if needed.
3. There are attempts at a regular, systematized continuation.

NUMBER WORDS: FROM BODY PARTS TO ABSTRACT NOUNS

The vast majority of anthropological findings converge to the statement that the origin of number words is closely related to names for parts of the human body. A detailed study may be found in a report about the use of counting systems in Papua New Guinea ([8]). There are however exceptions: the word *sorok* for 40 in Russian refers to to a certain, standardized bundle of fur.

Primitive words used in counting were at first tied to concrete groups of objects but in thousands of years they entered the abstract stage in which the group ceases to be a factor. We think of five no longer as of a certain group of objects (five fingers) but as of a word in a succession, following four and preceding six. It is acceptable that number words have developed from abstractions, using words for a concrete object and then loosing the original meaning of the word. Awareness of this process comes from historic times; in the development of units for measures, as foot and ell (elbow) the evolution is still visible. The evolution of number words may have gone along a similar path, as is exemplified by languages which use terms such as *hand* for five, *two hands* or *half a man* for ten, *one man* for twenty. There is even some linguistic evidence that *eye* might have been the original term for two. In due time, such terms lost their meaning and we think of them as abstract measures; nowadays we can do arithmetic without looking outside the number concept.

Counting, together with the words for it, seems to have developed on the basis of a radix, the most popular being the radix ten, but the oldest seems to be radix two, and three is often conceived as a large number, at a time when the fingers of a hand were not yet recognized as a base for counting. Examples are abundant in primitive languages: in some Aztec languages number names mean *one stone - two stones - three stones*, whereas counting in Niue goes *one fruit - two fruits - three fruits* and in Java *one grain - two grains - three grains*. See [12] for an extensive discussion of the variable radix, among which some very peculiar ones, which is used in many primitive and modern languages.

In modern mathematics to dispose of symbols as a shorthand for the study of concepts is of crucial importance. In particular, symbols representing numbers are needed to achieve the construction of an abstract number concept, and the introduction of words for numbers may be seen as a first step toward symbolisation. Grammatically this step corresponds to the shift of the adjectival status of number words to the status of a noun, which according to Wilder ([13] p.41), had never been realized without the previous introduction of some kind of ideographs for the representation of numbers.

A particular problem in the development of number which requires thorough understanding of the cultural background is the existence of an upper bound to counting. A boundary to counting exists in all languages, except in mathematics, but this boundary increases with time as languages develop and become appropriate to express more intricate concepts. The expression ``boundary to counting" may be interpreted ([5],p.22) as the practical boundary determined by the smallest number which has no name in a given language or counting system or alternatively as the situation where no clear way to continue counting is available. Some boundaries are rather unexpected: according to Menninger the number eight may have been an old boundary to counting in Japanese; the same is said of Finnish where number words above 8 are borrowed from other languages.

A possibly related phenomenon is the existence of jumps in the usage of number words: in Indo-European languages jumps from 10 to 100 to 1000 to one million, etc. occur. This is different from f.ex. Japanese which has jumps of 10,000. A jump may have been an old boundary to counting which became obsolete ([11],p.450).

BIBLIOGRAPHY

[1] B.BERLIN, *Tzeltal Numeral Classifiers*, The Hague (1968).
[2] J.CHADWICK, *Documents in Mycenaean Greek*, Cambridge (1973).
[3] R.H.CODRINGTON, *The Melanesean Languages*, Oxford (1885).
[4] L.L.CONANT, *The Number Concept*, MacMillan, New York (1896).
[5] J.N.CROSSLEY, *The Emergence of Number*, Publ. World Scientific, Singapore (1987).

[6] P.DAMEROW \& W.LEFEVRE, *Rechenstein, Experiment, Sprache*, Publ. Klett-Cotta, newline Stuttgart (1981).

[7] S.DESOUTER, *De Gebroken Lans* (The Broken Lance), Ed. ACT, Brussels (1992), in Dutch.

[8] Edit.\ D.F.LANCY, *The Indigenous Mathematics Project*, Papua New Guinea Journal of education, Vol. 14, Special Issue (1978).

[9] L.LEVY-BRUHL, *Les fonctions mentales dans les societes inferieures*, Paris (1926).

[10] A. MARSHACK, *Notation dans les gravures du paleolithique superieur* , Publ. Inst. Prehistoire Univ. de Bordeaux, M\'{e}m. No. 8 (1970).

[11] K.MENNINGER, *Number Words and Number Symbols*, Publ. MIT Press, Cambridge (1969).

[12] D.E.SMITH, *History of Mathematics*, Vol.I, Dover Publ. New York (1958).

[13] R.L.WILDER, *Evolution of Mathematical Concepts:* An Elementary Study}, Publ. Wiley, New York (1968).

CONSTRUCTIVISM AND THE ADULT LEARNER: MARIEANNE'S STORY

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When women return to study mathematics they frequently have to overcome obstacles, both long-term and short-term, in order just to be present at classes. Voluntary attendance at mathematics classes can provide both mental stimulation and an escape from the reality of everyday existence. It can also provide a support to sagging morale when self-esteem is under attack elsewhere. Further it can be emancipatory when a woman discovers that she is capable of actively helping her children learn mathematics in a productive manner, making it an enjoyable experience for all. This paper will discuss the positive effects of a constructivist mathematics classroom one woman's life.

A SHIFT TOWARDS SOCIAL CONSTRUCTIVISM?

At a recent conference on constructivism Lerman (1993) suggested that there is a need for radical constructivism to shift to a sociocultural view of learning, which takes into account, among other things, that subjectivity is constituted through social practices, and that cognition and affect are not separable. This is supported by Ernest (in press) who claims that the greatest strength of radical constructivism is in its detailed account of the psychological processes of individual construction of knowledge, but notes that there is a need for it to accommodate the complementarity between individual construction and social interaction. He states that radical constructivism is evolving and adapting to accommodate the criticism it has met, especially in the social and cultural dimensions of learning and knowledge. Cobb (1990) asserts that while the cognitive context includes the constructs of scheme and reflexive abstraction, it should also include sociocognitive constructs such as obligations, expectations and beliefs, as "teachers and students modify their context as they act within their contexts" (p. 206). He argues that it is essential to include a mathematico-anthropological context, since "learning is an interactive as well as a constructive activity" (p. 209).

AN EDUCATIONAL RESEARCH PARADIGM

Ernest (in press) distinguishes four components of an educational research paradigm: an ontology; an epistemology, comprising theories of knowledge from both psychological and philosophical perspectives (where the relative strengths of knowing and believing are juxtaposed) emphasising both individual and shared knowledge; a methodology; and a pedagogy. He notes that there may be some overlap in these categories, and although he sees that the last may be problematic he justifies its inclusion on the grounds that a central concern of constructivism in mathematics is to improve pedagogy.

Values

Since Ernest (in press) believes that pedagogy is predicated on a set of values, it is likely to be eclectic, being a theory of techniques for achieving the ends of offering selected knowledge or experiences to learners. In fact he suggests that the model may be extended to include a set of values as a fifth component.

The question of values was raised, following the paper by FitzSimons (1993), as to whether teaching from a constructivist perspective could not be replaced by simply teaching from a set of principles based on respect for the learner. Teachers following any of a variety of paradigms, even the most absolutist, may justify them on the grounds of their own values, but a constructivist perspective provides a coherent framework for pedagogical choices. It provides a philosophical basis for the myriad of decisions a teacher makes in every class and is a referent to analyse the learning potential of any situation, according to Tobin and Imwold (1993).

This paper will illustrate how teaching from a radical constructivist perspective, accommodating a sociocultural view of learning, has helped to change one woman's image of mathematics, and improve her self image.

CHANGING THE PUBLIC IMAGE OF MATHEMATICS

Ernest (1992?) describes the public image of mathematics as cold, abstract, inhuman, and related to absolutist philosophies of mathematics. He argues that the image should be changed to a fallibilist one, which views mathematics as the outcome of social processes, and "rejects the notion that there is a unique, rigid and permanently enduring hierarchical structure" (p. 7). According to Ernest, many people believe that "only a fallibilist view can provide the epistemological basis for critical mathematics education" (p. 9).

One way of changing the image is to raise public consciousness about the nature of mathematics, and an example of this is to be found in mathematics classes at the Centre for Women's Education, Box Hill College of TAFE. A constructivist approach to teaching is consistent with the philosophy of the Centre, and is given the full support of the Head of Department. According to Pateman (1993), administrative support is necessary for teachers attempting to implement a different approach to teaching. Not only do the mathematics classes raise the consciousness of women who attend, they also impact on the family and friends. Women, especially those with children, communicate powerful messages in being seen as a person who takes a vital interest in things mathematical, and with whom productive conversations can take place. The women themselves have commented, in private conversations, on their increased self-esteem.

THE CLASSROOM

Sociocultural context. Belenky, Clinchy, Goldberger and Tarule (1986, p. 4) observed that women students frequently speak of "problems and gaps in their learning, and so often doubt their intellectual competence." They make recommendations for educators of women, consistent with a constructivist perspective, such as emphasising connected knowing, understanding, acceptance, and collaboration; allowing time and respect for knowledge that emerges from first-hand experience; and encouraging students to evolve their own patterns of work.

Adults returning to study are in transition, in two senses: between phases of life, and between schools. Spanard (1990) proposed that adults continuing with their education do not see this as an end in itself, but as a means to a future change or coping with life changes that have already occurred. With their last formal mathematics classes likely to have been many years earlier the experience of discontinuity can be significant, and almost always accompanied by anxiety. Many women, even those with tertiary study in other areas, have little self-confidence, and ask to start "right at the beginning."

The social context of the classroom critically influences the attitudes of women returing to study. An environment that is uncomfortable, humiliating, or authoritarian can destroy any nascent self-confidence. According to Nordstrom (1989) a major theme in the continuing education literature is that adult learners are concerned with maintaining a positive self-concept. Although adult learners tend to be highly motivated to learn, they lack confidence in their ability to do so, and require continuing encouragement. One way of learning that is consistent with the recommendations made above, and which happens in most other subject areas at the Centre for Women's Education, is to have discussions about the subject matter. In fact discussions take place on several levels: the content of mathematics, its historical development, theories of how mathematics is learned, and so on.

Mathematical discussions

Richards (1991, p. 46) asserts that language "is the means by which a student is brought into a community," and believes that the best way of assisting students to construct their own mathematics is to provide a classroom designed to teach the language of mathematical literacy. According to Richards a consensual domain must be established where participants are prepared to respond to each other and to learn. While the teacher can be a facilitator and moderator at times, there is also a need for her to be a genuine participant in discussions. Cobb (1990) believes that mathematics is learned through the construction of a consensual domain, as learners participate in the process of negotiating and institutionalising mathematical meanings.

The teacher's role

Students need to be supported by the teacher in discussion so that they do not feel vulnerable as they disagree with their peers or ask for clarification. Lampert (1990) acknowledges that the teacher's role involves managing the tensions between individual learners and social conventions, as she offers vocabulary, encouragement, suggestions, and support for revisions; and modelling a way of knowing mathematics which depends on reasoning, not authority, for its veracity.

MARIEANNE'S WRITING ABOUT HER MATHEMATICS LEARNING

Personal History of Mathematics

As part of the assessment requirements for the course women are asked to write their personal history of mathematics learning as a homework task. One reason is for them to reflect on their experiences and beliefs about mathematics teaching and learning, as these issues are discussed throughout the course, in addition to the specific content and processes of doing mathematics. A second reason is that the history provides useful information for the teacher - as much as the students feel comfortable about sharing, although for some this can evoke painful memories. It can give greater insight into the obstacles that students have faced in order to be in a situation to both need and take up the option of further study in mathematics, as well as inform the teacher of strategies which have had negative consequences in the past.

This is Marieanne's personal history:

> I really don't remember that much about maths in the early days.
> After emigrating from Holland my first recollections were of being very frightened amongst people I could not understand. I was always very quiet and was too afraid to ask if I did not understand. Looking back I did grasp the basics but when it came to algebra and geometry, I just couldn't see the point in it at all. I left school at 15 as my parents did not see the point in educating a mere girl.
> I now find I enjoy maths and at last feel that the things I learnt as a child in maths I am starting to understand now. The knowledge I had as a child is still there and it is fun to bring it to the surface.

Although this history is short it has much to say about Marieanne, especially after one has worked with her for some time. Not being able to remember much about the early days is not surprising, given that she was so frightened. This is in contrast to another woman's history (see FitzSimons, 1993), where memories are happy. Marieanne, like so many women returning to study mathematics, is intelligent, and takes a real delight in learning, especially when she confirms her new knowledge, but also when she affirms her previous knowlege. Yet she was disadvantaged by a school system which failed to make mathematics appear relevant, and by parents who discriminated against her, curtailing her education, because of her gender. In 1993 Marieanne's education has been interrupted yet again, by family responsibilities. She has missed many classes on account of illness suffered by her children and by her aging close relatives. While her husband is supportive of her study, he is not in a position to assist with family responsibilities during class time.

Journal writing

Students are encouraged to keep journals to reflect on what they are learning each week, the progress they are making, and their feelings about the class. Recently their own evaluation of their journals has been included in assessment requirements.

The following are examples of Marieanne's journals:

> *18/2/92* The work today has really made me think. It is not what it at first seems with numbers and powers into the negative. (See FitzSimons 1992 for the full transcript of a class discussion on powers of ten and two.) I took for granted that the power numbers would be the same as 10^2 e.g. 10^2 = 100, 2^2 would be 4, then 2^-2 would be .4, instead it is 1/4 or .25. Once this was realised I could then do the other numbers.

12/5/92 I love doing fractions. I feel very confident with them. I am still not sure about the calculator. The process does not seem to stick in my mind.

19/5/92 I was frightened of starting negative numbers. Using the red and the black squares I feel I am understanding them better. Homework did not take long as I used the squares to do the additions. I feel fairly confident of addition of negative numbers.

26/5/92 What a blow! I feel completely lost, so much for my new confidence. I could not grasp negative numbers today - or rather subtracting negative numbers. For once I felt better using a calculator.

Hooray I have sat and worked through the sheet. Although I have most of the answers wrong, by going over them with the calculator I am starting to grasp what is happening. I shall try to keep practising and see if it works. I am also at last starting to understand what the number line is for! I actually used it to work out some of the answers or rather corrections together with the calculator. I worked through the sheet again and this time got only 6 wrong. I feel be*tter about it.*

These journal entries reflect several things about the class and about Marieanne. Discussion is clearly an important aspect of all classes, and the journals capture some of an ongoing dialogue between Marieanne and the group. She made reference to the use of multiple representations, such as algebra blocks and the number line, in the study of negative numbers, as well as the integral use of calculators, which provide yet another representation. The journals reflect the link between cognition and affect, and the use of schemes and reflective abstraction. They show Marieanne's positive attitude towards learning, and her persistence in the face of initial setbacks. She is able to express her uncertainty without shame or fear of reprisal.

The Constructivist Learning Environment Survey

During the first few weeks of class Marieanne completed the Preferred and the Perceived forms of this survey (see Taylor & Fraser, 1991). Three of the four scales were: Autonomy (the extent to which students control their learning and think independently), Prior Knowledge (the extent to which students' knowledge and experiences are meaningfully integrated into their learning activities), and Negotiation (the extent to which students socially interact for the purpose of negotiating meaning and building consensus). A fourth scale has not proved satisfactory and is expected to be replaced (Peter Taylor, personal communication). With a maximum score of 35 in each scale, allowing for question reversals, Marieanne's perceptions were very close to her preferences in Negotiation (27/26) and Prior Knowledge (30/30), but in Autonomy her perception exceeded her preference (22/16). Typical statements in the Autonomy category were: "I decide how much time I spend an an activity" and " I do investigations in my own way." Marieanne is not yet fully confident about her own autonomy as a learner, but is fairly supportive of a constructivist classroom.

AN INTERVIEW WITH MARIEANNE

In order to gain a more complete picture of events which Marieanne had alluded to in class I conducted an interview with her in her own home. The following is compiled from notes taken.

> Before I enrolled in the computer class and the maths class I felt depressed. I was working part-time, but looking for something more. All the kids were at school, and I wanted to do something for myself. I was scared of not being able to help them in computers and maths.
>
> When I started back at work I was terrified at first. On the first day I panicked, felt ill. My husband literally pushed me out the door. I built up my confidence after being back at work, but I needed to build up some more confidence through study. You lose your confidence really quickly, when you are not working.
>
> On my first day at the Centre for Women's Education I nearly turned back. My parents were no help. They said: "What do you want to do maths for?" Only my husband and kids gave me support. As I came in, I was talking to myself: "What's the worst they can do? Make me look stupid? Ask me to leave if I was not up to the standard?" I came in because I could not forgive myself for not trying, and would forever wonder if I could do it.

I have kept on coming over the last three years because I feel accepted. My input helps me to understand. I am not made to feel foolish or put down. There are lots of ways of explaining things. I feel comfortable asking questions now - you get more out of it if you do. It took about a year to have the courage to ask, I felt shy. At first I used to ask my husband - now I show him what I do. He boosts me up. Being with much the same group over the three years helps - you get to know each other, and trust is built up.

Personally I have gained confidence from coming to class. The main benefit has been in helping my children. It gives me a great deal of pleasure. We get out the whiteboards and everyone gets excited. Michael (7 years) learned the 11 times tables in grade 2 "the easy way." We started with 2 (digit) numbers, then 3, and so on up to the millions. I used to say: "I wonder what if ..." It felt good to let them think for themselves. We also used a calculator to check.

It was good to see Catherine actually understand fractions. She felt a real glow of pride. She was one of the few in her class who actually understood them, and went out to the board to demonstrate. She was glowing when she came home. I used to trace round plates for circle shapes, and we'd cut out 1/2, 1/4, 1/8. Also we would fold computer paper sheets. I would start with one child, and all the others would come round to listen and watch. We all enjoy doing it. I set work for the children, and tell them to show all their working out, as this helps their thinking. Sometimes the work is still on the boards when my husband Ken comes home, and he is most impressed.

I started to help the neighbour's children, but Catherine was upset, and asked me not to tell anyone else. She enjoyed having the advantage. She said that I was the only Mum who knew fractions - I can't understand why that is. Knowing that there are many ways of explaining things is useful. Also it's good for girls to have a role model. In any case, Ken isn't home until 8, and then it's too late to help them anyway.

I enjoy maths now, but it was a chore at school. I enjoy finding out new things, exploring, and feeling satisfaction when I understand something that I never knew before. Now it's clicking, dropping into place. At school I was frightened, I would groan, and was fearful when we had maths, because of not understanding. I didn't want my kids to feel that way. Maths was closed for me before, but now it's opened up; there are different paths. I wonder how much I would have learned if I had stayed on at school. Now I will attempt to answer new questions.

CONCLUSION

Marieanne's interview underlines the link between cognition and affect. Not only is her self-concept strengthened (she confided that it had helped her during a period of harrassment in the workplace), she is pleased to be able to constructively assist her children, and they too have gained in self-confidence in mathematics. The strategies she uses with her children are based on those used in our mathematics classroom. Bliss (1990) states tht collaborative learning and parental involvement show significant potential. This paper has attempted to demonstrate the positive impact that constructivism can have on adults as learners and teachers of their children.

REFERENCES

Belenky, M. F., Clinchy, B. McV., Goldberger, N. R., & Tarule, J. M. (1986). Women's Ways of Knowing. New York, NY: Basic Books.

Bliss, J. (1990) The nature of communication in early childhood in the contexts of home and school. In L. P. Steffe, & T. Wood (Eds.), Transforming children's mathematics education (pp. 294-303). Hillsdale, NJ: Lawrence Erlbaum.

Cobb, P. (1990). Multiple perspectives. In L. P. Steffe, & T. Wood (Eds.), Transforming children's mathematics education (pp.200-215). Hillsdale, NJ: Lawrence Erlbaum.

Ernest, P. (1992?). Images of mathematics: A philosophical perspective.

Ernest, P. (in press). The one and the many. In ??

FitzSimons, G. (1992). Constructivism in vocational and further education. In M. Horne, & M. Supple (Eds.), Mathematics: Meeting the challenge (pp. 77-82). Melbourne: Mathematical Association of Victoria.

FitzSimons, G. (1993). Constructivism and the adult learner: Jane's story. Paper presented at Constructivism: The Intersection of Disciplines conference, Deakin University, Gèelong.

Lampert, M. (1990). Connecting inventions with conventions. In L. P. Steffe, & T. Wood (Eds.), Transforming children's mathematics education (pp.253-265). Hillsdale, NJ: Lawrence Erlbaum.

Lerman, S. (1993). Constructivism and intèrsubjectivity: Parallel or shared meanings in mathematics? Paper presented at Constructivism: The Intersection of Disciplines conference, Deakin University, Geelong.

Nordstrom, B. H. (1989). Non-traditional students: Adults in transition. Flagstaff, AZ: Northern Arizona University.

Pateman, N. H. (1993). Can constructivism underpin a new paradigm? In J. A. Malone, & P. C. Taylor (Eds.), Constructivist interpretations of teaching and learning mathematics (pp. 69-79). Perth: Curtin University.

Richards, J. (1991). Mathematical Discussions. In E. von Glasersfeld (ed.), Radical constructivism in mathematics education (pp. 13-51). Dordrecht: Kluwer.

Spanard, J. A. (1990). Beyond intent: Reentering college to complete the degree. Review of Educational Research, 60(3), 309-344.

Tobin, K., & Imwold, D. (1993). The mediational role of constraints in the reform of mathematics curricula. In J. A. Malone, & P. C. Taylor (Eds.), Constructivist interpretations of teaching and learning mathematics (pp. 15-34). Perth: Curtin University.

VIDEOTAPES AND MATHEMATICS LEARNING: METHODOLOGICAL AND TECHNICAL ISSUES

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Videotaped records can have distinct benefits over other methods of data collection. The technique also has limitations and frustration can be encountered in its use. Inherent in its apparent simplicity is the temptation to use the medium inappropriately. Judicious consideration of the research question and related methodological and technical issues will determine whether videotape is the suitable choice. In this paper we describe three quite different mathematics learning situations in which videotaped records were used. Common to the three situations was the recognition that no other data collection technique would provide the information sought as effectively. Both visual and auditory information were essential. Yet in each case the researchers were faced with a unique set of interacting difficulties before, during and after videotaping. Our discussion also outlines some consequences associated with using and retaining videotaped information.

Videotape is an intuitively appealing means to collect observational data. The medium has distinct benefits over other research techniques. The most obvious advantage of having primary data in this form is that it can be reviewed and re-examined repeatedly and extensively (Adams, 1971; Mehan, 1979). However, the limitations of videotape and its vulnerability to inappropriate application are often not appreciated or discussed. In this paper we present some methodological considerations associated with the use of videotape and examine three different mathematics research projects in which videotape was utilised. Technical and other allied difficulties are highlighted.

A characteristic of most forms of qualitative research is an emphasis on conducting studies in `natural settings'. Often participant observation is included amongst the methods of data collection (Jacob, 1988). Field notes are the most common means by which ethnographers record data. However, several factors can influence the selection of the most appropriate method. The choice "depends very much on one's purpose, the nature of the setting, and the financial resources available" (Hammersley & Atkinson, 1983, p.145). Audiovisual techniques can "provide a much more accurate and detailed account of events than can be provided in notes" (Hammersley & Atkinson, 1983, p.157). In particular, if the researcher is looking at the detail of both verbal and non-verbal interpersonal interactions, "electronic recording is probably essential" (Hammersley & Atkinson, 1983, p.157). Videotape should be seen as supplementing and not replacing the participant observer taking field notes (Hammersley & Atkinson, 1983) since "much of the additional information available to the on-site observer is lost" (Hamilton & Delamont, 1974, p.9).

Videotape can also be used in studies based in more systematic observational paradigms. When a range of methods for coding observational data were compared, Kieren and Munro (1985) found that the on-site coding method produced less than 50% of the number of units recorded from methods based on videotape records. The categories into which the data were coded were also found to be the most discrepant. In a review of a classroom study where student-teacher interactions were analysed from videotaped records, Hart (1992) commented:

> Particularly, the use of videotapes allowed for a more detailed view of teacher-student interaction and clearer evidence of reliability than has been possible in many previous studies using .systematic observation. (p.81)

The use of videotapes has enabled the merging of two research paradigms: process-product and sociolinguistic (Peterson et al., 1984). The analyses within each paradigm suggested relevant and important implications for research in the other.

Financial considerations may determine whether videotape is a feasible alternative for recording information. Analysis of videotape can be very tedious and time consuming. Adams (1971) reported that "on an average, each hour of classroom tape was subjected to twenty hours of analysis" (p.107). Kieren and Munro (1985) found that producing transcripts from videotapes which involved three people interacting took a minimum of two and a half hours for ten minutes of videotape.

In classroom settings, technical limitations such as the type and sensitivity of the microphone and camera, and their positioning, can restrict the sound and picture quality of recordings. Students `performing' to the camera, background noise, the narrow view through the camera lens, and visual obstructions can be biasing aspects of using videotape. There are also associated ethical issues. Videotape is an intrusive medium producing permanent records. Confidentiality of names, faces, and school uniforms can be maintained in written reports emanating from videotape analysis, but not from the video records themselves. Participation and permissions associated with research projects involving videotape may also be more difficult to obtain. Consent for student involvement may not be granted or, if approved, may be coupled with considerable restrictions.

STUDY OF SMALL GROUP WORK

Developing positive attitudes towards mathematics has increasingly been viewed as an important aim of mathematics education. According to Schoenfeld (1992), new methodologies were needed in research associated with the affective domain to integrate meaningfully perspectives on cognition and affect in mathematics learning. Kiesler, Collins and Miller (1969) contended that:

> Social scientists have, almost without exception, settled on pencil and paper or interview techniques for the measurement of attitudes while retaining a theory that specifies behavioral implications for attitudes (Kiesler et al., 1969, p.23).

As early as 1934, Lapierre (1967/1934) found a mismatch in the relationship between attitude and intended versus overt behaviour. Studying humans behaving in relevant contextual settings is one way to develop a better understanding of "the relationship between feelings and inner expressions on the one hand and observable behaviour on the other" (Eiser, 1987, p.3).

In this study of five year 7 students working co-operatively on a group mathematical task, the aim was to examine the interaction of students' cognitive engagement in the task and their attitudes towards mathematics and themselves as learners of the subject.

Some data on students' attitudes were gathered using more traditional pencil-and-paper means. The affective variables of interest were derived from a number of models put forward to explain gender differences in mathematics learning (see Leder & Forgasz, 1992). Students completed a number of instruments and responded to items in both closed and more open formats. These data were regarded as providing measures of students' beliefs and as indicators of intended behaviour. Of interest, however, was the study of actual behaviour in a relevant contextual setting, engagement in a mathematical task. Affective behaviours could then also be compared with the pencil-and-paper indicators.

In order to infer students' attitudes and their cognitive engagement from what they said and from what they did, it was important to capture clearly all verbal interactions between the students as well as to have an unobstructed view of each student at work. Videotaping the students was considered the best means by which the aims of the project could be met. Field notes were kept to supplement the videotaped records. Since the camera and microphone were focused on the activities of the group, there were times, for example, when group members moved around the room out of camera view and instances when the teacher spoke to the class and was not clearly audible on the tape. In these situations, the advantages of having the participant observer were clear.

It was recognised that the close proximity of the camera and the placement of the `floor microphone' in the centre of the cluster of tables on which the students worked were factors which might constrain and inhibit the `normal' flow of conversations and the pattern of work. This called for early familiarisation with the presence of the camera in the classroom. Several weeks prior to the small group work, the camera was brought into the classroom. For two lessons the tape was not switched on. By the third lesson the class' curiosity had been alleviated and `performing to the camera' had virtually ceased. When the time came to film the small group at work later in the year, "camera shyness" was quickly overcome. The mathematical task set also stimulated the group's interest and the students set about tackling it enthusiastically. Some of the verbal exchanges picked up on tape indicated that they seemed to have forgotten that their words and actions were being recorded!

Videotape analyses of the lessons proceeded at several levels: an overview of the students' behaviours over the whole sequence of eight lessons, individual lesson summaries, and fine-grained analyses of critical episodes within lessons for which transcripts were central. The transcription phase was extremely labour-intensive and took an inordinate amount of time; approximately three minutes of videotape were transcribed per hour. Yet the data obtained were rich and most informative. To describe the students' affective behaviours, operational definitions had been developed which used transcript excerpts to illustrate the variables under investigation. The details of the operational definitions and analyses have been reported elsewhere (Leder & Forgasz, 1992).

Whilst the school, the teacher and the students had been most co-operative and enthusiastic throughout the conduct of the study, permission was not given for short videotaped excerpts to be used to illustrate aspects of the findings and share them with the wider educational research community.

WORKING WITH SINGLE STUDENTS

Monitoring student learning in a regular classroom setting and exploring the implications of the data obtained were the main goals of the second study to be described. The sample comprised students in grade 3. Selected mathematics lessons which contained substantial oral explanation segments were videotaped. The earlier general comments about the advantages and disadvantages of this method of data gathering and transcription of the material, are also pertinent to this study.

As described in more detail elsewhere (Leder, 1990a; 1990b), key components of the lesson were selected for replay and discussion with students in a one-to-one setting in a quiet room available for this purpose. A structured interview format was used to probe student interpretations of teacher explanations and their apparent understanding of the work covered. In this more intimate and relaxed setting students' beliefs and practices could be explored in greater depth than is often possible in a whole-class environment where questions need to be addressed to a larger group.

The interviews, and the examples students were encouraged to attempt during them, confirmed that students had been engaged in the work, had listened to the teacher, and that they had learnt from the lesson. However, their interpretations were often at variance with those intended by the teacher through her carefully sequenced series of steps that seemed most appropriate within *her* cognitive framework. Creative adjustments were made by some students in their struggle to make the teacher's explanations consistent with *their* conceptual framework. For example, reasons for students' confusion between subtraction and division procedures were not only identified but were able to be recognised as sincere attempts by students to achieve consistency with their earlier learning. Replay of selected sections of the lesson during the interviews and eliciting students' reactions in considerable depth, were invaluable tools which facilitated exposure of students' thought processes.

FOCUS ON A TEACHER

The central concern of the third study was to focus on teachers' work. That is, to find a way effectively to capture and describe the complexity of the planning, choices and actions that make up a typical teaching day, and to do so without oversimplifying, and without allowing an observer alone to decide the meaning of classroom events.

It was therefore essential that the teacher be given every opportunity to explain and comment on the planning and execution of the observed lessons. Since this clearly could not be done during class without unacceptable disruption to normal procedures, videos were chosen as a major means of data collection. It was felt that videotape offered the particular advantages of allowing events to be replayed "close to their original form" (Jacob, 1987, p.20) and then discussed.

The following data became available for examination:

- transcripts of the videotapes
- transcripts of the audiotaped discussions made before the lessons took place
- transcripts of the audiotaped discussions made while watching the videotapes
- teacher's planning notes
- observer's field notes

Through these data the unravelling of the layers of reality through the perceptions of both observers and participants could begin. The method of analysis is not discussed here. However, a number of comments follow regarding the technique.

The transcription of the six hours of videotape took an enormous amount of time. While Kieren and Munro (1985) argued that "use of the video increased the level of comprehensiveness as well as accuracy, even without a transcript" (p.16), transcription was considered necessary in this study. Avoidance might have lead to selectivity and to limitations on what was described. Since conveying the complexity of classroom life was a stated objective, it would have been unjust to oversimplify the research procedure.

But it would have been naive to think that in transfering the video to paper, no interpretation might intervene. On the contrary. The meanings of many sentences cannot be conveyed without the addition of adjectives to describe tone. For instance, the apparently clear 'What do you think, John?' may be a test of John's comprehension, it may be admonitory , or it may be a way of determining the success of a lesson by using an average student as a barometer. The transcriber chooses a word to describe the tone and in so doing has interpreted the event. At least with videotape the interpretation can be reviewed and in this particular study the teacher had the opportunity to offer her own explanations.

Even this posed its own challenges. It was necessary for the teacher to find the time soon after the recorded lesson had taken place to watch the tape and comment on it. At least as much time as the recording was needed to allow for discussion. Apart from the fact that teachers are fully occupied during the school day, their time is often required after school hours. Scheduling an extra task was not easy.

There were, however, clear benefits for the teacher and the researcher in having a tape. Accepting a colleague into the classroom is not an everyday event in schools and therefore was a brave and generous action in a potentially threatening situation. A certain security existed for the teacher in being able to review and explain her work while watching it happen. For both the researcher and the teacher there was also the benefit of having the opportunity to reflect on and discuss issues with a fellow professional.

CONCLUSION

Researchers should not be seduced by the latest advances in technology which might appear to present simple solutions to difficult research situations. As discussed in this paper, videotape as a medium has distinct advantages over other data gathering techniques. But costs are also incurred. Adopting videotape as a research method should be a measured decision linked to the goals of the project and to funding constraints. At the same time, it must be recognised that while videotape may seem an easy way to replace the field observer or to immortalise 'reality', its limitations ensure that this is not the case.

In summary, videotape can benefit research in the following ways:

- Recordings can be replayed, slowed down, skipped over and reviewed. These strengths enable more accurate entries when observational schedules are followed, provide the means for participants to comment and explain behaviours, and allow the researcher to hear and see more than participant observation of the events would allow.
- Material available for transcription and analysis is more detailed than can be gathered from audiotape or field notes. Detailed and focussed verbal exchanges can be determined, accompanied by gestures, movements and

facial expressions. General interactions, the setting and the atmosphere are more vividly captured by the video camera than by other means.

However, just as the computer is unlikely ever to eradicate pen and paper completely, video cameras neither serve all ends nor are without weaknesses. The machine should not be thought the answer to the problem of bias. The camera has its own blind spots. It is but `one eye', and it can miss sections of the room and some voices.

One of the main advantages of videotape is also one of its difficulties. A large quantity of data is collected. Transcription is often required before analysis can begin. Those without access to considerable research funds should take heed. Delegating the task of transcription is also undertaken at some risk. Descriptions of tone, actions and even dialogue are subject to interpretation.

There are also ethical considerations of confidentiality, and difficulties may be associated with permissions and participant consent. Finally, videotape is open to inappropriate and ethically questionable research practice. For example, permission may have been given for a particular research project to be undertaken. Some time later, the researcher (perhaps short of research funds), wants to investigate a different research question. The temptation to revisit earlier recorded tapes is there. The status of time limitations on the re-examination of `old' videotaped material is often unclear.

There is little doubt that videotape adds possibilities to research not previously available. We have argued, however, that the medium is not without limitations and that even its advantages have associated difficulties. It is, nonetheless, a valuable and powerful tool in many research settings.

REFERENCES

Adams, R. S. (1971). A sociological approach to classroom research. In I. Westbury & A. A. Bellack (Eds.) Research into classroom processes (pp.101-117). New York: Teachers College Press.

Eiser, J. R. (1987). The expression of attitude. New York: Springer-Verlag

Hamilton, D., & Delamont, S. (1974). Classroom research: A cautionary tale, Research in Education, 11, 1-15.

Hammersley, M., & Atkinson, P. (1983). Ethnography principles in practice. London: Routledge.

Hart, L. E. (1992). Two generations of feminist thinking. Journal for Research in Mathematics Education, 23(1), 79-83.

Jacob, E. (1988). Clarifying qualitative research: A focus on traditions. Educational Researcher, 17(1), 16-19, 22-24.

Kieren, D. K., & Munro, B. (1985). The observational recording dilemma. Ottawa (Ontario), Canada: Social Sciences and Humanities Research Council of Canada. (ERIC Document Reproduction Service No. ED 297 021).

Kiesler, C. A., Collins, B. E., & Miller, N. (1969). Attitude change: A critical analysis of theoretical approaches. John Wiley & Sons, Inc..

Lapierre, R. T. (1967). Attitudes versus action. In M. Fishbein (Ed.). Readings in attitude theory and measurement, New York: John Wiley & Sons Inc. (pp.26-31). Reprinted from Social Forces, 13, 230-237, 1934.

Leder, G. C. (1990a). Talking about mathematics. The Australian Researcher, 17(2), 17-27.

Leder, G. C. (1990b). Does teaching equal learning? Keynote address to the 13th Biennial Conference of the Australian Association of Mathematics Teachers. Hobart, Tasmania.

Leder, G. C., & Forgasz, H. J. (December, 1992). Inside the mathematics classroom. Paper presented at AARE/NZARE Joint Conference, Deakin University, Geelong.

Mehan, H. (1979). Learning lessons: Social organization in the classroom. Cambridge, Massachusetts: Harvard University Press.

Peterson, P. L., Wilkinson, L. C., Spinelli, F., & Swing, S. R. (1984). Merging the process-product and the sociolinguistic paradigms: Research on small-group processes. In P. L. Peterson, L. C. Wilkinson, & M. Hallinan (Eds.), The social context of instruction: Group organization and group processes (pp.125-152). Orlando: Academic Press, Inc.

Schoenfeld, A. H. (1992). Learning to think mathematically: Problem solving, metacognition and sense making in mathematics. In D. A. Grouws (Ed.), Handbook of research on mathematics teaching and learning (pp.334-370). New York: Macmillan Publishing Company.

COMMUNICATING MATHEMATICS: HOW LANGUAGE FORMS MAKE IN/ACCESSIBLE MATHEMATICALLY IN/APPROPRIATE CALCULUS CONCEPTUALIZATIONS

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The study reported here was designed to investigate student learning in calculus with a focus on language use and the ways truth and validity are determined. Results reported here are those related to students' processes of construction of particular mathematics conceptualizations as a result of exposure to three different approaches to calculus instruction: technique-oriented, concepts-first and infinitesimal instruction. When students used infinitesimal language and used it in conjunction with everyday language they generally did so as a foundation by which to construct mathematically valid problem responses. This finding indicates that instruction emphasising connections between everyday and technical language is likely to guide students to build mathematically appropriate inter-connected conceptualizations. Also, the use of infinite magnification in a variety of problem situations by students who received infinitesimal instruction demonstrates that instruction emphasising visual interpretations can influence students' conceptualizations.

A whole body of research in mathematics education in the last decade has focussed on students' interpretations of mathematical concepts and processes. This has included documenting how "students bring pre-mathematical experiences into the classroom which affect their understandings of the mathematics" (Tall, 1990; p.49). Researchers have found that students hold "mini-theories" about mathematical ideas and that they learn mathematics in ways that are "personally reasonable and sensible" (Confrey, 1992; p.122). That is, students' mathematical models, though not necessarily in congruence with those of a teacher or researcher, are reasonable to themselves. As alternative perspectives these models are viable and legitimate within a certain range of situations and applications. (For reviews of this literature, see Confrey, 1990; Perkins and Simmons, 1987; Driver and Easley, 1978).

Whether calling students' mathematical models personal conceptions, mini-theories, alternative theories, inadequate beliefs or misconceptions, it is evident that "the theories and their forms of argument must be addressed if students are to come to a more acceptable understanding of the concept" (Confrey, 1992; p.121). Consequently, as teachers and researchers attempt to develop learning experiences that will help students construct mathematically appropriate conceptualizations, they must be informed as to how mathematics might be effectively communicated. The inherent ambiguities of communication must be considered, particularly in relation to the symbolic and verbal language forms that are prominent elements of communication within mathematics classrooms. Thus, there is a need for mathematics educators to consider how students' constructive processes and related conceptualizations are mediated by various mathematical symbols, technical and everyday language.

In relation to students' mathematical representations, an area of mathematics education in which researchers have identified a persistent and recurring phenomena is that of calculus learners' "misconceptions" of calculus concepts. The term "misconception" is used here and in the upcoming discussions, rather than alternative conception, personal conception or some other phrase, because it is the term employed most commonly in the research literature. Examples of student misconceptions in calculus have been noted with learners on several continents and across a range of topics. In particular, investigations with calculus students have documented misconceptions present in their understandings of limits, infinity, continuity, tangents, derivatives and integration (for example, Artigue, 1986; Cornu, 1981; Davis and Vinner, 1986; Orton, 1977, 1983a, 1983b; Schwarzenberger and Tall, 1978; Sierpinska, 1987; Tall, 1989; Tall and Vinner, 1981; Williams, 1991). These investigations of student learning in calculus have given insight into students' misconceptions. What is needed now is research into how instruction can better guide and support student learning in calculus. The results of the study reported here have practical implications in relation to this last point.

DESCRIPTION OF THE STUDY

Research Setting

The research was a naturalistic study involving three undergraduate calculus classes located at three different post-secondary institutions in Western Canada. These institutions included a large university and two small private colleges. The course at the university was representative of introductory calculus courses in its content and an emphasis upon learning techniques for differentiation, integration, graphing and problem solving. In comparison, one of the colleges used a "concepts-first" approach to instruction in which concepts are explored intuitively before introduction of their formal definitions and proofs and before skill development is emphasised. The second college used an instructional approach which develops concepts intuitively while using infinitesimal methods related to nonstandard analysis as analytic and computational tools. Infinitesimal methods are the tools by which Newton and Leibniz first developed calculus in the late 1600's.

The best way to demonstrate how an infinitesimal approach to instruction differs from the use of methods in real analysis and in particular from technique-oriented and concepts-first instruction is to provide some specific examples of its use. Two appropriate examples are the following:

(1) Limits and their precise $\varepsilon-\delta$ definition are replaced by the more intuitive notion of "rounding off", denoted by $\sim>$ (an idea students have used since elementary school). (See Robinson (1966) for a complete and mathematically rigorous account of the development of calculus using infinitesimal numbers, called nonstandard analysis).

(2) The derivative is not introduced via rotating secants which in the limit become a tangent line at a point on a graph. Rather, the value of the derivative at a point is the slope of the tangent line at that point (if the tangent line exists). This concept of derivative is introduced after tangent lines (and where they do and do not exist) have been introduced via the intuitive notion of magnification.

Research Methods

Task-based interviews with 17 students were the primary method of inquiry into the nature and role of students' language use. These in depth interviews involved students in oral and written responses to a number of calculus problems focusing on calculus skills and concept interpretations. Students' language use was initially examined on the broad level of an entire problem response. This examination was done by counting occurrences of a student's use of symbols and technical or everyday language terminology that were not given in the problem statement. In relation to symbol use, manipulations or operations with symbols present in the statement of a problem were distinguished from a student's use of a symbolic representation not present in the problem statement. Results of the counts of students' use of symbols, technical and everyday language terms were used to determine the nature of their language use, while the role of their language use was determined from more extensive examination of what they said or wrote and what this language reflected of their calculus conceptualizations. It is this latter extensive examination that is reported in this paper.

The range of instructional settings allowed partial examination of the impact of different approaches to instruction on the nature and role of students' language use, although the small number of students interviewed at each institution did not permit statistical analysis or definitive answers on the effect of instruction upon students' calculus learning. The examination of students' problem responses did however give insight into the potential impact of each instructional approach on all students' learning.

RESULTS AND RELATED DISCUSSION

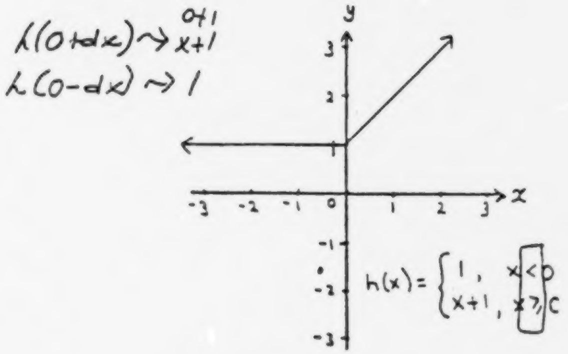
Students' problem responses revealed some important features of their language use, including: (1) conceptualizations built using infinitesimal language displayed features different from conceptualizations built using traditional calculus language, and (2) whether speaking with traditional or infinitesimal language, students used terminology as tools by which to conceptualise their descriptions and explanations and in these constructions pre-calculus language knowledge was prominent. In relation to point (1) it must be noted that students who received infinitesimal instruction, including those students who had studied some calculus previously using traditional language and symbols, used symbols and words particular to infinitesimal calculus. In particular, it will be seen as this discussion proceeds that infinitesimal symbols served students as objects that could be concretely represented on a graph and referred to and used as tools for construction of an explanation or justification.

Although symbols did not form a large component of students' language use, with more than half the students using symbolic representations in one third or less of their problem responses, students who received infinitesimal instruction made more use of symbols. Unlike most of the other students, most of the students who received infinitesimal instruction were able to give symbolic justifications or explanations of continuity and differentiability. Furthermore, the symbols they used and their corresponding verbal language (both everyday and techincal) were particular to an infinitesimal approach to instruction. An exampleof one of these responses is the following:

(Tanya)

(Problem 5)

[5. For each function given below, determine if it is continuous or discontinuous. Give reasons for your answer.]



That any, I'll kind of do it this way. y at x. And these two x's are the same. Ah. If you take any x point and go a little bit to the left or a little bit to the right an infinitesimal amount, it rounds off to y at that x on the y-axis.

Figure 1. Tanya's Response to Problem 5

In this extract Tanya uses infinitesimal language (words and symbols both) to explain the relationship between the behaviour of a graph and the corresponding notion of continuity. In doing so, infinitesimal notation serves as a primary tool for construction of an explanation. It is a key tool in that interpretation of dx as an infinitesimal number provides Tanya with something fairly concrete to work with. She easily visually locates on a graph what dx corresponds to and how the position of dx relates to the behaviour of the graph. The role of her use of infinitesimal notation is both to build and to justify her response.

In relation to technical language use, students who received infinitesimal instruction used technical language to about the same degree as students at the other two institutions, but they used everyday language more. Although this finding distinguishes them from the other students, what distinguishes them more is the content of their technical and everyday language use and the role of this language in describing, explaining or justifying calculus ideas. These features will now be discussed, pointing out the nature and role of language in students' interpretations of calculus problems.

Students who received infinitesimal instruction generally integrated everyday language more with symbols or technical language than did the other students. In comparison, students who received technique-oriented or concepts-first instruction, although they often gave valid explanations of situations using everyday language, did not as frequently use technical terms or symbols for further, more detailed or precise justifications. In particular, unless specifically asked to do so, they did not make use of language and ideas related to limits. There were also occasions when they used technical terms or symbols but were unable to explain their connections to everyday language explanations.

Another aspect of infinitesimal instruction which was displayed in students' problem responses and which appeared as important in these responses was the notion of magnifying a curve. At some point in their interviews all students who received infinitesimal instruction spoke of infinitely "magnifying" or "blowing up" the graph of a function. In an infinitesimal approach to calculus instruction magnification is a means by which a function can be examined "up close". In this process infinitesimal language plays a role in students' interpretations by orienting them to construct descriptions of a magnified curve. Non-infinitesimal language related to the slope of a tangent line also served to orient students to descriptions of a curve. However, these descriptions, justifications and conclusions seldom made use of limit-related language or processes. In comparison, the notion of infinite magnification has limiting processes built into its use. This feature distinguishes it from traditional slope and tangent line notions in more than one way. First, it is a dynamic rather than static method for interpretation of graphs. Second, magnification makes the limit concept of "close to" accessible. That is, the visual mechanism of blowing up or infinitely magnifying a curve serves as a visual, physically accessible means by which to examine related limiting notions.

The traditional limit concept also has visual interpretations but these were not regularly used by students who received technique-oriented or concepts-first instruction. In fact, the general absence of use of limit notation or terminology by students who received these approaches to instruction, unless it was specifically requested, indicates they did not integrate their limit conceptualizations into other calculus conceptualizations. For example, their responses included explanation of the derivative as the limit of slopes of a sequence of secant lines, but the relationship of limits and derivatives was then not applied in other problem responses. Use of the notion of magnification was more regularly applied by students as a tool by which to construct calculus conceptualizations.

In particular, this study found differences in the nature of student problem responses that related to whether or not they made use of infinitesimal numbers or the notion of infinite magnification. A feature of the problem responses of students who received infinitesimal instruction was that when they used magnification and related terminology they did not construct the same misconceptions present in problem responses of students who did not use infinitesimal terminology (including incidents when students who received infinitesimal instruction did not use infinitesimal terminology). For example, students who did not use infinitesimal language tended to interpret the technical language term "continuous" using everyday language phrases such as "no breaks", "no jumps", "existing", "being defined" and "not changing". Many of their notions associated with these everyday language phrases were valid interpretations of situations, although they were not necessarily valid mathematical interpretations. The interpretations therefore tended to guide students to construct mathematically incorrect justifications or justifications that were used inconsistently. For example, interpretation of the technical language term "continuous" in terms of "existing" led students to construct mathematically incorrect justifications. Further, although interpretation of "continuous" as "no breaks" usually oriented students to mathematically correct notions related to continuity, it did not do so for all of them. For example, Doug believed that a "break" in the way a function is defined constitutes a discontinuity.

Another misconception displayed by students who did not use the notion of magnification was that non-uniqueness rather than non-existence of a tangent line implies non-differentiability. For example, students said such things as:

(Cindy)
(Problem 9)
[9. The graph of y = F(x) is given below. At which points does the function not have a derivative? Why?]
Because this is undefined [at a cusp]. Because a derivative means you're taking the slope of a tangent. But the tangent, it could be here, it could be here, it could be here. It could be anywhere. And we don't know where it is.

(Annabel)
(Problem 9)
Derivative is suppose to ah, on a graph the derivative is supposed to be a tangent line that touches the graph at only one spot. And at a sharp point or an endpoint there it touches it, it can do that in many different places. So you cannot define any one derivative.

The students to whom the above interview excerpts belong gave the correct conclusion that no derivative existed at a particular point, but their justifications for nondifferentiability were mathematically incorrect. In comparison, students who employed the notion of magnification to determine differentiability said such things as:

(Gordon)
(Problem 9)
The line has to be continuous. So you wouldn't have one [a tangent] at the endpoint. [pause] If you blow that up, infinitesimally you still have that. You can't draw a tangent to that. Then you can't have a derivative.

(Tanya)
(Problem 9)
Right at this point if you magnify it. You're magnifying the point and you still have a straight line. In order to have a derivative you need a line. You don't need a point and a line to the left or right of it. You need a line where you can draw a tangent line and a slope to it. Here, like I said, a derivative just to the right of it exists [at a point of discontinuity]. Left, sorry. Just to the left it exists. Infinitesimally. Right at that point it doesn't exist.

(Nadine)
(Problem 9)
. . . you take the point and you blow it up an infinite amount. And if you see a straight line there's a derivative. . . . You'll still see this. You blow it up and you'll still see a V [at a cusp]. And at this point there is no derivative.

The above excerpts show how magnification of a curve generally served to focus students' perceptions and subsequent justifications upon non-existence rather than non-uniqueness of a tangent line.

In summary, students who received infinitesimal instruction used symbols and words particular to infinitesimal calculus as primary tools to explain or justify calculus ideas. They used infinitesimal language and related visual notions such as infinitesimal closeness and infinite magnification so that infinitesimal symbols served them as objects that could be concretely represented on a graph and referred to in construction of an explanation or justification. Students who used infinitesimal language, more frequently than the other students, appropriately integrated technical and everyday language and did not display the same misconceptions present in other students' responses. All students often gave valid physical interpretations for visually oriented notions such as continuity, slope or size, but students who used infinitesimal language had both the tools and mechanisms by which to make different constructions that did not lead to the same misconceptions. In this way, since students who used infinitesimal language did so as a means to regulate their actions, infinitesimal language made accessible mathematically appropriate thought.

IMPLICATIONS

The implications for calculus instruction of this study's findings are twofold:

(1) When students used infinitesimal language and used it in conjunction with everyday language they generally did so as a foundation by which to construct mathematically valid problem responses. This finding indicates that instruction emphasising connections between everyday and technical language is likely to guide students to build mathematically appropriate inter-connected conceptualizations.

(2) The use of infinite magnification in a variety of problem situations by students who received infinitesimal instruction demonstrates that instruction emphasising visual interpretations can influence students' conceptualizations. It can guide students to use physical, bodily experiences as a means by which to articulate abstract ideas and construct related conceptualizations. Most importantly, construction of calculus meanings from the physical experiences of magnification and closeness did not make accessible to students the misconceptions present in the conceptualizations of students who had received technique-oriented or concepts-first instruction. That is, use of infinitesimal language did not facilitate expression of mathematically inappropriate or inaccurate conceptualizations.

The implications of these points for mathematics education at all levels, not just for the teaching and learning of calculus, are both relevant and useful. Practical use can be made of the notion that students can be guided to build mathematically appropriate inter-connected conceptualizations by instruction emphasising connections between everyday and technical language. Specifically, the design and implementation of instruction should provide students with opportunities to construct conceptualizations through elaboration of their everyday language meanings. Use of everyday language interpretations to describe, explain and justify particular mathematical situations could then be connected to more abstract symbolic mathematical representations.

More importantly, practical use can be made of the notion arising from this study that students will model mathematical concepts using language forms that reflect their instructional experiences. In particular, mathematics teaching at all levels should consider that some instructional language forms and related experiences are more conducive to facilitating students' construction of mathematically correct or appropriate thought, while rendering inaccessible certain misconceptions. That is, it is possible to design instructional experiences which are language based and which do not provide students with the vocabulary to frame certain misconceptions. The experiences and related vocabulary, by inhibiting certain misconceptions, facilitate construction of more mathematically appropriate conceptualizations.

REFERENCES

Artique, M. (1986). The notion of differential for undergraduate students in the sciences. In *Proceedings of the Tenth International Conference of Psychology of Mathematics Education*, University of London, London, 229-234.

Confrey, J. (1990). A review of research on student conceptions in mathematics, science and programming. In Cazden, C. (Ed.). *Review of Research in Education*, 16, American Education Research Association, Washington, D.C., 3-56.

Confrey, J. (1992). Learning to listen: a student's understanding of powers of ten. In von Glasersfeld, E. (Ed.). *Radical Constructivism in Mathematics Education*. Kluwer Academic Publishers, Dortrecht.

Cornu, B. (1981). Apprentissage de la notion de limite: modeles spontanes et modeles propres. In *Actes due Cinquieme Colloque du Groupe International P.M.E.*, Grenoble, 322-326.

Davis, R. & Vinner, S. (1986). The notion of limit: some seemingly unavoidable misconception stages. *Journal of Mathematical Behavior*, 5, 3, 281-303.

Driver, R. and Easley, J. (1978). Pupils and paradigms: a review of literature related to concept development in adolescent science students. *Studies in Science Education*, 5, 61-64.

Orton, A. (1977). Chords, secants, tangents and elementary calculus. *Mathematics Teaching*, 78, 48-49.

Orton, A. (1983a). Students' understanding of integration. *Educational Studies in Mathematics*, 14, 1, 1-18.

Orton, A. (1983b). Students' understanding of integration. *Educational Studies in Mathematics*, 14, 3, 235-250.

Perkins, D. and Simmons, R. (1987). Patterns of misunderstanding. An integrative model of misconceptions in science, math, and programming. In Novak, J. (Ed.). *Proceedings of the Second International Seminar on Misconceptions and Educational Strategies in Science and Mathematics,* Ithaca, 381-395.

Schwarzenberger, R. and Tall, D. (1978). Conflicts in the learning of real numbers and limits. *Mathematics Teaching,* 82, 44-49.

Sierpinska, A. (1987). Humanities students and epistemological obstacles related to limits. *Educational Studies in Mathematics,* 18, 371-397.

Tall, D. (1989). Concept images, generic organizers, computers, and curriculum change. *For the Learning of Mathematics,* 9, 3, 37-42.

Tall, D. (1990). Inconsistencies in the learning of calculus and analysis. *Focus on Learning Problems in Mathematics,* 12, 49-63.

Tall, D. & Vinner, S. (1981). Concept image and concept definition in mathematics with particular reference to limits and continuity. *Educational Studies in Mathematics,* 12, 2, 151-169.

Williams, S.R. (1991). Models of limit held by college calculus students. *Journal for Research in Mathematics Education,* 22, 3, 219-236.

THE PROFESSION, THE PUBLIC, AND SCHOOL MATHEMATICS

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Shared meanings and constructions are necessary if purposes embedded in documents such as the National Statement on Mathematics are to be realised. One starting point is to establish where the profession and the public stand in relation to values and goals.

This paper builds on a previous study and reports on perceptions of the role and purpose of school mathematics as obtained respectively from samples of community members ($N=662$) and mathematics teachers ($N=97$). Viewpoints were sought by structured interview or questionnaire across the thematic areas of content, individual values, additudinal aspects, and folklore.

The data from the study suggest that public and professional views in relation to school mathematics have more in common with the competency movement than with the broader ideals of mathematics education.

BACKGROUND

The publication of the National Statement on Mathematics for Australian Schools, Australian Education Council (1991) provided guidelines for the learning and teaching of mathematics across a variety of content, emphases, and approaches.

Subsequent publication of The Finn Review (1991), Mayer Committee Report (1992), and the Carmichael Report (1992) has moved the national agenda firmly in the direction of competency based training. Intersections between the concepts of education in mathematics as training for vocational needs, and as a broad and rich intellectual enterprise carry inevitable tensions.

Three major players can be identified in the present climate. These are firstly government agencies which may be taken to include the concerns of business and industry in a current climate of economic rationalism.

Secondly there is the general public who provide the consumers of education in all its forms. Thirdly there is the education profession and in particular, in relation to this paper, the mathematics teaching fraternity.

One perpetual challenge to curriculum change is the move from rhetoric to implementation. The change literature is replete with examples of programs that have not achieved their goals. Reports are compiled and curricula written by highly committed professionals working from a superior knowledge base and vision. However recommendations must be realised in the real world - the world of the ordinary citizen and the average teacher; not the rarefied world of expert knowledge, total commitment, and high professional skill.

The knowledge, attitudes, and values of the community as employers, workers, parents, students, taxpayers, and consumers defines the context in which educational change is sought.

What values does the community have in regard to its expectations and perceptions of school mathematics? What perceptions do teachers of mathematics have regarding the public image of school mathematics and its role and purpose?

Answers to questions such as these are needed if a shared construction of the role and purpose of school mathematics is to be achieved. Without a shared construction initiatives for change, from whatever perspective, will have their energy dissipated.

Among the recent efforts to relate aims and achievements in mathematics education to societal contexts and national needs, factors such as gender differences, problems of motivation and attitude, the changing needs of the workforce, and the influence of government, receive continued attention.

The magnitude of the issues involved are inherent in the stimulating dialogue between Apple and Romberg (Apple, 1992 (a) and (b); Romberg, 1992). The substance of this paper addresses but one aspect of the challenge.

The aim of the study is to identify commonalities and differences between community perceptions of the role and purpose of school mathematics, and the corresponding perceptions of practising mathematics teachers.

METHODOLOGY

The paper extends an earlier study, Galbraith and Chant (1990). Further details of the rationale and an extensive literature base are included in that paper.

Influences identified and elaborated from the literature were classified into four areas according to the following criteria.

Content oriented items (C) either required a mathematical response, or involved a perception of what is important or typical about mathematics.

Value oriented items (V) sought views about what individuals believed concerning the role of mathematics, about desirable emphases in teaching, or about its importance.

Attitude oriented items (A) focussed on aspects actually experienced by individuals in their own learning of mathematics; these included perceptions of bias, personal attitudes within themselves, and related perceptions in respect of other students.

Folklore items (F) were chosen to sample popular beliefs that have typically appeared in the public media - both historical and emerging, and/or are a part of traditional views expressed about mathematics.

The items were presented in the form of a structured questionnaire which required a mix of likert type responses, open ended responses, and qualitative comment.

In all instances members of the community sample responded in terms of their own individual positions. For the teacher sample this was also the major form of response. However in some cases the teachers were asked for their perceptions of community knowledge or views. Such items are marked with an asterisk in Table 1.

Community Sample

As described in the earlier paper community responses were obtained from approximately 660 members of the public. The sample was representative, although not random, and was stratified by gender (females (334), males (326)); and by age (21-30 years (226), 31-45 years (222), above 45 years (212)). Maximum educational levels were represented respectively as (years 9-10 (272), years 11-12 (146), tertiary (240), unknown (2)). The main occupational categories were distributed as follows: (professional (145), technical/clerical (149), tradespeople/skilled workers (109), home duties (122), student (84), miscellaneous (54)).

The distribution of occupations identifies the sample as a representative urban Australian one, with a slight bias towards the professional category. If anything this may tend to increase the number valuing mathematics relative to the population as a whole.

Teacher Sample

The teacher sample was drawn from practice teaching schools used in the Diploma in Education program. Eleven state high schools (62 teachers), and nine independent schools (35 teachers), participated in the study. The sample was chosen to be representative with respect to gender (females (45), males (52)); responsibility (subject head/supervising teacher (54), neither (38), undefined (5)); and teaching experience (0-5 years (15), 6-10 years (15), 11-20 years (43), above 20 years (24)).

RESULTS

Selections from the responses are summarised and compared in Table 1. They have been selected as representative of the four thematic areas and chosen for their interest in relation to the purpose of the study.

TABLE 1

Percentage Distribution of 'Content' oriented items

C1.*	What meaning do the following statements have for you?		Correct	Incorrect/ Don't know			
	(a) pi	P	50.94	49.06			
	$\chi^2 = 25.34$ (n = 432, df = 1, $p < .001$)	T	30.56	69.44			
	(b) triangle with two equal sides	P	47.61	52.39			
	$\chi^2 = 15.36$ (n = 377, df = 1, $p < .001$)	T	47.22	52.78			
	(c) ½ x base x altitude	P	44.49	55.51			
	$\chi^2 = 6.10$ (n = 450, df = 1, $p = .01$)	T	72.22	27.78			
	(d) opposite over adjacent	P	17.05	82.95			
	$\chi^2 = 45.64$ (n = 228, df = 1, $p < .001$)	T	2.78	97.22			
C2.	It was enough to get the right answer at school; you didn't have to understand why.		SA	A	N	D	SD
		P	8.11	30.35	4.16	32.85	24.53
	$\chi^2 = 51.26$ (df = 4, $p < .001$)	T	0.00	0.00	5.56	37.50	56.94
C3.*	What math do you use in your daily life? (multiple responses accepted)		Arithmetic	Algebra	Geometry	Computing/ statistics	Other
			70.65	5.92	7.63	10.68	5.12
			88.26	0.76	1.89	3.03	6.06
C4.	What math should all pupils be able to do on leaving school? (multiple responses accepted)		Arithmetic	Algebra	Geometry	Computing/ statistics	Other
		P	59.16	9.98	9.66	15.51	5.69
		T	77.51	4.07	7.32	4.88	6.23

Percentage Distribution of 'Values' oriented items

			SA	A	N	D	SD
V1	It is more important to be able to solve problems than to do routine calculations. $\chi^2 = 17.27$ (df = 4, p = .002)	P	20.58	39.09	10.19	24.74	5.41
		T	9.72	38.89	25.00	25.00	1.39
			SA	A	N	D	SD
V2	All mathematics subjects taught at school should be useful to students in their lives. $\chi^2 = 34.94$ (df = 4, p < .001)	P	29.52	33.26	3.95	28.69	4.57
		T	6.94	20.83	8.33	51.39	12.50
			SA	A	N	D	SD
V3	Industry is entitled to expect school leavers to be able to do job calculations. $\chi^2 = 15.46$ (df = 4, p = .004)	P	41.37	45.32	4.16	8.52	0.62
		T	19.44	59.72	9.72	11.11	0.00
			SA	A	N	D	SD
V4	It is more important for boys to go on in mathematics than it is for girls. $\chi^2 = 19.10$ (df = 4, p = .001)	P	2.29	10.81	7.28	35.76	43.87
		T	0.00	1.39	5.56	23.61	69.44
			SA	A	N	D	SD
V5	People use school mathematics marks to decide how smart someone is. $\chi^2 = 13.44$ (df = 4, p = .01)	P	14.97	54.68	7.48	17.88	4.99
		T	4.17	75.00	8.33	11.11	1.39

Percentage Distribution of 'Attitude' oriented items

			Yes	No	Inapplicable
A1	Do you recall sex bias being shown by your teachers during mathematics lessons? $\chi^2 = 1.08$ (df = 2, p = .58)	P	17.26	53.64	29.11
		T	22.22	51.39	26.39
			Yes	No	
A2	Do you recall bias shown in maths lessons for or against pupils of different abilities? $\chi^2 = 3.41$ (df = 1, p = .07)	P	55.09	44.91	
		T	66.67	33.33	
			Positive	Negative	
A3*	What feelings come to mind when you recall school mathematics *lessons*? (multiple responses)	P	42.71	57.29	
		T	18.10	81.90	

A4*	What feelings come to mind when you recall school mathematics *teachers*? (multiple responses)		Positive	Negative			
		P	50.83	49.17			
		T	35.93	64.07			
A5	It is important that students enjoy mathematics.		SA	A	N	D	SD
		P	38.88	46.78	7.90	5.61	0.83
	$\chi^2 = 4.60$ (df = 4, p = .33)	T	30.56	59.72	5.56	4.17	0.00

Percentage Distribution of 'Folklore' items

F1	Mathematics is a subject that you can either do or you can't.		SA	A	N	D	SD
		P	13.51	32.85	6.65	36.80	10.19
	$\chi^2 = 11.13$ (df = 4, p = .03)	T	4.17	29.17	15.28	41.67	9.72
F2	Mathematics should be compulsory because it is good training for the mind.		SA	A	N	D	SD
		P	20.58	41.58	11.85	22.45	3.53
	$\chi^2 = 4.33$ (df = 4, p = .36)	T	13.89	40.28	19.44	22.22	4.17
F3	Do you think that the mathematics of school leavers is better today than it was in the past?		A lot better	Better	The Same	Worse	A Lot Worse
		P	5.82	29.11	28.07	26.61	10.40
	$\chi^2 = 18.56$ (df = 4, p = .001)	T	0.00	23.61	44.44	31.94	0.00
F4	Allowing the use of electronic calculators will lead to students becoming poorer at mathematics.		SA	A	N	D	SD
		P	22.45	36.17	11.23	23.70	6.44
	$\chi^2 = 63.14$ (df = 4, p < .001)	T	2.78	13.89	6.94	52.78	23.61
F5	The use of computers means that mathematics as we know it will disappear.		SA	A	N	D	SD
		P	8.73	29.11	13.10	34.10	14.97
	$\chi^2 = 73.73$ (df = 4, p < .001)	T	0.00	1.39	0.00	51.39	47.22

Discussion

Responses to the *content* items suggest that basic results, several at the level of general knowledge, are not well remembered by members of the public. It is also interesting that teachers do not expect them to be. Does this mean that much of the

DISCUSSION

Responses to the *content* items suggest that basic results, several at the level of general knowledge, are not well remembered by members of the public. It is also interesting that teachers do not expect them to be. Does this mean that much of the material taught at school is not seen as relevant to ordinary citizens, even when it falls within the scope of what may reasonably be called basic mathematical knowledge?

Both the community and the teachers have an overwhelming view of mathematics for the majority as arithmetic, with this view more evident in the profession than among the public sample.

The *value* items provide interesting response patterns across the groups. There is a higher valuing of problem solving among the community members than among the teacher respondents, the former group strongly values utilitarian aspects of content where teachers do not, both groups agree that school mathematics should prepare students for the workforce, that mathematics is important for the prospects of both females and males, and that mathematics performance is used as a proxy for general intelligence.

With respect to the *attitude* items, the perceptions of bias based on ability is much more evident among the public respondents than bias based on gender. The teachers also agree that ability bias is the more prevalent. While the public is somewhat negative in their recollection of school mathematics lessons they are equally balanced in relation to their teachers. By contrast the teacher group is very negative in their perception of public views in relation to themselves and their subject. The self image appears decidedly low. However both groups of respondents agree that mathematics should be an enjoyable subject.

Finally the *folklore* items focus attention upon some persistent beliefs. There is substantial fatalism in the community that mathematics is a subject that a person can either do or not do. The teacher group is less strongly inclined although one third subscribe to this belief.

Support for the concept of mental discipline lingers on with over sixty percent of the community, and more than fifty percent of the teachers, believing in compulsory mathematics as training for the mind. The community and the profession are both divided as to whether current school leavers are better equipped mathematically than in the past, but the teacher group has a more positive expectation of technology in relation to the learning of mathematics than the general public.

In relation to future intentions and potential developments as promulgated in publications such as the National Statement and the Finn, Mayer, and Carmichael reports, the following observations may be made.

There is an overwhelming propensity among both the public and the teaching profession to view school mathematics primarily as arithmetic, both in terms of what is learned and what is useful. This appears to present a fertile environment for the advance of a competency based emphasis, less so for the higher ideals embedded in the National Statement.

Further, if the teacher sample can indeed be generalised, the profession as a whole has a low self image. It does not expect its students to remember what has been taught, nor to harbour positive feelings about the subject or their teachers.

A substantial amount of fatalism resides both in the community and with the teachers. Mathematics is still viewed as a subject beyond the reach of many, and as a proxy means of labelling ability rather than as an enriching study. There is, however, a positive view among mathematics teachers that educational technology has a productive future in enhancing the teaching and learning of the subject.

This paper set out to explore an aspect of the context of school mathematics believed to be an important component in the future of curriculum and attitude change. Based on the restricted samples in this study it would be risky to forecast unbridled joy for those with a vision of a future nirvana for school mathematics. The evidence we have obtained, suggests levels of understanding and value in the community, and in the profession, that have more in common with the competency reviews than with the higher ideals of the National Statement.

REFERENCES

Apple, M.W. (1992a). Do the standards go far enough? Power, policy, and practice in Mathematics Education. *Journal for Research in Mathematics Education*, 23(5), 412-431.

Apple, M.W. (1992b). Thinking more politically about the challenges before us: a response to Romberg. *Journal for Research in Mathematics Education*, 23(5), 438-440.

Australian Education Council and Curriculum Corporation (1991). *A National Statement on Mathematics for Australian Schools*. Carlton, Vic: Curriculum Corporation.

Australian Education Council Review Committee (Chair: Finn, B.) (1991). *Young People's participation in post-compulsory education and training*. Canberra: Australian Government Publishing Service.

Employment and Skills Formation Council (1992). *The Australian Vocational Certificate Training System*. Canberra: National Bureau of Employment, Education, and Training.

Galbraith, P.L. and Chant, D. (1990). Factors shaping community attitudes to school mathematics: implications for future curriculum change. *Educational Studies in Mathematics*, 21, 299-318.

Romberg, T.A. (1992). Further thoughts on the standards: a reaction to Apple. *Journal for Research in Mathematics Education*, 23(5), 432-437.

The Mayer Committee (1992). *Employment-related key competencies: a proposal for consultation*. Melbourne: Mayer Committee.

AN EXPLORATION OF THE DIFFERENT STRUCTURE OF THE MATHEMATICS REGISTER IN ENGLISH AND ASIAN LANGUAGES: SOME CONSEQUENCES FOR THE TEACHING OF ESL IN PREPARATORY PROGRAMMES.

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This study investigates the linguistic problems ESL post-secondary students have when communicating through the content area of mathematics. The majority of these students have the mathematics register in their own tongue but need to transfer this to English so they can communicate effectively in tutorials, lectures, practicals and assignments.

To investigate these linguistic problems, students from a pre-tertiary (UNIPREP) course were interviewed and their mathematics pretests and the problems from mathematics vocabulary tests were analysed.

Findings from the pretests suggest there are a number of levels of understanding and types of methods used by students in solving the problem. The results of the vocabulary study outline the frequency of occurrence of the types of mistakes students make. Preliminary findings from student interviews and research suggest that while some of the mathematics learnt through their native tongue is easily translatable into English other features are more complex and further research needs to be undertaken in this area. The overall results of this paper clarify the problems faced by ESL students and can be used for ESL educators in their curriculum design and assessment.

Non-English speaking overseas students undertaking tertiary studies in English speaking Universities may have what is described as cognitive academic language proficiency (CALP) (Cummins, in Dale and Cuevas 1987) in their native tongue, but do not possess this in English. Moreover, they may have the English skills to communicate effectively for social purposes, but when they are faced with precise academic language, their ability to communicate effectively is reduced.

In mathematics, which is rich in cognitive academic language, students may have the language skills in their native tongue but there are translation difficulties due to the precise nature of the mathematics language (i.e. the mathematics register). It is not just the individual words in mathematics, but the semantic, syntactic and pragmatic features which create problems for the ESL student (Spanos, Rhodes, Dale and Crandall, 1991). ESL students' level of mathematical thinking may help to overcome some of the linguistic problems. These have been categorised by Kessler (1987) (from Schoenfeld (1985)): (i) the cognitive resources students already have, (ii) the heuristics the students use to help solve the problems (iii) the control processes or metacognitive resources students may be able to utilise and (iv) the belief systems students bring with them.

Much research has been conducted in primary and secondary education in the areas of language and mathematics for ESL students (McGregor 1990, Cuevas 1984, Dale and Cuevas 1987). However, limited research has been conducted in the area of post-secondary education. For the majority of students who have the mathematics register in their own tongue, the problem is to develop their abilities to the extent that they can communicate effectively in tutorials, lectures, practicals and assignments.

This topic will investigate the problems ESL students have and the thinking students use to overcome the problem when communicating through the content area of mathematics. This includes an investigation into some of the typology of the mathematics language i.e. a comparison of the mathematics register in English and other Asian languages (Chinese, Korean and Japanese). The data were collected through (A) interviews (B) pretest results and(C) vocabulary quiz mistakes.

THE SUBJECTS

All students used in the testing were overseas students who had either completed an ELICOS programme at university or had an equivalent English standard, and were enrolled in a preparatory programme.

THE INSTRUMENTS

The 1993, 13 Semester 1 ESL students sat for a pretest (designed to identify the mathematical mastery of any prospective student). Students were asked to circle words or phrases that they did not understand. As well some students were interviewed on their maths pretest ensuring a more complete picture of the linguistic difficulties. Students must pass nine vocabulary and mathematics mastery quizzes during the semester (Arithmetic (3), Algebra (2), and (1 each) Measurement, Graphing, Probability and Statistics). Each student sits with the tutor when the quizzes are marked and the student is expected to talk through each problem with the tutor. This process, together with extensive individual interviewing with a Korean, a Tiawanese and a Japanese student clarified the extent of the linguistic transfer difficulties. As well, the most common mistakes on the vocabulary quizzes for the previous three semesters were summarised.

THE RESULTS

For the pretest (See Appendix A) each question for each student was analysed and categorised thus:

		ENGLISH	
		UNDERSTOOD	NOT UNDERSTOOD
MATHS	INCORRECT	E_uM_i	E_nM_i*
	CORRECT	E_uM_c	E_nM_c

*some students picked up the right cues but the maths was incorrect; others picked the wrong cues and the maths was correct.

A number of questions from the test can be used to exemplify the mathematics register categories used by Spanos et. al. (1991) and the mathematical thinking processes outlined by Kessler (1987).

While generally students were able to solve questions involving syntax such as 4d (Appendix 1), they had more problems with the Semitics, e.g. where new meanings were placed on words like 'rounding', 'subject' and 'sum of'. Students also displayed lack of cognitive resources e.g. in Q9 and 11b the concept was new and in Q3 and 10b the process had been forgotten.

If students had the cognitive resources, then questions such as 2d posed few problems. If they didn't know the word "simplify" or "ratio", but understood the symbol 24:14, they knew its alternative form was 12:7. However, using these mathematical heuristics for problem 1f led students to give the exact arithmetic answer. However, to solve the problem they must understand the word 'estimate' and probably 'rounding'. For ESL students with an ordinary translation dictionary this would not be difficult since estimate is a natural word and rounding usually has a mathematical translation in the dictionary. Question 5b. poses more problems since knowing 'rearrange' and 'formula' sometimes gives a solution $3 + y - 5x - 4 = 0$. Although some metacogintive "control" process intuitively allow students to rearrange the formula a specific knowledge of "x...the subject" is usually necessary.

The pragmatic and discourse features of the mathematics could be found in the last part of question 11b but as students did not understand scattergram, parts (ii) and (iii) were generally not attempted. Students who did attempt part (b) took the cues from the data and attempted to draw a column or a trend graph.

The vocabulary tests completed by the students emphasised the common problems faced by ESL students. These could be divided into 10 main types (Table 1). Given the nature of their languages, this would be expected. For example, it is possible to say the phrase 3 x 7 in Chinese, Japanese and Korean in more than one way, but in Mathematics it is usually expressed one way only. While there is extensive use of prepositions in these languages it is not as common in mathematics. For example in Japanese they may say Ju o ni de waru (10 with two divide) it is more common to say Ju waru ni (10 divide two). In Mandarin they usually say 7 gen 4 (7 take 4) not 4 bay 7 gen (4 is from 7 take). Mandarin especially does not have plurals or suffixes, tenses, or conjugations and sometimes does not use verbs (see Newnham 1987). Hence the large proportion of mistakes seen in the vocabulary tests. Some interesting findings from the interviews with students include the literal translation of the vocabulary. While our derivation of the word parabola, quadrilateral or rhombus come from Greek morphemes or roots, their meaning is not widely known, but the word throw something up, four corner shape or stage shape conveys a meaning. In scientific language this is more dramatic - a sphygmomanometer would be much better described as a blood pressure calculator as is the Chinese translation. A general detailed analysis of the typology of Mandarin can be found in Li and Thompson (1981) and a more detailed analysis in the mathematical field is currently being investigated.

TABLE 1: MOST COMMON MISTAKES ON VOCABULARY TESTS

	percent of all errors
(i) not putting the correct ending on the word, e.g. equal(s), round(ed), fifth(s), factor(s);	24%
(ii) not putting the correct preposition and the ending, e.g. multipli(ed by) , add(ed to);	23%
(iii)not including the preposition of, e.g. inverse (of), likelihood (of), square root (of)	5%
(iv) misuse of the preposition in, as, on, with, etc.;	5%
(v) misplacement of words eg for $-\sqrt{2}$ or $\sqrt{-2}$;	4%
(vi) misuse of the articles 'the' and 'a';	8%
(vii)problem with comparasion, e.g.as big as; is greater than;	10%
(viii) new technical vocabulary;	9%
(ix) discourse features, e.g. describing power; statements with brackets;	9%*
(x) saying numbers correctly, e.g. seven hundred (and) five.	3%

* as well, most students has some problems forming sentences e.g. with words such as sixth, sixths and sixes.

DISCUSSION

The preconception in the past that mathematics is relatively language free is a myth. In the contextually reduced realm of much of the mathematics, language can be merely an unimportant embellishment to the mathematics problem, but often single words or phrases can cause confusion or misinterpretation. This could be due to some of the typology of many Asian languages. Sometimes slight rewording can improve the interpretation of the problem for the ESL student, or allowing a student to use an ordinary translation dictionary. While English is rich in lexical alternatives, it does not become a problem for understanding unless the key words are not the main indicators of the problem.

A further problem arises when there are many new technical words to be learnt which are different in their component parts (or morphemes) from English. Mandarin generally has fewer morphemes than most other languages, in the realm of mathematics new words have been developed combining morphemes. While English uses Greek or Latin morphemes, Chinese have used current morphemes to form these new words, hence making the new word more practically understandable. However, as mathematics becomes more complex English words are often used to explain the idea or phenomena - this is more prevalent in Japanese (Fujii, 1991), e.g. the word percento. More research is being undertaken to investigate the effect of this more natural use of language to help in the understanding of mathematics.

Other research (as in McGregor and Moore 1991) has outlined some of the problems faced by the ESL student when learning mathematics in English. While many of the problems may be the same pre-tertiary students' understanding is much greater because of the greater levels of mathematical thinking they have at their disposal. Their ability to communicate is greater, but it is often unclear and stifled and not of an acceptable university standard. If ESL teachers are aware of the common mistakes students make and why they make them, they can incorporate more suitable strategies into a structured course (Galligan and Surman 1992) which will provide a suitable environment for ESL students to improve their communication skills at tertiary level.

Notes

1 Japanese, Korean and Chinese have the same regular word structure for numbers (Dunkin and Shire, p212)

2 ELICOS = English Language Intensive Course for Overseas Students

REFERENCES

Cuevas G. (1984) "Mathematics Learning and English as a Second Language" Journal for Research in Mathematics Education V.15 No.2 March pp 134-144.

Dale T. and Cuevas G. (1987) "Integrating Language and Mathematics Learning" in Crandall et al. (eds.) ESL through content area instruction: Mathematics Science and Social Studies Language in Education: Theory and Practice No 19 (Prentice Hall N.Y.)

Durkin K and Shire B. (eds.) (1991) Langauage in MAthematical Education Open University Press Milton Keynes Philadelphia

Fujii, Noriko (1991) Historical Discourse Analysis of Grammatical Subject in Japanese. Mouton de Gruyter.

Galligan L. and Surman (1992) "Developing Communication Skills in Mathematics Using Problem Solving, Group Work and Individualized Learning" Paper presented at the Asian Perspectives in Mathematics Education COnference, University of Western Sydney, July.

Kessler C. (1987) "Linking Mathematics and Second Language Teaching: Paper presented at the 21st Annual Convention of Teachers of English to Speakers of Other Languages, Miami Beach, Florida, April 21-27

Li C. and Thompson S. (1981) Mandarin Chinese: A Functional Reference Grammar University of California Press, Berkley.

McGregor M. and Moore R. (1991) Teaching Mathematics in the Multicultural Classroom University of Melbourne.

McGregor M. (1990) "Reading and Writing in Mathematics" in Bickmore-Brand (ed) Language in Mathematics (Carlton South, Vic Australian Reading Association) pp 100-108.

Newnham R. (1987) About Chinese Penguin.

Spanos G. Rhodes N. Dale T. and Crandall J. (1991) "Linguistic Features of Mathematical Problem Solving" in Cocking R. and Mestre J. Linguistic and Cultural Influences on Learning Mthematics. Hillsdale N.J. Lawrence Erlbaum.

APPENDIX A

DIAGNOSTIC TEST

	E_nM_1	E_nM_o	E_uM_1	E_uM_o	T
Ques.1					
(a) Place these numbers in order from the smallest to the biggest 3 -2 1.5 -21/4 $\sqrt{5}$ $-\sqrt{2}$	3	5	2	3	13
(b) Give three values of p which satisfy the inequality $p \geq -5$	1	7	2	2	12
(c) Round 3256 to the nearest 100	4	1	2	5	12
(d) Round -0.0652 to its leading digit	5	1	1	4	11
(e) Complete 24+☐ = -6	-	1	-	12	13
(f) Estimate 56+23x9246+125 by using appropriate rounding	12	-	-	1	13
Ques.2					
(a) Express 5/6 as a fraction equivalent with a denominator of 24	3	2	3	2	10
(b) express 1/8 as a decimal	3	1	1	7	12
(c) Find 8% of 330ml	-	-	3	9	12
(d) Simplify the ratio 24:14	1	2	2	8	13
(e) Simplify 1/4- 5/6+3/4-5/2x4/3	-	-	8	5	13
Ques.3					
(a) Evaluate $27^{5/3}$	3	-	7	1	11
(b) Simplify $(-2)^5 \times (-2)^6$	-	-	7	5	12
(c) Simplify -3(2-8)	-	-	1	12	13
(d) Evaluate $\{20-3[3+(12/4)^2]\}^3$	-	-	6	7	13
(e) Evaluate 2^0	-	-	3	10	13
Ques.4					
(a) Simplify $2x^2+5x^{-1}+2y^2+12x^2-4x^{-1}$	-	-	6	7	13
(b) Simplify $\frac{4(ab)^3b^{-1}}{2ab^4}$	-	-	3	9	12
	-	-	3	9	12
(c) Simplify $(-3x^2+4)(3-2x)$	-	-	4	8	12
(e) If $z=3x^2+2xy-y^3$, find z when $x=2$ and $y=-1$	-	-	3	10	13
Ques.5					
(a) Solve $7x+6=9x-5$	-	-	5	8	13
(b)Rearrange to make x the subject of the formula:$3+y=\surd 5x-4$	10	2	-	-	12
(c) If $x_1=10$, $x_2=8$, $x_3=4$, $x_4=-9$, $x_5=-3$ find $\sum x_i + \sum x_i^2$	-	-	4	2	6
Ques.6					
(a) Express 9250g as kg	3	1	1	7	12
(b) Find the sum of 0.005m and 3932mm expressed as cm.	5	1	4	2	12
(c) Convert 0.00385 into scientific notation.	6	-	1	1	8
(d) Evaluate $\sqrt{9.5\times10^3+37\times10^4}$	-	-	9	3	12
(e) If a=900, b=0.09 and c=90000 find (i) $a^2+b^2+c^2$; (ii)$\sqrt{a}+\sqrt{b}+\sqrt{c}$	-	-	6	5	11
Ques.7					
For each graph below state whether the graph represents a linear, quadratic or exponential relationship.	4	4	-	-	8

Ques.8					
(a)Write an equation for a straight line with slope of -4 and y-intercept of -3	3	1	4	2	10
(b)Write an equation for the straight line below	3	-	5	3	11
Ques.9 Use your calculator to find					
(a) $\log_{10}$ 1496	-	2	3	4	9
(b) e^x when $x=-1.2$	1	-	4	4	9
(c) $\ln(x+1)$ when $x=409.8$	-	-	7	4	11
(d) $e^x + e^{3x}.e^{-2x}$ when $x=4$	-	-	10	1	11
Ques.10					
(a) Sketch the graph of $y=-x/2 + 2$	2	-	4	6	12
(b) Draw the graph of $y= x^2+7x+6$.	-	-	9	1	10
Use the graph to predict the y value when $x=-2.5$	-	-	7	5	12
Ques.11					
(a) Twenty prople are given a short test. There were 20 possible marks and the results were: 18, 16, 15, 20, 13, 16, 16, 17, 16, 14, 13, 15, 16, 16, 16, 18, 20, 20, 16, 20. (i) Organise this data into a frequency distribution table	10	-	-	1	11
(ii)Construct a histogram to represent the class results.	6	1	-	-	7
(iii) Find the mode and the mean.	3	-	2	3	8

(b) The Queensland Cities Rental Vacancy Rates for April and July 1989 are given below

	Rental Vacancy Rate (%)	
city	April 1989	July 1989
Sunshine Coast	5.36	6.43
Gold Coast	5.05	6.27
Cairns	7.79	3.83
Toowoomba	0.00	1.19
Rockhampton	0.00	1.58
Brisbane	2.90	3.40

(i) Draw a scatterplot of the data	6	-	-	-	6

(ii) Draw a line through the data points of the scattergram to show the trend in rental vacancy rates.

(iii) If another Queensland town showed a vacancy rate of 4% in April 1989, what would you predict it to be in July 1989.

EARLY CHILDHOOD MUSIC AND MATHEMATICAL ACHIEVEMENT - COUNTING THE SHARPS AND FLATS OF EMPIRICAL RESEARCH

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This study explores how music might provide an effective aid to higher achievement in mathematical development. An experimental group of 35 preschool children was chosen from children enroled in a community music program. A comparison group consisting of 39 preschool children with limited musical background was employed as a contrast group. By testing both groups on early number concepts, initial results indicated that the experimental musical group performed better than the non-musical comparison group. However, post-hoc analysis indicated that musical experiences to be found in the home, and other pre-existing differences, were more likely to contribute to the experimental group's higher mathematical achievement than the music program treatment.

Research investigating the development of number concepts and processes in early childhood has been influenced largely by the constructivist view of learning (Steffe, 1990; Wright, 1992). However, although references to constructivist approaches are pervasive, practical descriptions of such approaches have not been readily accessible (Clements & Battista, 1990; Brophy, 1986). In consonance with the constructivist paradigm, early childhood education portrays children as active thinkers, who construct sense and meaning out of personal practical experiences (Wood, Cobb, & Yackel, 1989). Making sense is the purpose of education; to encourage children to look for similarities, oppositions and connectedness in the sensory inputs they are receiving (Dienes, 1987). This ideology in education underpins the global, integrated nature of educational experiences in early childhood.

MUSIC AND MATHEMATICS

Efforts to integrate mathematics with music, are rare indeed (Kleiman, 1991). As teaching in early childhood focuses more and more on process-oriented, holistic, and collaborative approaches to learning, integration of mathematics and music could find an educational climate in which to flourish. Research is indicating that in addition to developing concepts in music, children are being introduced to concepts which are the foundation for other subject areas including mathematics (Bridges, 1980; Covell, 1984; Gregory, 1988; Kalmar, 1989a).

In the learning of mathematics, the ability to solve problems is considered one of the most important skills for children to develop. In early childhood, because "real life problem solving is really creative problem solving in that it requires a wide range of creative, conceptual and logical thinking abilities" (Feldhusen and Treffinger, 1984, p. 1), creativity and problem solving combine into a single concept. Music provides a wealth of opportunities for children to solve problems with creative thinking in mathematical contexts. Many people believe that the learning of mathematics and the learning of music are related but there is little evidence to make such convictions persuasive arguments. A search of the literature indicates there has been little analysis of music and its effects on achievement in mathematical development in early childhood.

RELATED RESEARCH

Research has implied that a group of children with extra music training provide more creative, original and complex ideas and a higher level of abstraction than those with the usual amount of music (Kalmar, 1989b; Bridges, 1980; Hoermann and Herbert, 1979). Other research (Gregory, 1988; Neufeld, 1986) has shown that children who were involved in musical instructional programs demonstrated significant gains in achievement in mathematics.

METHOD

Overview
The study reported in this paper was designed to investigate the effects of a musical program on the mathematical achievement of young children who were about to commence their first year at primary school. The study compares the understandings of early number concepts of a group of children involved in a structured music program with those understandings of a group of children who have no musical training and limited musical background. The mathematical concepts tested were (1) concepts of relative magnitude, (2) counting skills, (3) calculational skills, (4) knowledge of conventions, and (5) number facts.

Subjects
A sample of preschool children in Bathurst was selected using a parent questionnaire. An experimental group of 39 children and a comparison group of 40 children were selected, delimited on the following criteria: (1) age, (2) socio-economic status, and (3) parental input. As the study aimed to contrast the mathematical development of musical children against non-musical children, all the comparison group subjects had limited musical background, and all the experimental group subjects were actively involved in the Central West Music Centre preschool music program. At the time of testing, the experimental group consisted of 35 children;

(4 children dropped out of the music program), and the comparison group consisted of 39 children as 1 child left the community.

Procedure
This study did not use random assignment. A static-group comparison design was used. This involved two groups: one received the experimental music treatment and both groups were posttested. The posttest scores of the two groups were then compared.

Experimental Treatment
Subjects in the experimental group participated in a year long music program at the Central West Music Centre. The experimental treatment was an 'in-house' music program based on Kodaly techniques and sequenced to teach concepts of pitch, dynamics, duration, timbre and form; as well as skills such as moving, playing, listening, singing and organising sound.

The Measure of Achievement
The instrument chosen for this study was the TEMA-2: Test of Early Mathematics Ability-2 developed by Ginsburg and Baroody (1990).

RESULTS
Table 1 shows the means and standard deviations of the mathematics score by the experimental group and the comparison group, and level of significance of the difference in the means.

Group	Mean	SD	*t*
Experimental Group	19.9714	5.721	0.016
Comparison Group	16.6410	5.829	

Table 1
Means and standard deviations for experimental group and comparison group on the TEMA-2 test.

As can be noted from the results the mean score was higher for the experimental group (mean=19.9) than for the comparison group (mean=16.6) with a significant difference between groups ($p<0.05$).

Post-hoc Analysis

The results indicated that the experimental group performed better than the comparison group. However, it was apparent that musical experiences other than the experimental music treatment (in particular music experiences in the home), as well as other pre-existing differences between groups, may have contributed to group differences. Post-hoc analysis was done to re-categorise the experimental group into two subgroups: Subgroup 1 ('No Homus' - those with limited home musical experiences), and Subgroup 2 ('Homus' - those involved in musical experiences at home). The subjects with limited music background experiences (Comparison group and 'No Homus' group) were then compared on the relationship of mathematical scores using a *t*-test. Similarly, the subjects who had done the music program ('No Homus' group and 'Homus' group) were compared on the relationship of mathematical scores using a *t*-test.

Results of Post-hoc Analysis

Table 2 shows the means and standard deviations of the 'No Homus' and comparison groups, and level of significance.

Group	Mean	SD	*t*
Experimental Group (No Homus)	17.3125	5.082	0.689
Comparison Group	16.6410	5.829	

Table 2

Means and standard deviations for the 'No Homus' experimental group and comparison group on the TEMA-2 test.

As can be noted from the results in Table 2 the mean score was higher for the 'No Homus' experimental group (mean=17.3) than for the comparison group (mean=16.6) but was not significant.

Table 3 shows the means and standard deviations of the 'Homus' experimental group and the 'No Homus' experimental group on the relationship of mathematical scores, and level of significance. As can be noted from the results in Table 3 the mean score was higher for the 'Homus' experimental group (mean=22.2) than for the 'No Homus' experimental group (mean=17.3) and with a significant difference between groups ($p<0.025$).

Group	Mean	SD	*t*
Experimental Group (Homus)	22.2105	5.360	0.009
Experimental Group (No Homus)	17.3125	5.082	

Table 3

Means and standard deviations for the 'Homus' experimental group and the 'No Homus' experimental group on the TEMA-2 test.

The results provided in Tables 2 and 3 were an indication that the difference in mathematical achievement might be connected to the musical background experiences rather than the music program. To ascertain what aspects of home background might be significant in contributing to higher mathematical scores, responses to questions on the questionnaire were subjected to a series of *t*-tests. Three questions focusing on home musical background provided substantial results (see Table 4). The three questions were: (1) Does (the child) listen to his/her own music collection very often? (2) Does (the child) learn music? (3) Does anyone in the family sing to (the child)? The mean maths scores, standard deviations and *t* according to the way in which parents answered the above three questions from the questionnaire are shown in Table 4.

Question	Response	Mean Score	SD	*t*
Listens to own music collection?	Yes	19.6	5.71	0.014
	No	16.2	5.84	
Learns music?	Yes	19.9	5.72	0.016
	No	16.6	5.82	
Any family member sing to child?	Yes	20.0	6.42	0.030
	No	16.9	5.38	

Table 4
Responses, means and standard deviations for questions in relation to scores on the TEMA-2 test.

SUMMARY

Initial findings in this study indicated that the experimental music group showed a significant difference in mathematics achievement compared to the non-musical comparison group. However, post-hoc analysis indicates that a structured music program alone is less likely to contribute to higher achievement in early number concepts than musical experiences generated from the home environment or other pre-existing differences. Accordingly, further research utilising random assignment is required in order to control for pre-existing differences.

REFERENCES

Bridges, D. (1980). Outcomes of a developmental music program. *KEIA Journal*, 5 (1), 6-13.

Brophy, J. (1986). Teaching and learning mathematics: Where research should be going. *Journal for Research in Mathematics Education*, 17 (5), 323-346.

Clements, D.H. & Battista, M.T. (1990). Constructivist learning and teaching. *Arithmetic Teacher*. September, 34-35. Reston, Virginia: NCTM.

Covell, R. (1984). *Stop the Rot!*. Paper delivered at the Australian Society for Music Education; Fifth National Conference. Sydney; ASME.

Dienes, Z. P. (1987). Lessons Involving Music, Language, and Mathematics. *Journal of Mathematical Behaviour*, 6 (2), 171-181.

Ginsburg, H.P. & Baroody, A.J. (1990). *TEMA-2 Test of Early Mathematics Ability*. Texas: Pro-ed.

Gregory, A.S. (1988). *The effects of a musical instructional technique on the mathematical achievement of third-grade students*. EdD Thesis. Tuscaloosa: University of Alabama.

Hoermann, D.B. and Herbert, G.F. (1979). *Report and Evaluation: A developmental programme of music education for primary school (Kodaly-based)*. Sydney; Dominie.

Kalmar, M. (1989a). *Effects of early musical training on the mental development of nursery school children (in press)*.

Kalmar, M. (1989b). The effects of music education on the acquisition of some attribute-concepts in preschool children. *Canadian Music Educator*, Research edition, 30 (2), 51-59.

Kleiman, G.M. (1991). Mathematics across the Curriculum. *Educational Leadership*. October, 1991, 48-51.

Neufeld, K.A. (1986). Understanding of Selected Pre-Number Concepts; Relationships to a Formal Music Program. *The Alberta Journal of Educational Research*, 32 (2), 134-139.

Steffe, L.P. (1990). Action group A1: Early childhood years. In L.P. Steffe & T. Wood (Eds) *Transforming children's mathematics education: International perspectives* (pp. 3-15). Hillsdale, New Jersey: Lawrence Erlbaum Associates Publishers.

Wood, T., Cobb, P. & Yackel, E. (in press) The contextual nature of teaching: Change in mathematics but stability in reading. *Elementary School Journal*.

Wright, B. (1992). Number Topics in Early Childhood Mathematics Curricula: Historical Background, Dilemmas, and Possible Solutions. *Australian Journal of Education*, 3 (2), 125-142.

EXPLAINING MATHEMATICS

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This paper argues, with examples, that as a result of how mathematics is presented, especially in textbooks, explanation is highly implicit. Students are given mathematical problems and are told how to do them, but the whys - the underlying principles and connections - are understated. Thus students are left to infer these from examples, visual representations and manipulations, with the result that mathematics makes little sense to many learners. A strategy for determining at what points explanation breaks down is suggested.

As a way of getting your attention, I'd like to suggest that teaching mathematics and consulting a doctor have a lot in common. From a description of a patient's symptoms and clinical signs, a doctor must:

(a) identify what the problem is;
(b) know how to treat it;
(c) know why this is the best course of action; and these days,
(d) explain to the patient the what, how and why.

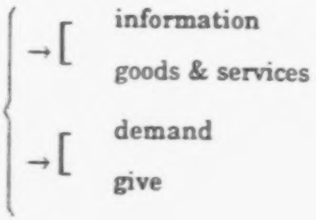
Mathematics instruction aims to enable students

(a) to identify what the problem is from linguistic and mathematical symbols;
(b) to know how to solve the problem;
(c) to understand why the solution works; and according to recent curriculum documents,
(d) to explain what they've done and why.

Knowing and explaining **why** a solution works proves problematical for many learners. This is in part because explanation in mathematics teaching is often implicit rather than explicit. Underlying principles, precepts and especially relationships between mathematical phenomena are presented but **not** in so many words, indeed often not in words at all.

Some may say 'so what?'. Mathematics is not language; mathematics is a meaning system distinct from language. I don't agree. Mathematics is as grounded in language as Literature or English. To illustrate what I mean by this, I will show you first an abstraction from my work and then one from mathematics.

Illustration 1:



demand: info = Question
give: info = Statement
demand: g & s = Command
give: g & s = Offer

This configuration is called a system network. It denotes the fact that speakers have a choice of giving or demanding. That which they give or demand is goods and services or information. So if a speaker demands information, she is asking a Question (e.g. What's this bird on about?) If she gives information, a Statement is being made (e.g. She's prattling on about language). A Command is a demand for goods and services (e.g. Shut up and listen). And should she give goods and services, she's making an Offer (e.g. I'll just move this transparency up and focus it better).

The graphics and terms of the above network are meaningful but their meaning could not be revealed to you *except through language*. The same is true of mathematics, which is represented through graphic and linguistic symbols.
Ilustration 2:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

To explain the meaning of this mathematical representation, language is needed. Mathematics is constituted of language no less than any other domain of knowledge. Without language there is no mathematics.

This latter statement - without language there is no mathematics - is both a truism and a problem. It is a problem because of the way mathematics is presented in teaching texts. Such texts, both spoken and written, make explicit the 'how to' much better than the 'why'. This claim and others made in this address are based on research exploring the language of classroom mathematics conducted on behalf of the Disadvantaged Schools Program, Metropolitan East Region, in Sydney over the last two years.

Mathematics instruction typically includes five phases:

1. Introduction
2. Activity
3. Information Giving
4. Instruction
5. Practice.

The **Introduction** provides an orientation to the topic at hand.

Example 1a:

> A girl has made this beautiful patchwork quilt.
> It could cover a bed, or hang on a wall. The
> amount of space it covers is called its **area**.
> (Barry, Booker, Perry and Siemon, 1984:172)

Example 1b:

> Not long ago in Australia we used the Imperial system of measurement. In this system we had no consistent base for grouping units. For example there were 12 inches in 1 foot, 3 feet in 1 yard and 1760 yards in 1 mile. The system we now use is the International System of Units known as the SI system. This system, like our number system, is based on groupings of ten and powers of ten. This makes caculation much simpler.
> (Jones and Couchman, 1982)

The **Activity** phase involves manipulating concrete materials as part of the learning experience. This is most prominent in lower and middle primary school.

Example 2a:

> Cover the front of your maths book with envelopes and then with playing cards. Count how many are used each time. (Barry, et al, 1984:172)

Example 2b:

> Tchr: Now this first stencil that I'm going to give you has the solids on the left hand side of the sheet and the nets on the right hand side. So what I want you to do is to try and imagine. What a net is is what the solid would look like if you could make it out of cardboard and you unfolded it. Alright, if you made a cube out of cardboard and if you

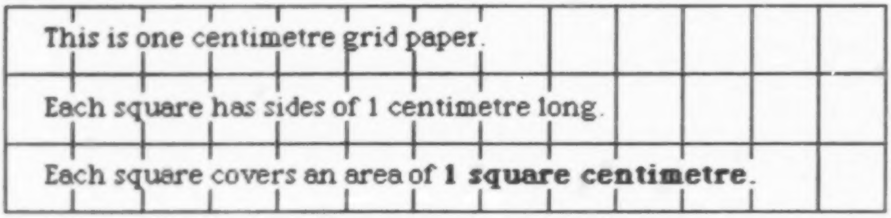
> could unfolded it and lay it flat what would it look like. Alright, so what I want you to do on that right hand side of the page is have a little bit of an experiment with what the nets of the solids would look like.
>
> (Maths in Society: Year 11, NSW)

These two examples are significant not only for illustrating **Activity**, but also for indicating a major difference in what constitutes *doing* mathematics in primary and secondary school. In the research database on which the paper is based, primary school students actually engaged in drawing, measuring, laying flat, cutting, folding and taping together nets. In the secondary school, the doings were all mental, happening only in the learners' imaginations - "try and imagine", "if we were to unfold that cube, lay it out flat". If our observations are correct, primary school mathematics students are involved in physically manipulating concrete materials and secondary school students in mentally manipulating symbolic representations. Whether there is an appropriate transition from doing mathematics materially in the primary school to doing it mentally and symbolically in secondary school is currently under investigation.

The third phase of mathematics instruction is **Information Giving**.

Information Giving involves introducing technical terms, defining these and providing formulae where appropriate. These comprise the most explicitly stated relevant mathematical concepts.

Example 3a:



This is one centimetre grid paper.

Each square has sides of 1 centimetre long.

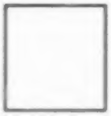
Each square covers an area of **1 square centimetre**.

(Barry, et al, 1984:175)

Example 3b:

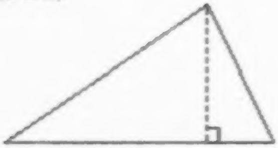
Formulae can be used for finding the area of many plane figures instead of counting squares on graph paper.

1. Square $A = l^2$

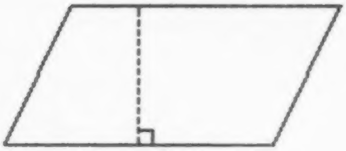


2. Rectangle $A = l \times b$

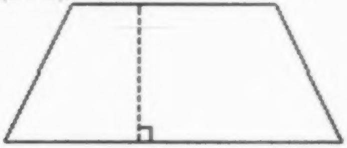
3. Triangle $A = 1/2\ (b \times h)$



4. Parallelogram $A = l \times h$



5. Trapezium $A = 1/2\ (a + b)h$



(Lynch, Parr and Keating, 1987:325-26)

As someone not overly gifted in mathematics, I have to accept the above formulae on faith, because they are presented as fact without explanation.

So I may be able to calculate the area of a triangle without knowing why the area of a triangle is half that of a rectangle (any old rectangle?), just as I may be able to invert and multiply fractions to calculate 1/3 - 4/9 without knowing why I'd ever want to divide fractions in the first place, and without knowing why one inverts and multiplies.

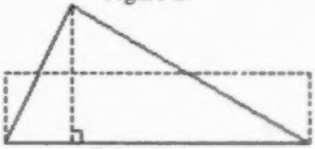
The triangle example above is highlighted deliberately, because I've been having trouble with the area of a triangle since starting to analyze the language of classroom mathematics. The first 'explanation' I found was as follows.

> Starting with the formula for a rectangle the other shapes
> below can be dissected and rearranged to form rectangles.
> When this is done the truth of each formula is demonstrated.
>
> (Jones and Couchman, 1982)

Figure 1:

Rectangle
$A = b \times h$

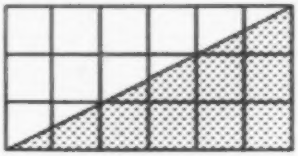
Figure 2:



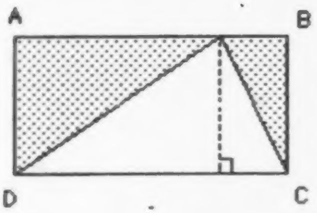
Triangle
$A = 1/2\ (b \times h)$

I can see that the first part of this statement is true. But I fail to see how dissecting and rearranging Figure 2 demonstrates the truth of (proves?) the formula. In this instance the explanation is assumed in the visuals provided. This is very common in mathematics instruction. Visual representations carry much of the weight of explanation. Accompanying verbal explanation is apparently thought unnecessary.

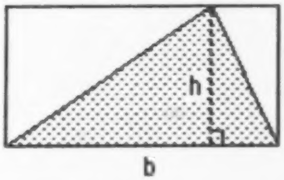
I thought my problems were finally over when I came across the explanation provided in a NSW Year 7 textbook (Jones and Couchman, 1978:101):



What is the area of the rectangle?
What is the area of the triangle?
Is the area of the triangle half the
area of the rectangle?



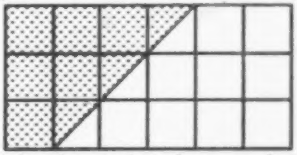
Copy the diagram onto a piece ofpaper and cut out the rectangle ABCD.Cut off the two shaded triangles and fit them on triangle DPC. Do they fit exactly?
Does this mean that the area of triangle DPC is half the area of the rectangle ABCD?



Since the area of a triangle is half of the rectangle containing it, the area of a triangle is given by A = 1/2 (b x h) where *b* is the number of units of length in the base and *h* is the number of units of length in the perpendicular height.

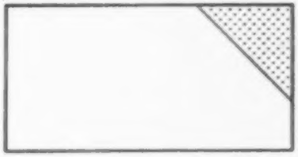
This explanation is convincing as far as it goes. But what if I want to divvy up my rectangle as follows? Is this allowed?

Figure 3:



If every triangle, not just the ones shown above, have a 'rectangle containing it', am I on safe ground if I draw my proof as follows:

Figure 4:

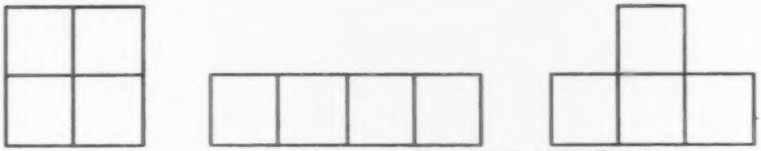


Answers of 'no' to the above two questions suggest that there is a hidden agenda, something that still isn't being explained.

In the fourth phase of mathematics teaching, **Instruction**, sample problems are solved step by step. These steps entail giving the parameters of the problem (quantities); a task - that is, the problem to be solved; and a solution, in which the values specified as parameters are plugged into the formula provided.

Example 4a:

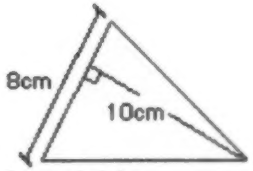
Here are some shapes with an area of $4cm^2$. They have different shapes but they have the same area.



(Barry, et al, 1984:176)

Example 4b:

Find the area of

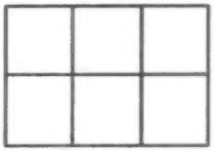


Solution: Area = 1/2 (base x height)
= 1/2 (8 x 10) cm^2
= 40cm^2

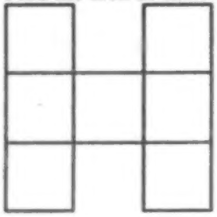
(Haese, et al, 1983:137)

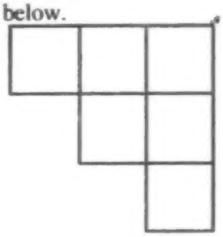
Then students are given **Practice** exercises to do.

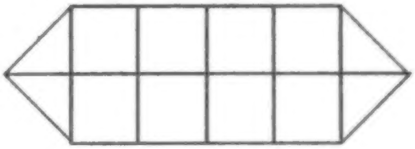
Example 5a:



This shape has an area of 6cm^2
Find the area of each shape below.



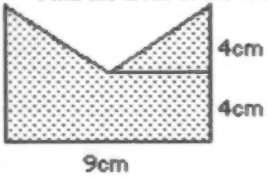


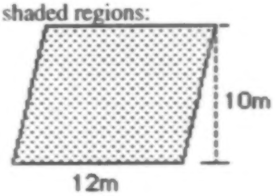


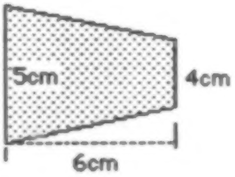
(Barry, et al, 1984:177)

Example 5b:

Find the areas of the following shaded regions:



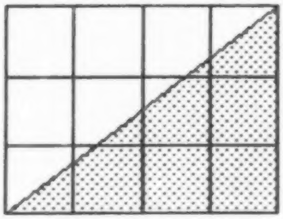




(Haese, et al, 1983:138)

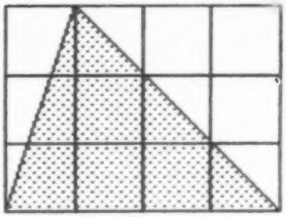
I would like to return briefly to the **Instruction** phase. This phase is usually given the most attention and time/space in mathematics teaching. I see two major problems here for learners. Firstly, explanation at this stage is primarily procedural. Students are shown 'how to' solve given problems (ones already identified for the students), but the connection between the 'how to' and the 'why' is left for the learner to infer. Moreover, universal principlies are taught through particular instances. Students must determine, with little guidance, which bits of an example apply to all instances of the phenomenon unders focus and which apply to the particular instances demonstrated. Consider again the area of a triangle.

Consider the diagrams on the left. Notice that:



1. The base of the triangle is equal to the length of the rectangle.

2. The height of the triangle is equal to the width of the rectangle.



3. The area of the triangle is equal to half the area of the rectangle.

From this we can see that:
The area of a triangle = half the base x the height.
(Hollands and Moses, 1980:50)

The learner can see that in this case the base of the triangle is equal to the length of the rectangle, etc., but is this always so? How is the learner to know? After all, it isn't in Figure 4!!

If the data analyzed are representative, it seems that mathematical principles and their relationships to each other are often not made explicit, are not stated in so many words at any phase. Instead, students are left to infer these for themselves from (1) manipulation of concrete materials in primary schools, (2) diagram/formula pairs as in Figures 1 and 2 above, and/or (3) exemplification using particular instances. What principles, for example, *am* I to understand when confronted with Figures 1 and 2 above? What connection do they have to each other? To me?

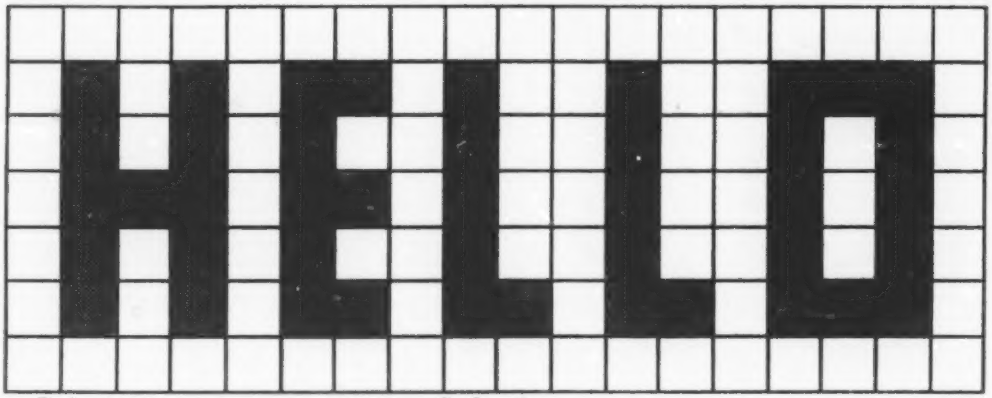
Exemplification, so prominent in the **Instruction** phase, is necessary for showing students how to solve problems, but is not sufficient to ensure understanding of the principles involved. Thus students are often baffled when confronted with word or real life problems. They do not even know what the problem *is* because they are not able to recognise that the new problem is in principle the same as one they've already met. In other words, when basic principles are not understood, because they are not spelled out in so many words, learners cannot generalise from the exemplified instances to new instances. Each problem is seen as novel and unfamiliar.

Mathematics teachers may not be aware of the extent to which learners are required to infer concepts and connections in mathematics. Certainly writers of mathematics textbooks seem unaware of the need to **explicitly** explain. Because mathematicians understand the principles involved, inferencing is automatic and the gaps in teaching texts may not be at all obvious. But if a learner says "I don't get it", chances are some principle has not been adequately explained.

Perhaps explanation implicit in manipulation, diagrams and examples is sufficient; perhaps discursive explanation would be superfluous. To check these possibilities, small groups of NSW Year 4,6,9 and 11 students were given sets of tasks for which the answers were provided. The students were asked to explain why the answers were as they were. The problems were taken from work done earlier by the students so should have been familiar.

Year 4 example:

Perimeter is the distance around the outside of an object or shape.
Surface area is the amount of space something covers.
Pretend your friends don't know anything about perimeter or surface area. Teach them how to complete the table below.



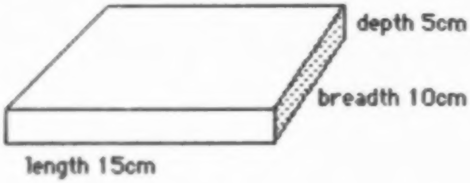
Perimeter	Surface Area
H = 24cm	H = 11cm^2
E =	E =
L =	L =
L =	L =
O = 16cm	O = 12cm^2

The group of six immediately split into two camps, one of which read and considered the definitions and answers provided, the other which launched straight into deriving formulae for calculating perimeter and surface area. The former subgroup had no difficulty linking the definitions and answers provided, and using these to derive accurate answers for the perimeter and surface areas of letters E and L. The latter group decided that to calculate perimeter, one must add top plus bottom to sides times 2: (t + b) + (s x 2). So their calculation for the perimeter of E was 4 + (5 x 2) = 14, and for L was 3 + (5 x 2) = 13. Notice that this 'formula' will work for O, but not for H, a fact the sub-group overlooked. To calculate surface area the second group decided that the formula was 'down plus across'. Using this formula, the surface area of E was calculated to be 11 (5 down and 6 across) and for L was 7 (5 down and 2 across). One member wasn't happy with this and suggested a formula wasn't needed, that just counting the number of squares coloured in would suffice "cause that tells ya surface area".

Year 6 examples:

The Year 6 group was unable to resolve or explain any of its three tasks.

1. The surface area of this prism is



$\text{SA} = (150 + 75 + 150 + 75 + 50 + 50)\text{cm}^2$
$= 550\text{cm}^2$
This is the same as
$2(150 + 75 + 50)\text{cm}^2$
$= 550\text{cm}^2$

See if you can discover a rule or formula for finding the surface area of prisms like this.

2. A swimming pool in the shape of a rectangular prism is 20 metres long, 10 metres wide and 2 metres deep. Explain why it would take 400 cubic metres (400,000 litres) of water to fill the pool.
3. What is the volume of the prism in Question 1? How would you explain the difference between volume and surface area to a Year 4 student?

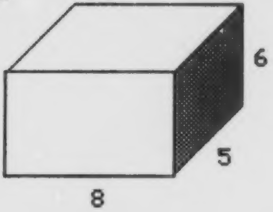
Year 9 examples:

A group of three Year 9 girls had no difficulty explaining the first answer below but took 30 minutes to resolve question 2 and gave up completely on question 3.

1. A plasterer is paid 96 cents per square metre. For plastering two ceilings, one 5.2 metres by 4.5 metres, and the other 6.0 metres by 5.6

metres, he charged the customer \$57.00. Explain why the account came to this amount.

2. The surface area of this rectangular prism is $236cm^2$. Explain why the formula for calculating the surface area of a rectangular prism is: S.A. = 2(lb + lh + bh).



Explain why the volume of this same rectangular prism is $240cm^3$. Work out the formula for calculating the volume of a rectangular prism.

3. If the surface area of a rectangle is S.A. = b x h, explain why the surface area of a triangle is S.A. = 1/2 (b x h).

Year 11 examples:

The three tasks comprising the Year 11 set came from a revision unit and caused the students little difficulty. It is interesting to note how similar these tasks are to those set for much younger learners.

1. How would you explain to a Year 7 student why a carpet 4 metres by 3.5 metres laid in a room 6 metres by 4.25 metres leaves an area of floor $11.5m^2$ uncovered?
2. A 5 metre length of fencing timber costs \$8.00 and fence posts cost \$5.00 each. If 9 lengths of timber are needed to fence a triangular paddock and a fence post is needed for each metre of fence, explain whether the fence timber or the fence posts would cost more and why.
3. A tiler charges \$9.60 per square metre for labour. He has charged your granny \$56.00 to tile her bathroom floor which measures 2 metres by 2.5 metres. Granny feels that she has been over-charged. How do you help her check?

The students involved in the above tasks varied in their abilities to explain. Several commented that they were used to 'finding the answers' and explaining how they got their answers to teachers, but were not accustomed to being given solved problems to explain. These comments reflect their understandings of how mathematics instruction is presented. Mathematics teaching texts, I have argued, are good at defining problems and explaining how to solve them, but leave the explanation of why much too implicit. Written mathematics especially has become very cryptic and hieroglyphic, requiring sophisticated translation and interpretation skills on the part of both teachers and learners. I suspect it will take a long while for written mathematics to become discursive. Meanwhile, teachers need to be aware that mathematical definitions, manipulations, diagrams, formulae and examples are by no means self-explanatory. They need to be aware of how implicit explanation in mathematics is, and need to make mathematical concepts and connections explicit.

Students can help teachers with this. Engaging students in the kind of explanatory exercises of the kind described above is useful in several ways. Firstly, it is useful diagnostically. Inability to explain suggests that principles and/or connections have not been made sufficiently explicit and so are not understood. Secondly, it provides opportunity for learners to make explicit for themselves what they understand when they do understand. And thirdly, students' explanations are often more intelligible to peers than adult explanations. Students-as-teachers often appreciate better what their peers-as-learners need to hear and how than we do. Thus, while responsibility for knowing what needs to be explained and explaining it falls firstly and primarily to teachers, learners also have a significant role to play in explaining mathematics.

REFERENCES

Barry, B., Booker, G., Perry, B., and Siemon, D. (1984): *HBJ Mathematics 4*. Sydney: Harcourt Brace Jovanovich Group.

Haese, R.C., Haese, S.H., Webber, B., Danielsen, F.G. and Burch, B.S. (1983): *Mathematics for Year 9 (Second Edition)*. South Australia: Haese Publications.

Hollands, R. and Moses, H. (1980): *Headway Maths 3 (Australian Edition)*. West End, Queensland: Coterie Publication.

Jones, S.B. and Couchman, K.E. (1978): *New Series Modern Mathematics 7*. Sydney: Shakespeare Head Press.

Jones, S.B. and Couchman, K.E. (1982): *Modern Maths Core and Lobe Year 9 (Advanced)*. Sydney: Shakespeare Head Press.

Lynch, B.J., Parr, R.E., and Keating, H.M. (1987): *Maths 9 (Second Edition)*. Melbourne: Longman Sorrett.

IDENTIFIED PROBLEMS IMPEDING EFFECTIVE EVALUATION OF TERTIARY BRIDGING MATHEMATICS PROGRAMS

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Mathematics bridging programs and related assistance schemes for mathematically underprepared students have become essential features of many tertiary institutions. Often unstated, but nevertheless intended, the principal goal of such programs is to provide a service which gives the mathematically underprepared student the same opportunity to satisfactorily complete their chosen tertiary qualification as the mainstream student. Evaluating the effectiveness of each bridging program therefore necessitates determining whether this goal is being reached. Long-term or longitudinal studies are strongly recommended. However, there is little evidence in the literature that formal evaluation of such programs is occurring to any great degree. Why is this so?

This paper takes up this question and reports on one aspect of a large study which involved the synthesis of research evidence concerning bridging mathematics programs in the USA and Australia. Several important reasons for the limited nature of evaluations are identified. The paper concludes by suggesting that evaluation, in the traditional sense, may be incompatible with the successful conduct of tertiary mathematics assistance programs.

Many tertiary students find themselves mathematically underprepared for their chosen studies. This problem occurs primarily for three reasons. First, the mathematical skills of the student are deficient. Second, the depth of treatment (exposure) to various mathematics topics is limited. Third, there is an increasing reliance on mathematics techniques and concepts in subjects and courses not traditionally mathematically orientated. Tertiary institutions have responded in a myriad of ways to this issue, with varying degrees of formality. However, the responses may be broadly classified in terms of preparatory, bridging or concurrent programs.

To provide a home for such programs, some institutions have established a physical facility: a unit or centre, which conducts some or all of these services, and provides "drop-in" assistance. In Australia, these programs and centres are referred to collectively as "bridging mathematics". They are now so prolific as to have specific forums in place to discuss relevant issues: for example, the *Australian Bridging Mathematics Network Conference* . The United States of America has also a well-established and extensive "developmental mathematics" scene, based mainly at the two-year colleges. Where specific centres exist in the US, mathematics is frequently not the major emphasis. Instead a large range of services is covered in a whole-person approach. Although this is true of some centres in Australia, most centres (variously named Mathematics Learning Centres, Remedial Mathematics Units, etc) are restricted to mathematics. A similar situation exists in New Zealand. Variety again exists in the strength of administrative ties between each program or centre and the relevant institution's mathematics department.

Much assistance is provided concurrently with the students' 'regular' undergraduate (or postgraduate) coursework. Formal means include interventionist strategies where the students attend compulsory or voluntary sessions dealing with background mathematics. Informally, 'drop-in' centres allow students to arrive as and when the need for assistance arises. Short intensive bridging courses prior to semester commencement are a common form of assistance, particularly in Australia. Semester-long preparatory courses and distance education versions of these have also been established in this country, in an attempt to pre-empt some of the mathematics difficulties students face and to lessen their impact. In any case, the emphasis is on individualised and small group attention, and is very much student-centred. A variety of teaching resources and modes are employed in any one session or contact. Flexibility is of the essence.

EXISTING EVALUATION

The need for evaluation

The rationale for evaluating bridging mathematics programs lies partly in the identification of sound educational practice. That is, identifying effective techniques in terms of producing optimal results in students, possibly with respect to time-effectiveness. Other purposes include: the notion of feedback and review aspects which the evaluative process should provide; and, the justification of the viability of the programs to those in higher authority.

Forms of evaluation

Types of evaluations which have been taking place include those which focus on the results of the program itself, or the results of the first mathematics subject, and those which consider the completion rate of the tertiary course for which the student was aiming. Evaluations have often been of an informal nature, for example, student evaluation questionnaires, simply observing mathematics results of students, and feedback from staff who teach mathematics-based subjects.

Methodology

The most common method of formal evaluation has been of a pre-test/post-test nature. This has potential validity problems. To rely on this form of evaluation alone is often misleading (Budig 1986, in Tomlinson 1989). Regression of test scores towards the mean is clearly a confounding factor. This is especially true when mature age students, who have been absent from formal mathematics instruction for some time, are considered. An alternative approach, which overcomes this problem, uses the fact that some students do not take advantage of remedial/preparatory programs when indicators suggest that they should. Comparisons are then made with students in the programs.

Results

No definite trend on the effectiveness of bridging programs can be discerned from the literature. At an individual level, students have satisfactorily completed the programs and continued on to complete tertiary studies. However, when results of a program are viewed overall, there is disagreement as to the effectiveness of the programs in the studies reported. The increase in students' mathematical prowess for the group as a whole may well have improved but the level of their mathematical ability may still be below that of mainstream students with whom they are to be compared. Barling & Jones (1991) for instance, found that it took a time-period of around eighteen months before students generally reached the standard of their initially better prepared peers. Students entering the programs who had seen the material previously, and/or were of relatively high placement levels on entering tertiary education have been seen to fare better. However, the evidence is not conclusive in some cases, and not comparable in others.

IDENTIFIED PROBLEMS/ISSUES

Introduction

Insight into some reasons why evaluations are not up to standard or are lacking in critical features has been provided by Bers (1987):

> In many places, uncertainty about program goals precludes identifying criteria for measuring 'success'. Faculty and staff are more concerned with meeting students' needs than with evaluating the results of their efforts. Reliance on 'soft money' or limited institutional funds further impedes evaluation activities ... The uncertain status of remedial programs and faculty in many institutions makes staff weary of submitting to evaluations which might suggest programs are ineffective, particularly when there is fear that program continuation is at stake ... Where programs are under pressure to provide evaluations, researchers or program staff might be tempted to select a particular variable ... and use this as a measure of program success or failure.
>
> (pp.2-5)

This paper now looks at these and other issues confronting mathematics bridging educators attempting evaluation of their programs.

Clarity of real aims

An essential task which should be carried out for each mathematics bridging program is the definition of the program's aims and objectives, otherwise "success" cannot be determined. Is the program intended: to improve a student's skills in specific topics; to prepare a student for entry to certain regular courses; to improve a student's chances for the current regular course; and/or, to enable a student to have the same chance of satisfactory completion of the regular course as the mainstream student? It is the latter aim which is often not clearly defined and yet is the long-term goal of many programs (e.g., Ross & Roe 1986, Wepner 1987).

Students' perceptions of needs may undergo a number of changes as they progress through both the assistance program and their regular course, and at any one time may be quite unlike their real needs. Their urgency is to cope with the current or imminent mathematical skills with which they are presently confronted. The staff, on the other hand, wish to ensure the students grasp the concepts and gain wider perspectives on the topics. Thus the students are actually aiming for different results compared to the staff, causing conflict and frustration in both. 'Success' for students is in coping satisfactorily with current or imminent work and not to be constantly floundering. 'Success' for staff is that students have gained some insight so they have the grounding needed to do 'different' questions and a basis upon which to build. However, 'success' for the programs must be measured on what the program staff are actually aiming for, if only implicitly: the long-term goal of mainstream levels of successful course completion. Importantly, this is similar to the aim of those in higher authority concerned with the extent of increase in student retention and graduation (Akst 1985). Since the objectives of a program should determine its evaluative process, evaluating this type of success indicates that longitudnal studies are a necessity (e.g., Wright & Cahalan 1985). Unfortunately, this type of evaluation is not generally undertaken (e.g., Atweh 1981, Wepner 1987), and has been, even more rarely, built-in to the program itself.

Spontaneous pragmatism of some programs

The establishment of some mathematics bridging programs, particularly in the past, has been of a hurried nature, in response to an immediate need. Little provision, if any, has been therefore possible in the initial stages of development, for the building-in to the program of an evaluative process. The instruction techniques used have been dependent on that which was available at the time, and able to be put into effect immediately. Changes have been made as and when possible, both from a time aspect, and from the point of view of funding. Such spontaneous responses to the mathematics underpreparation problem have contributed to the variety and uniqueness aspect of the programs overall.

Flexibility of teaching/learning modes

Another evaluation problem stemming from a less than concrete base is the flexibility for which such programs are renowned. Since a student-centred approach is of priority in these programs, the teaching techniques used tend to be dictated by the individual student's needs including background mathematics and learning styles. Thus for each student in any one session a variety of resources and instruction modes may be used. These include guided reading and solution to problems, Computer Enhanced Learning, video- or audio-tapes, one-to-one tutoring, small group discussions, and sometimes (usually infrequently), lectures.

Comparability across programs

Due to the uniqueness of almost every program, and the flexibility of instruction modes which may have been used, it is almost impossible to compare programs in a traditional investigative manner. The uniqueness occurs in a number of ways: namely, whether the type of approach is for skills alone or also for concepts; the type of evaluative technique which was most applicable for each of the programs; the duration and timing of the assistance given; and, the students themselves. The students are of a much less homogeneous mathematical background than in other educational settings. Part of the reason is the predominance of mature-age students who have a wide range of mathematics backgrounds and length of time since recent mathematics practice. How the program actually attracts students often determines where in the range that program's students fit. An additional aspect is the attraction to (particularly preparatory) programs of would-be students who would never have considered attempting further studies had they not been exposed to this opportunity. The students of various programs also differ in their requirements with regard to content to be covered, depending on their intended, or current, tertiary course.

The duration and timing aspects are of major concern in attempting comparability between programs. In a comparison between preparatory and concurrent assistance programs, such issues as: the lead-time between assistance and the commencement of the regular course; and, the length of time devoted to remediation during the course, are potentially quite dramatic.

Comparing the effectiveness of programs from one country to another is made more difficult by different topics and levels of mathematics being taught in secondary and tertiary institutions and by different purposes for the programs.

Complexity within programs

The huge variability of students has been mentioned previously. The individualised approach to student assistance within just one session makes any traditional approaches to evaluating instructional techniques impractical at the very least. More than one method of evaluation is necessary since there is usually more than one objective. Statistical testing is thwarted by the interaction or confounding of treatments, since isolation is impossible (e.g., Bers 1987). Therefore, any tests on just one variable will be invalid. A lack of generalisation to the group as a whole plagues even phenomenological investigations.

Attrition

One of the major problems for researchers comparing their results, is the necessity to be precise in their descriptions of the characteristics of the target group of students and any comparison groups. Researchers need to define carefully any terminology and assumptions used, including how they have accounted for attrition. Kulik , Kulik & Schwalb (1983), in conducting a meta-analysis of evaluations of (US) programs for high-risk students, were enormously restricted by the inadequacies of reports from this perspective. There are two main aspects to attrition of students from bridging mathematics programs. Where more than one course of preparation/remediation is conducted within the program, reports made concerning the second course (which then reflect on the whole program) frequently neglect to mention that the "success" rate does not take into account those students who "dropped out" before the second course commenced. Comparisons with other programs are then invalid. Precise details of the setting in which the evaluations being reported took place are essential.

The second aspect provides a disturbing vagueness to the entire bridging mathematics scene. When students leave after a remediation or preparation program, and are not able to be traced again at that institution, have they merely not continued, and therefore not succeeded in completing a tertiary course, or have they in fact enrolled elsewhere? (and did they complete the tertiary course satisfactorily?) (Bers 1987). Such questions must largely remain unanswered, and so the effectiveness of these programs can never be determined with a high degree of certainty.

Ethical considerations

Comparison of groups of students by teaching technique introduces concerns about the ethics of providing suspected "better" services to one group over another. However, some investigators have attempted carefully designed comparisons. A major ethical problem not unique to bridging mathematics is that of withholding some form of assistance from a control group. Where control groups have been used in evaluations on bridging mathematics programs, the students in the control groups are those students who should have taken advantage of the program and did not (e.g., Budig 1986, in Tomlinson 1989). This is a very restrictive process.

Since students have a limited time in which to remediate or prepare for their mathematics requirements, it seems unreasonable to expect any additional time constraints to be placed upon them, as may occur with evaluation techniques. The staff suffer time limitations, also since their availability to students is of paramount importance. Therefore, they usually cannot afford time on additional projects, unless alternative staffing arrangements are made.

Staffing issues

A high proportion of staffing is on a fractional basis, with a leaning toward females. Major sources of staff are the mathematics department (and this may be provided reluctantly) and secondary teachers. The requirement of a teaching background in mathematics may often assume less importance than having empathy for the students' predicaments. Specific training is more 'on-the-job' than formal, although some response to this void is now being made, particularly in the US. This background, and the added problem of students needing assistance *now*, makes staff availability low for such activities as research, including evaluative studies. The low image which these programs have in the academic world is reflected in how poorly the worth of research in the area is considered. This is evident in the minimal extent to which research on bridging mathematics is published. Therefore, the staff struggle with the obvious need to evaluate their programs and the practical priority on which such evaluation appears to be placed.

Funding

One of the major reasons why staffing is limited is the difficulty in attracting sufficient funds to satisfactorily conduct the programs. Part of this is due to the methods in which funding is obtained and apportioned throughout the institutions. Another reason is the poor image which bridging mathematics often has in a tertiary institution (Godden & Pegg (in press)).

The lack of available funding causes further troubling repercussions. In fact, it creates and is part of a rather vicious circle. Without the funding to carry out substantial evaluations on the effectiveness of the programs, both as sound educational practice and for justification to those in higher authority, only minimal evaluative efforts are made by most of the educators. Thus the programs, not adequately evaluated, may whither due to lack of noticeable effectiveness. A corresponding reduction in funding due to the perceived low viability ensues. Optimal teaching/learning techniques remain unidentified, thus the chance of being time- and effort-effective, which is so necessary under the limited conditions of these programs, is lost.

CONCLUSION

One of the major strengths of bridging mathematics programs is their flexibility and student-centred approach. Unfortunately, it is this very strength which is a major downfall in terms of evaluating their effectiveness. Such a weakness cannot be ignored. Evaluations are considered essential. However , this paper has shown that obstacles to substantial evaluation exist at every turn. Perhaps the challenge to overcome some of the obstacles is actually defeating the purpose of the great strengths of these programs. That is, traditional evaluative techniques employed in secondary and tertiary educational settings are just not possible. The sacrifices are too great and contrary to the aims of the programs. In attempting to minimise bias, maintain validity, and strive for some degree of replicability, the educator risks losing the essence of the support and assistance so necessary for these students. Why compromise a set of processes which for the *individual* student is often quite rewarding and in a number of cases, given time, produces results akin to that of mainstream students? The individual 'success' may well be more important than some statistical improvement in test scores of a large group.

Given that the desire and/or necessity to evaluate the programs is strong, perhaps what is really needed is a new set of procedures specifically tailored to the area of bridging mathematics. These procedures would need to take into account the unique conditions which are the mainstay of the programs and the constraints applicable to them. Kinsler & Robinson (1990) concluded, about developmental education generally, that "Research in the field is new and the area is wide open" (p.354). We would add that not only is the research aspect "new" but that the techniques should be 'new' also.

REFERENCES

Akst, Geoffrey. (1985). Reflections on Basic Math Programs in the Two-Year College. In Albers *et al* (Eds.), *New Directions in Two-Year College Mathematics.* (Proceedings of Sloan Foundation Conference on Two-Year College Mathematics, Menlo College, Wharton, Calif., July, 1984), p.142-158.

Atweh, W. (1981). Research and Evaluation in Remedial Programs. In William Atweh (ed.). *Remedial Mathematics in Tertiary Institutions.* (Proceedings of a seminar, QIT. Educational Resource & Development Unit, QIT, Brisbane.), p.8-15.

Barling, Chris & Jones, Peter. (1991). *Bridging the Gap. An Intervention Strategy for Improving the Mathematical Competencies of Students in First Year Applied Science and Engineering.* Final Report to the Sponsor: The Victorian Education Foundation, September.

Bers, T.H. (1987). Evaluating Remedial Education Programs. *Association for Institutional Research Professional File.* Association for Institutional Research, Tallahassee. ED282492.

Budig, J.E. (1986). An Evaluation of a Junior College Developmental Education Program. *AIR 1986 Annual Forum Paper.* Association for Institutional Research, Indianapolis, Ind. ED280393.

Godden, Gail L. (1992). *Synthesis of Research on Facilities Provided by Tertiary Institutions for Mathematically Underprepared Students.* Unpublished Masters Dissertation. University of New England.

Godden, Gail L. & Pegg, John E. (in press). Confronting Image Issues Which Restrict the Effectiveness of Bridging Mathematics Programs. *Proceedings of Australian Bridging Mathematics Network Conference, Brisbane, 1993.*

Kinsler, Kimberley, & Robinson, Andrew. (1990). Research in Developmental Education. In Robert M.Hashway (ed.), *Handbook of Developmental Education.* Praeger. New York. p.335-355.

Kulik, C-L.C., Kulik, J.A., & Schwalb, B.J. (1983). College Programs for High-Risk and Disadvantaged Students: A Meta-Analysis of Findings. *Review of Educational Research,* 53 (3) Fall, 397-414.

Ross, E.P. & Roe, B.D. (1986) The Case for Basic Skills Programs in Higher Education. *Fastback 238. Phi Delta Kappa.* ED273166.

Tomlinson, Louise M. (1989). *Postsecondary Development Programs: A Traditional Agenda with New Imperatives. Report No.3.* School of Education and Human Development, The George Washington Uni., Washington D.C.

Wepner, Gabriella. (1987). Evaluation of a Post-Secondary Remedial Mathematics Program. *Journal of Developmental Education,* 11 (1) September, 6-9. EJ358376.

Wright, Douglas A. & Cahalan, Margaret W. (1985). Remedial/Developmental Studies in Institutions of Higher Education. Policies and Practices, 1984. Presented at American Educational Research Association Conference, 1985. National Center for Education Statistics. ED263828

OBSERVING MATHEMATICAL PROBLEM SOLVING: PERSPECTIVES ON STRUCTURED, TASK-BASED INTERVIEWS

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Structured, task-based individual interviews are one aspect of an ongoing mathematics education research study of elementary school children at Rutgers University. The goals of the interviews are to observe complex, mathematical problem-solving behavior in detail, and to draw inferences from these observations about the children's thinking and development. This paper discusses the scientific underpinnings of the methodology, the role of cognitive theory in structuring an interview, constraints and limitations imposed by the social/psychological context of an interview, and the interplay between task variables, observed behaviors, and inferred cognitions. Some principles of interview design and construction are suggested for consideration by the mathematics education research community.

The structured clinical interview is finding increasing acceptance today as a research method for the study of mathematical learning and problem solving. In general, such structured interviews are for the twin purposes of (a) *observing* mathematical behavior, usually in a problem-solving context, and (b) *drawing inferences* from the observations about the problem solver's cognitions and/or affect. The ideas in the present paper grew out of several earlier studies making use of task-based, individual interviews (Goldin, 1985, 1986; Goldin and Landis, 1985, 1986; Bodner and Goldin, 1991a,b; DeBellis and Goldin, 1991). Presently a group of us at Rutgers University are developing a series of such interviews in the context of a three-year, longitudinal study of individual children's mathematical development (Goldin, DeBellis, DeWindt-King, Passantino, and Zang, 1993). The perspectives presented here are helping to shape this development; thus comment and criticism are invited.

Adoption of the task-based interview as a principal research tool in our study raises a series of questions: (1) In what sense do such interviews permit a *scientific* investigation? What are the implications concerning replicability of results, comparability of outcomes, or generalizability from observations? (2) What is the role of *theory* in structuring an interview? To what extent are the observations made contingent on (tacit or explicit) theoretical assumptions underlying the interview? How does theory guide the drawing of inferences about cognition and/or affect from these observations? What is the interplay between *task variables*, observed behaviors, and the inferences one can draw? (3) What constraints or limitations are imposed by the social and psycho- logical *context* of the interview? (4) What general *principles* of interview design and construction, if any, are appropriate for the mathematics education research community to adopt? Is it possible through such principles to optimize the information gathered through a task-based interview?

The intent of this paper is to raise these questions for discussion, to offer a few illustrative examples from the study currently in progress, and to propose some preliminary and partial answers for consideration.

A RESEARCH STUDY IN PROGRESS

In the study now under way, the mathematical development of an initial group of 22 children is being observed over three years; we hope half or more of the children will remain in the study for the full term. Subjects at the outset, in 1992-93, were in the third and fourth grades (ages 8 to 10) in a cross-section of schools in New

Jersey. Their teachers are participants in a mathematics education reform partnership (MaPS: Mathematics Projects in Schools) sponsored by the Rutgers Center for Mathematics, Science, and Computer Education, and directed by Carolyn Maher and Robert Davis. One component of the study consists of six task-based, individual inverviews with each child over the three years, during which the children's problem-solving is videotaped.

The purpose of the study is thus purely *exploratory* and *descriptive*--subjects are not a random sample, and no general hypotheses are being tested. Overall we hope to describe individual mathematical development in as much detail as possible, focusing not on standard, discrete skills or algorithmic problem solving, but on the growth of complex, internal representational capabilities. The framework for describing these capabilities is based on a model for mathematical problem-solving competency embodying five kinds of systems of internal, cognitive representation (Goldin, 1987; 1992): (a) a verbal/syntactic system (use of language); (b) imagistic systems (visual/spatial, auditory, kinesthetic encoding), (c) formal notational systems (use of mathematical notation), (d) planning, monitoring, and executive control (use of heuristic strategies), and (e) affective representation (changing moods and emotions during problem solving). Of particular interest are interactions among these processes, and the interplay between the children's (internal) representations and external representations that they use or construct during the interviews.

Since the study is longitudinal, a major focus is how over a period of time systems of representation *develop* in the child. In this respect the theoretical model incorporates three main stages: (a) an inventive/semiotic stage, in which internal configurations are first assigned "meaning", (b) a period of structural development, driven by the meanings first assigned, and (c) an autonomous stage, in which the representational system functions flexibly and in new contexts.

Though the analysis of outcomes is theoretically-based, we seek not only to observe and draw inferences from *expected* processes, but also to search for unanticipated occurrences. The hoped-for results include: (1) a set of detailed, descriptive case studies of individual mathematical development, with accompanying videotapes, protocols, analyses and interpretations; (2) improved capability for observing and drawing inferences from mathematical behavior; (3) further refinement and development of the theoretical model for problem solving, including identification of inadequancies, and progress toward an assessment framework; and (4) conjectures for wider investigation through experimental studies.

Design of the interviews themselves is tied to these goals. It includes the following steps: (a) planning in relation to mathematical content and strucutre, anticipated observations, and inferences; (b) the creation and critique of an interview script; (c) pilot-testing and revision of the script; and (d) rehearsal and training of clinicians. As this is written, two of the six interviews have been completed (Spring 1992 and Fall 1993) and the third is under design to be administered next month (Spring 1993). The following are some elements of the interview scripts (see Goldin *et al.*, 1993 for more detail; the full scripts as they are completed are available from the authors):

Interview #1 [55 pages]: An additive structure is embodied in an arithmetic sequence, represented *via* a geometric arrangement of dots. The first three cards in the sequence are presented: "Here is the first card, here is the second card, and here is the third card."

A series of exploratory questions follows, with contingencies based on the nature of the child's responses; special emphasis is placed on exploring the student's pattern-construction and use of external representations.

Interview #2 [38 pages]: A series of questions explores "one half" and "one third" in many embodiments. "When you think of 'one half' [subsequently, when you think of 'one third'], what comes to mind?" Included are requests to take one half and one third of a number of objects (12 apples), various two-dimensional shapes (square, circle, six-petaled flower), an array, and a solid wooden cube. Then the child is asked to think about painting the cube red and cutting it in a number of different ways, describing the pieces that would result. A multiplicative structure is embodied in cutting across different dimensions. Special emphasis in this interview is placed on exploring visualization by the child.

Interview #3 (under design): Two different problems are presented successively: (1) cutting a birthday cake (without or with frosting) to share equally among two or three children, and (2) a problem about moving colored M&M candies back and forth between two jars. Both problems embody symmetry and coordination of conditions--the first in the context of volume and area, the second in a numerical context. Emphasis is placed on exploring the child's affect, as well as metacognitions about the two tasks.

Interviews are planned to take approximately 45 minutes (less than one class period). In all three interviews, alternative embodiments for external representation are provided: paper and pencil, markers, cards, chips and/or other manipulatives, paper cut-outs, etc., in accordance with the task. The questions tend to increase in difficulty, so that each child begins with a level of comfort, but even mathematically advanced children encounter some questions that are challenging before the interview ends. Free problem solving is encouraged wherever possible, with (specified) hints given or suggestions made only after the child has had the opportunity to respond spontaneously. All responses are accepted by the clinician (with occasional, specified exceptions), with "wrong" and "correct" answers treated similarly. Follow-up questions are without overt indication of the correctness of earlier responses.

Two videocameras are in operation simultaneously during each interview--one focusing on the clinician and the child, and the second focusing on the student's work with his or her hands (paper and pencil, or manipulatives). An observer also makes notes during the interview on a copy of the script. Subsequently the videotapes are transcribed, viewed, and analyzed.

The latter three interviews remain to be developed for administration during 1993 and 1994. Interview #4 will place special emphasis on exploring the child's strategic and heuristic thinking. In interview #5 we plan to include an interactive computer environment. Interview #6 will return to selected mathematical ideas from the earlier interviews.

As this is written the first two sets of interviews are being transcribed, and analysis is commencing. The purpose of this paper is not to report on the outcomes to date, but to discuss general perspectives on structured, task-based interviews of this sort, and to invite comment.

ON THE SCIENTIFIC NATURE OF TASK-BASED INTERVIEWS

This study is exploratory. Consisting as it does of a collection of individual case studies, its outcomes are not scientifically reproducible. Nevertheless we have devoted great effort to structuring the interviews ahead of time to be both *flexible* and *reproducible*. Flexibility by the clinician in such an interview is essential to allow for the enormous differences that occur in individual problem-solving behaviors, and (since a major goal is to observe processes the child uses spontaneously) to avoid "leading" the child in a predetermined direction.

Reproducibility, however, means that the clinician is not merely inventing questions as the child responds. It permits, to a certain imperfect degree, the "same" interview to be administered by different clinicians, to different children, in different contexts. To accomplish this sufficiently many contingencies must be anticipated, and the criteria for the clinician's choices of questions or suggestions must be made explicit *in advance* for each contingency. This is what we have sought to do in the process of interview design.

For example in interview #1, after a brief pause (to allow for the possibility of spontaneous responses to the three presented cards), the child is asked, "What do you think would be on the next card?" Contingencies then include "response" and "don't know"; if the child responds, the next contingencies include "offers a complete, coherent reason" or "has not yet given a complete, coherent reason", with or without having constructed a "coherent external representation." The definitions (from the directions in the interview #1 script) are as follows:

> "A complete and coherent verbal reason means one based on a described pattern. A coherent external representation means a drawing, picture, or chip model. It is *not* required that the 'canonical' fourth card (with 7 dots) be drawn, or the canonical pattern described, for a response to be considered a complete and coherent reason and a coherent external representation. An answer such as '7, because it's 2 more' is a coherent verbal reason, but not considered complete because it refers only to finding the next card and not to the basis for the pattern. An answer such as '7, because this card has 2 more than that one, so the next one has 2 more also' would be considered coherent and complete. If there is a discrepancy between the number of dots stated and the number in an external representation, the verbal reason is not considered 'coherent'. This is intended to describe the 'boundary' between responses that are and are not accepted as complete and coherent at this stage."

This is the level of detail at which many contingencies are considered. The clinician's next question or suggestion (e.g., "Why do you think so?" or "Can you show me what you mean?" leading if necessary to "Can you show me using some of these materials?") depends on the contingency which best describes the child's response. Such a level of description seeks to make *explicit* the usually *tacit* conditions that ordinarily influence a skilled clinician. In principle, a detailed structured interview description permits (a) *replicability* of the interview itself, though contextual and other factors may still vary from occasion to occasion; (b) *comparability* of interview outcomes between different children, among different populations of children, or across different conditions; (c) subsequent experiments to investigate the *generalizability* of observations made in individual case studies; (d) explicit *discussion and critique* of the contingencies, the criteria for the clinician's responses, etc.; and (e) an explicit basis for discussing the analysis of outcomes. For other perspectives, see Cobb (1986), Hart (1986), and Steffe (1991).

THE ROLE OF THEORY

The questions asked and the observations made during any scientific investigation depend heavily on the theory we bring to it. In my view, the main question is not whether theory *should* influence us in this enterprise--I would maintain that it always, inevitably does:

> ". . . perhaps the attempts to use the methods of science [in education] have failed because science has been misunderstood.
>
> In these attempts it had been assumed that science was primarily factual, that indeed it dealt almost solely in facts, that theory had no role in science. Careful observation of science reveals this to be false. It might be closer to the truth to say that 'facts'--at least interesting facts--are almost unable to exist *except in the presence of an appropriate theory* [emphasis in original]. Without an appropriate theory, one cannot even state what the 'facts' are." (Davis, 1984, p. 22)

The question pertaining to clinical interviews is the extent to which the influence of theory is tacit, through the unconscious assumptions of clinicians, researchers, and/or teachers, or explicit. Our goal in the present study is to be as explicit as possible.

The theoretical underpinnings of the series of task-based interviews include the concept of (internal) *competencies* and structures of such competencies, that develop over time in the child, and that can be *inferred* from observable behavior. The idea that competencies are encoded in several different kinds of internal representations, and that these *interact* with each other and with observable, external representations, is also fundamental. The key distinction between the child spontaneously bringing particular competencies to bear, or doing so only when prompted, is also theoretically-based: it involves the child's exercise of planning competencies to call on other competencies (verbal, imagistic, formal notational, etc.). These ideas have influenced the task-based interview development as follows: We pose tasks which permit the children to perform at each step spontaneously. We explore not only the child's overt behavior, but the reason the child gives for taking each step. Recognizing that competency structures may be partially developed, we provide hints or heuristic suggestions when blockage occurs--this often permits the child to demonstrate competencies that otherwise he or she would never "get to". We seek information about each kind of internal representational system--thus, not satisfied with a coherent verbal explanation only, we encourage the child to construct a concrete, external representation. We in- clude a "cross section" of questions exploring visualization, affect, and strategic thinking.

The distinction between external and internal representation means we must attend carefully to both. We regard the tasks posed as *external* to individual children; as embodying syntax, content, context, and structure variables that we select when we design the interviews. In particular the mathematical structures of the tasks (semantic structures and formal structures-- additive, multiplicative, etc.) are consciously chosen. The children's behaviors then result from interactions between the task environment and their internal cognitive and affective representations.

THE ROLE OF CONTEXT

Interviews do not take place outside of a social and psychological *context*. We observe that the child's expectations of an interview are influenced by the fact that it is conducted by a relative stranger (the clinician); it takes place in school (and thus might involve some kind of test that "counts" toward an evaluation, and the tasks are likely to have "right" and "wrong" answers); it involves tasks unrelated to a goal or purpose generated by the child; it may be taking place at a moment when the child is alert, tired, hungry, distracted, or excited; and so forth. Seemingly small, contextual aspects of the tasks themselves may have important effects. For example in presenting the three cards interview #1, we permit the child to see the cards being drawn from a stack of cards in a manila envelope. From this minor contextual feature the child may infer that there is a deck of cards larger than the three that are shown, and possibly that there is a pattern in the cards. Three cards presented wholly "out of context" might not so readily elicit this expectation.

Since so much that may occur during a task-based interview is context-dependent, how can we consider what we observe to be more than accidental, one-time events? One important condition is that the constructs we *infer* from our observations be reasonably *stable* against contextual variations. Thus, while a child's behavior may vary considerably from one context to another, when we infer particular competencies or structures of competencies from the behavior (such as the ability to visualize cutting a cube across two perpendicular directions, inferred from a coherent description of the component pieces with appropriate gestures) we are inferring aspects of the child's cognition that we expect to be fairly stable. Understanding the context dependence of the interviews also means recognizing how very difficult it is to establish advance criteria for the inferences about each child's cognition and affect that we want to draw from our observations. The plan is to make the best conjectures possible, and to try to be explicit about the reasons for these conjectures (including relevant contextual factors) as they occur.

PRINCIPLES OF INTERVIEW DESIGN AND CONSTRUCTION

The above considerations lead me to formulate the following tentative and partial principles of interview design and construction, in trying to establish the strongest possible scientific foundation and maximize the information gathered through a task-based interview: (1) *Accessibility:* Interview tasks should embody mathematical ideas and structures appropriate for the subjects being interviewed--so that they are able to represent task configurations, conditions, and goals. (2) *Rich representational structure:* Tasks should embody meaningful semantic structures (imagistic level) and strategies of some complexity (planning and executive control level), as well as formal, symbolic structures (notational level). (3) Subjects should engage in *free problem solving wherever possible* to allow observation of spontaneous behaviors and reasons for spontaneous choices *prior* to offering prompts or suggestions. Providing premature guidance results in loss of information. This may mean some sacrifice of the speed with which the subject understands the problem, or progresses through it. (4) *Explicit criteria:* All major contingencies should be clearly addressed in the interview design, as explicitly as possible; but *without distinguishing "right" and "wrong" responses.* Thus all responses should be "accepted", with structured questions designed to provide subjects with opportunities to self-correct in any contingency. (5) Various external representational capabilities should be provided, allowing for *interaction* with an observable learning or problem-solving environment.

REFERENCES

Bodner, B. L. and Goldin, G. A. (1991a), Drawing a Diagram: Observing a Partially Developed Heuristic Process in College Students. In F. Furinghetti (Ed.), **Procs. of the Fifteenth Int'l. Conference for the Psychology of Mathematics Education (PME),** Assisi, Italy, **Vol. 1.** Genoa, Italy: Diparti- mento di Matematica dell'Università di Genova, 160-167.

----- (1991b), Cognitive Obstacles of Developmental-Level College Students in Drawing Diagrams. In R. G. Underhill (Ed.), **op. cit.,** 8-14.

Cobb, P. (1986), Clinical Interviewing in the Context of Research Programs. In G. Lappan and R. Even (Eds.), **op. cit.,** 90-110.

Damarin, S. and Shelton, M., Eds. (1985), **Procs. of the Seventh Annual Meeting of PME-NA** (North American Chapter of the International Group for the Psychology of Mathematics Education). Columbus, OH: Ohio State Univ. Department of Educational Theory and Practice.

Davis, R. B. (1984), **Learning Mathematics: The Cognitive Science Approach to Mathematics Education.** Norwood, NJ: Ablex Publishing.

DeBellis, V. A. and Goldin, G. A. (1991), Interactions between Cognition and Affect in Eight High School Students' Individual Problem Solving. In R. G. Underhill, **op. cit.,** 29-35.

Goldin, G. A. (1985), Studying Children's Use of Heuristic Processes for Mathematical Problem Solving through Structured Clinical Interviews. In S. Damarin and M. Shelton, **op. cit.,** 94-99.

----- (1986), Comments on Structured Individual Interview Methods for the Study of Problem Solving. In G. Lappan and R. Even, **op. cit.,** 111-119.

----- (1987), Cognitive Representational Systems for Mathematical Problem Solving. In C. Janvier (Ed.), **Problems of Representation in the Teaching and Learning of Mathematics.** Hillsdale, NJ: Erlbaum, 125-145.

----- (1992), Toward an Assessment Framework for School Mathematics. In R. Lesh and S. J. Lamon (Eds.), **Assessment of Authentic Performance in School Mathematics.** Washington, DC: American Association for the Advancement of Science, 63-88.

Goldin, G. A., DeBellis, V. A., DeWindt-King, A. M., Passantino, C. B., and Zang, R. (1993), Task-Based Interviews for a Longitudinal Study of Child- ren's Mathematical Development. In **Procs. of the Seventeenth Int'l. Conference for the Psychology of Mathematics Education (PME),** in press.

Goldin, G. A. and Landis, J. H. (1985), A Problem-Solving Interview with "Stan" (Age 11). In S. Damarin and M. Shelton (Eds.), **op. cit.**, pp. 100-105.

----- (1986), A Study of Children's Mathematical Problem-Solving Heuristics. In L. Burton and C. Hoyles (Eds.), **Proceedings of the Tenth Int'l. Conference for the Psychology of Mathematics Education (PME), Vol. 1.** London: Univ. of London Institute of Education, 427-432.

Hart, L. (1986), Small Group Problem Solving as a Data Source for the Individual. In G. Lappan and R. Even (Eds.), **op. cit.**, pp. 120-127.

Lappan, G. and Even, R., Eds. (1986), **Procs. of the Eighth Annual Meeting of PME-NA: Plenary Speeches and Symposium.** E. Lansing, MI: Univ. of Michigan Dept. of Mathematics and Dept. of Teacher Education.

Steffe, L. P. (1991), The Constructivist Teaching Experiment: Illustrations and Implications. In E. von Glasersfeld (Ed.), **Radical Constructivism in Mathematics Education.** Dordrecht, The Netherlands: Kluwer, 177-194.

Underhill, R. G., Ed. (1991), **Procs. of the Thirteenth Annual Meeting of PME-NA** (North American Chapter of the Int'l. Group for the Psychology of Mathematics Education), **Vol. 1.** Blacksburg, VA: Virginia Tech. Division of Curriculum & Instruction.

METACOGNITIVE DECISIONS AND THEIR INFLUENCE ON PROBLEM SOLVING OUTCOMES

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In recent years, the role of metacognition in mathematical problem solving has begun to attract research interest, as metacognitive processes are considered to be an important factor influencing problem solving performance. The study described in this paper investigated the metacognitive strategies used by a pair of upper secondary school students while working on mechanics problems. The bulk of the data consisted of verbal protocols from uninterrupted think aloud paired problem solving sessions. Metacognitive decision points were identified in order to examine the monitoring contributions of each individual student, and the significance of student-student interactions. The main findings of the study were:

1. The subjects assumed differing, but complementary, metacognitive roles during problem solving.

2. The quality of the subjects' metacognitive decisions had an important influence on problem solving outcomes, but their decision making was sometimes adversely affected by the social interaction between them.

This paper reports on a study whose origins can be traced to the mathematics classroom, where the experience of teaching and observing upper secondary school students prompted questions such as:

> Why do students fail to use the knowledge they undoubtedly possess to help them solve mathematics problems that should be well within their grasp?

and

> Having chosen a problem solving strategy, why do students persist with the strategy whether or not it leads towards the desired goal?

Behaviours such as these can be explained by reference to the concept of metacognition, or awareness and regulation of one's own thinking (Brown, Bransford, Ferrara and Campione, 1983). Although it is important to be aware of one's mental state during problem solving, self-regulatory skills are considered to be even more crucial because the knowledge one possesses needs to be put to effective use. Regulation of cognition involves such activities as planning an overall course of action, selecting specific strategies, monitoring progress, assessing results, and revising plans and strategies if necessary (Garofalo and Lester, 1985).

Schoenfeld (1985a) identifies two broad types of control decision which may occur during the course of such activities, and which can influence problem solving outcomes. He argues that success is favoured if students:

1. exploit their knowledge to act on potentially useful information, and
2. discontinue inappropriate and unproductive strategies.

On the other hand, failure is virtually guaranteed by poor decisions like those made by the students mentioned above.

Early research on metacognition mostly involved reading or memory tasks (Brown et al., 1983), and it is only recently that the role of metacognition in the performance of mathematical tasks has begun to be studied. Much of this research has used either tertiary level or primary school students as the subjects (for example, Kroll, 1988; Venezky and Bregar, 1988); and studies which have attempted to train metacognitive strategies have tended to do so within separate "problem solving" courses (for example, Lester and others, 1989; Schoenfeld, 1985a), rather than treat metacognition - and problem solving itself - as a thinking process common to all branches of mathematics.

The present study differs from such research in two ways: the two subjects were senior secondary school students; and the problem tasks on which they worked, although challenging and unfamiliar, were similar to those they were likely to meet every day in their mathematics classroom. The aim of the study was to describe and qualitatively analyse the metacognitive strategies the two students used. Most of the data were obtained by instructing the students to think aloud while they worked cooperatively on these problems.

Two research questions were addressed:

1. What metacognitive strategies does each student use during problem solving?

The first question investigates the extent to which the subjects use the knowledge they possess to help them solve the problems they are set, and the manner in which they monitor and assess their progress towards the desired goal.

2. How does the presence, or absence, of metacognitive behaviour influence the outcome of problem solving?

The second question investigates Schoenfeld's (1983, 1985a) claim that the quality of metacognitive decision making can either promote or hinder problem solving success.

The remainder of the paper describes the conduct and results of the study, and discusses one important implication for mathematics teaching. However, as doubts are sometimes raised about the validity of verbal reports of thinking, and because think aloud methods have mainly been used with adults, rather than secondary school students like those who were the subjects of this research, it is appropriate to first outline the methodological decisions which influenced the design of the study.

VERBAL METHODS IN RESEARCH ON THINKING

Although the validity of verbal methods has been challenged on the grounds that subjects are unable to give accurate explanations for their behaviour (Nisbett and Wilson, 1977), verbal protocols do provide useful information if they are treated as data from which explanations should be inferred by the researcher, rather than the subjects (Genest and Turk, 1981).

The choice of suitable data collection procedures for this study was made by constructing three dimensions which can be used to characterise any verbal method. The dimensions are:

1. the time at which verbalisation is requested - concurrent or retrospective;
2. the degree of researcher intervention - from no intervention (in "think aloud" methods) to specific probes (in clinical interviews);
3. the instructions given to the subjects - to either report, or explain, their thinking (Ericcson and Simon, 1980; Genest and Turk, 1981).

After the advantages and limitations associated with variations along these dimensions were considered, it was concluded that the most accurate description of cognitive processes during task performance is obtained with concurrent verbalisation, no researcher intervention, and the instruction to report thinking. Nevertheless, two significant limitations remain: subjects may be unable to report all the cognitive processes of interest (incompleteness), and stress or task demands can distort cognitive processing (reactivity). For this reason it was decided to use Schoenfeld's (1985a, 1985b) pair protocol method, which has been designed to address these two limitations. Pair protocols are more likely to capture a complete record of students' typical thinking than single protocols because, first, two students working together produce more verbalisation than one and, second, the reassurance of mutual ignorance alleviates some of the pressure of working under observation. However, as incompleteness and reactivity are difficult to eliminate entirely, it is necessary to confirm inferences drawn from pair protocols by data from other sources, such as interviews and classroom observation.

METHOD

Subjects

The two subjects, "Rick" and "David", were Year 11 mathematics students in a large state run Senior College. The boys were sixteen years old when they took part in the study. Although both were described by their teacher as high ability students, David usually achieved better results in mathematics than Rick. This difference in status contributed an adversary flavour to their relationship; yet, the two remained good friends and consistently worked as a pair in the classroom. They were chosen for this study because they were highly articulate and accustomed to verbalising their thoughts as they worked together on mathematics problems.

Problems

To ensure that the problem tasks were relevant to the subjects' classroom experience, mechanics problems dealing with topics recently covered in their mathematics class were chosen for use in the think aloud sessions. For the purposes of this study, a task is considered to be a genuine problem for the student if progress is blocked at some stage, but merely an exercise for the solver who can call on a ready made solution schema (Silver, 1982). Most of the think aloud problems were therefore intended to be challenging enough to require, and elicit, metacognitive behaviour to remove any blockages. However, some routine exercises were also included to help put the subjects at ease at the start of each session.

Procedures

Three problem solving sessions, each of which lasted about one hour, were videotaped over a period of four weeks. The subjects attempted two problems in each session. Taping took place during the subjects' free time, in their regular mathematics classroom (which was otherwise vacant). The subjects were instructed to work together on the problems to produce one solution, and to say everything that came into their heads as they worked. They were informed that they would not be interrupted or given hints.

Retrospective interviews were used for supplementary data gathering if the pair protocols were found to be an incomplete record of the subjects' thinking. The interviews asked for a report, and then an explanation, of specific thinking processes so that the tentative inferences drawn from the pair protocols could be checked. Classroom observation (one and a half hours per week for ten weeks) and discussions with the subjects' teacher also allowed judgments to be made as to the typicality of the subjects' think aloud problem solving behaviour.

Data Coding and Analysis

Schoenfeld's (1985a) protocol analysis method was the basic tool for identifying the metacognitive strategies used by each student. The videotapes of each think aloud session were transcribed and the resulting protocols parsed into episodes representing distinctive types of problem solving behaviour: reading, analysis, exploration, planning, implementation and verification. However, a discussion of the episode parsing is beyond the scope of this paper, in which most interest centres on metacognitive decision points where new information was recognised or local assessments of specific aspects of the solution were made. Decision points were identified and classified in a way which extends Schoenfeld's analysis procedure. Unlike Schoenfeld's original scheme - which was designed to allow generalisations about the metacognitive behaviour of many pairs of students to be made - the analysis technique described below reveals the unique contributions made by two specific individuals, and the pattern of interactions between them.

Table 1. Scheme for Classifying Problem Solving Outcomes

Control Decision		Classification	Outcome
Discontinue inappropriate strategy	Exploit knowledge		
-	-	Control -ve	Bad decisions guarantee failure
+	-	Control neutral	Equivocal
+	+	Control +ve	Control decisions promote success
N/A	N/A	Expert	No need for control behaviour - the task is an exercise

Schoenfeld's new information points were renamed New Idea points (NI's), to describe places where previously overlooked or unrecognised information came to light or the possibility of taking a new approach was mentioned. NI's were classified according to who initiated them, how relevant or useful they were, and the nature and appropriateness of the response. Since any student's NI's may be either "useful" or "not useful", and his partner's response either "acceptance" or "rejection", it is reasonable to argue that if a good idea is rejected, or a bad idea accepted, then problem solving goes astray. In this way it was possible to trace studentstudent interactions, and their consequences for the solution process.

Local Assessments (LA's) of a particular aspect of the solution were classified according to who made them and what was assessed (procedure, result, task difficulty or knowledge), and were evaluated for appropriateness, effectiveness and influence on the solution.

The extent to which the presence, or absence, of metacognitive behaviour influenced problem solving outcomes was determined by classifying each protocol in terms of the control decisions it contained. The classification scheme summarised in Table 1 describes four types of solution attempt, and was derived from Schoenfeld's (1985a) discussion of the influence of control decisions on problem solving outcomes. Evidence for the two types of control decision, "discontinue inappropriate strategy" and "exploit knowledge", was provided by the incidence, function and quality of Local Assessments and New Idea points respectively.

RESULTS

This section summarises the results of analysis of four of the most interesting protocols - the PULLEY, GOLF, CRICKET and MASCOT problems. (Problem statements are given in the Appendix.) The findings are outlined by addressing the two research questions which guided this study. It should be noted, however, that these findings are results of a small case study and are not presented as conclusions which are generalisable from the sample to a larger population.

1. What metacognitive strategies does each student use during problem solving?

Table 2 shows the extent to which Rick (R) and David (D) exploited their knowledge and the manner in which they monitored their progress, by giving the numbers and types of New Idea points and Local Assessments initiated by each student across the four protocols.

Table 2. Summary of NI's and LA's - All Protocols

	PULLEY		GOLF		CRICKET		MASCOT	
Initiator	**R**	**D**	**R**	**D**	**R**	**D**	**R**	**D**
Total NI's	2	0	3	2	8	4	9	6
LA's								
procedure - accuracy	-	-	-	-	1	-	3	-
procedure - usefulness	-	1	-	-	1	4	4	2
result - accuracy	-	-	-	1	2	2	1	-
result - reasonableness	-	-	-	-	1	-	1	2
task difficulty	-	2	-	1	-	1	-	1
knowledge	-	-	-	1	-	1	-	3
Total LA's	0	3	0	3	5	8	9	8

Several inferences can be made about their metacognitive strategy use:

1. Rick consistently generated more new ideas than David.
2. David produced more Local Assessments than Rick (except in the MASCOT protocol).
3. Only Rick checked the accuracy of procedures as they were executed.
4. Only David evaluated task difficulty, and assessed what was known or not known.
5. Rick and David shared the responsibility for assessing the accuracy and reasonableness of results.

It seems, then, that Rick and David have differing, but complementary, metacognitive strengths. Rick played two roles during problem solving: he was both the idea generator and the checker of David's calculations. However, many of Rick's ideas were irrelevant or unworkable. Because he failed to assess the usefulness of his ideas, Rick was in constant danger of setting off on wild goose chases. The task of rescuing him from this fate fell to David, who effectively filled the role of procedural assessor in all but the MASCOT problem. (In the latter protocol it was Rick who made the majority of procedural assessments as he tried, unsuccessfully, to convince David that his strategy was wrong.)

2. How does the presence, or absence, of metacognitive behaviour influence the outcome of problem solving? Table 3 classifies each protocol according to the influence of control decisions on the outcome.

Table 3. Influence of Metacognitive Behaviour on Problem Solving Outcomes

PROBLEM	OUTCOME	DISCONTINUE STRATEGY	EXPLOIT KNOWLEDGE	CLASSIFICATION
PULLEY	Success	N/A	N/A	Expert
GOLF	Success	+	-	Control Neutral
CRICKET	Success	+	+	Control +ve
MASCOT	Failure	-	-	Control -ve

The PULLEY problem, presented as a warm up task at the start of the second videotaped problem solving session, was solved in a little over five minutes. It elicited very little metacognitive behaviour because the students were able to call on a ready made solution schema. Since control decisions had no effect on the outcome this task was merely an exercise for Rick and David, and the protocol is classified as expert.

Although the GOLF problem was also intended to be an exercise, Rick and David's progress was blocked for a time because their inappropriate strategy had led to an impossible result: 2.5 = cos ¨. Despite their previous experience with trigonometry, the students continued until their calculator's "Error" message alerted them to their mistake. After mechanical errors were eliminated as the cause of the difficulty, more careful reading and analysis pinpointed the source of the error and the incorrect strategy was discontinued. Nevertheless, the pair's failure to access their knowledge of trigonometry could have had serious consequences, and the GOLF protocol is therefore classified as control neutral: the outcome could have been either success or failure.

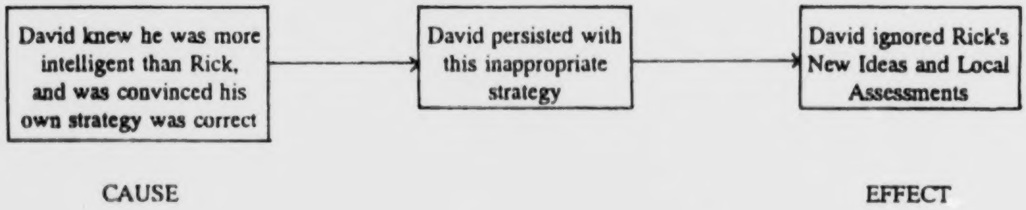
In the CRICKET problem, Rick and David effectively coordinated their respective roles of idea generator/calculation checker and procedural assessor. On discovering that they were stuck after only two minutes of working, David prevented Rick from going any further with his inappropriate strategy for calculating the time of flight of the ball. During the next few minutes, many new ideas were proposed, tested, and rejected as unhelpful. Relevant information was eventually recognised, and prior knowledge about projectile motion was accessed and used. Good control decisions promoted success; therefore the CRICKET protocol is classified as control positive.

The MASCOT problem was the only one that Rick and David were unable to solve. After analysing the problem statement they settled on an inappropriate and unproductive trigonometry based strategy, which they nevertheless pursued for more than twenty minutes. Evaluation of student-student interactions revealed many instances where good ideas were rejected or ignored, and bad ideas were accepted without assessment. Useful knowledge about the relationship between force and acceleration remained unexploited, and the wild goose chase was not curtailed. Because these poor control decisions contributed to Rick and David's failure to solve the MASCOT problem, the protocol is classified as control negative.

DISCUSSION AND IMPLICATIONS

The first finding of the present study, concerning metacognitive strategy use by secondary school students, adds to the limited knowledge that currently exists in this area. By coordinating their differing strategic preferences and metacognitive strengths, Rick and David were usually able to think their way around obstacles to their progress. This result illustrates the benefits of peer collaboration as an effective means of developing and practising selfregulation in the problem solving classroom (Schoenfeld, 1987).

However, further consideration of data related to the second finding suggests that caution is needed in forming collaborative groups. Although the quality of metacognitive decision making did indeed contribute to problem solving success or failure, just as Schoenfeld (1983, 1985b) claims it should, there is evidence that the outcome of the MASCOT problem was influenced by another factor - the social interaction between the subjects. In particular, Rick's useful New Ideas and appropriate Local Assessments were consistently rejected or ignored by his partner. A plausible explanation for Rick and David's failure to solve this problem could therefore involve the following chain of cause-effect relationships:


David knew he was more intelligent than Rick, and was convinced his own strategy was correct
David persisted with this inappropriate strategy
David ignored Rick's New Ideas and Local Assessments
CAUSE
EFFECT

In the MASCOT problem, the supposedly collaborative interactions between the students hindered, rather than promoted, metacognitive decision making.

Clearly, not all collaborative relationships are educationally valuable. Forman (1989) names three conditions for collaboration to be effective:

1. Students must have mutual respect for each other's perspective on the task.
2. There must be an equal distribution of knowledge.
3. There must be an equal distribution of power.

All three conditions were violated in the MASCOT protocol. First, David did not respect Rick's New Ideas or Local Assessments. Second, there were unequal distributions of knowledge and power: David was the more powerful student because he took charge of the course of the solution; but he was not, in this protocol, the more knowledgeable. David rejected Rick's attempt to usurp his own role of procedural assessor (see Table 2), and the continued existence of differentiated problem solving roles had an adverse effect on the solution process. Further research is needed on the conditions under which peer interaction fosters or interferes with metacognitive self-regulation, so that effective approaches for teaching problem solving in small group settings can be developed.

REFERENCES

Brown, A. L., Bransford, J. D., Ferrara, R. A. & Campione, J. C. (1983). Learning, remembering and understanding. In P. H. Mason (Ed.), **Handbook of Child Psychology, Vol. 3** (4th ed.) (pp. 77-166). New York: Wiley.

Ericcson, K. A. & Simon, H. A. (1980). Verbal reports as data. **Psychological Review, 87,** 215-251.

Forman, E. (1989). The role of peer interaction in the social construction of mathematical knowledge. **International Journal of Educational Research, 13,** 55-70.

Garofalo, J. & Lester, F. K., Jr. (1985). Metacognition, cognitive monitoring, and mathematical performance. **Journal for Research in Mathematics Education, 16,** 163-176.

Genest, M. & Turk, D. (1981). Think-aloud approaches to cognitive assessment. In T. V. Merluzzi, C. R. Glass & M. Genest (Eds.), **Cognitive Assessment** (pp. 233-269). New York: The Guilford Press.

Kroll, D. L. (1988). Cooperative problem solving and metacognition: A case study of three pairs of women. Doctoral dissertation, Indiana University. Dissertation Abstracts International, 49, 2958 A. (University Microfilms No. 8902580).

Lester, F. K., Jr. and others (1989). The role of metacognition in mathematical problem solving: A study of two grade seven classes. Final Report. Indiana University, Bloomington: Mathematics Education Development Centre. (ED 314 255).

Nisbett, R. E. & Wilson, T. D. (1977). Telling more than we can know: Verbal reports on mental processes. **Psychological Review**, 84, 231-259.

Schoenfeld, A. H. (1983). Beyond the purely cognitive: Belief systems, social cognitions, and metacognitions as driving forces in intellectual performance. **Cognitive Science, 7,** 329363.

Schoenfeld, A. H. (1985a). **Mathematical Problem Solving.** Orlando, Florida: Academic Press. Schoenfeld, A. H. (1985b). Making sense out of "out loud" problem-solving protocols. **Journal of Mathematical Behaviour, 4,** 171-191.

Schoenfeld, A. H. (1987). What's all the fuss about metacognition? In A. H. Schoenfeld (Ed.), **Cognitive Science and Mathematics Education** (pp. 189-215). Hillsdale, New Jersey: Erlbaum.

Silver, E. A. (1982). Knowledge organisation and mathematical problem solving. In F. K. Lester, Jr. & J.

Garofalo (Eds.), **Mathematical Problem Solving: Issues in Research** (pp. 15-25). Philadelphia: The Franklin Institute Press.

Venezky, R. L. & Bregar, W. S. (1988). Different levels of ability in solving mathematical word problems. **Journal of Mathematical Behaviour, 7,** 111-134.

APPENDIX

PULLEY

Two bodies of mass 4 kg and 3 kg are at rest on two smooth inclined planes placed back to back. The bodies are connected by a string passing over a smooth pulley at the top of the planes. If the 4 kg mass rests on a plane inclined at 35° to the horizontal, find the inclination of the other plane.

GOLF

A golfer hits a ball from a point on a level fairway, and 2 seconds later it hits the fairway 50 meters away. Find:

(a) the velocity and angle of projection of the golf ball

(b) the maximum height of the ball above the fairway.

CRICKET

A batsman hits a cricket ball "off his toes" towards a fieldsman who is 65 meters away. The ball reaches a maximum height of 4.9 meters and the horizontal component of its velocity is 28 m/s. Find the constant speed with which the fieldsman must run forward, starting at the instant the ball is hit, in order to catch the ball at a height of 1.3 meters above the ground. (Use g = 9.8.)

MASCOT

A mascot suspended from a car's rear view mirror hangs vertically when the car is moving with uniform velocity of 80 km/hr along a straight level road. The brakes are applied so that the car is stopped with uniform retardation. Find the angle through which the mascot is deflected if the car comes to rest 137 meters after the brakes are applied.

ATTITUDES OF SOME N.S.W. SECONDARY MATHEMATICS TEACHERS TO ALTERNATE METHODS OF ASSESSMENT IN MATHEMATICS.

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In a paper presented at Merga - 15 in 1992 at University of Western Sydney - Hawkesbury, N.S.W., the author reported on the beginnings of a study of teacher's attitudes to possible changes in assessment practices within secondary school mathematics classrooms within this state. It was reported how content laden the secondary mathematics curriculum is in N.S.W. and just how prevalent is the utilisation of timed pencil and paper tests in this state. Because these assessment procedures are so conservative, methodology employed within the classroom is similarly traditional and does not really reflect changes occurring in procedures in secondary mathematics classrooms in some overseas countries and interstate within Australia. Nonetheless, some changes are occurring.

This paper will report further on a series of questionnaires that have been conducted with practising secondary mathematics teachers on the subject of alternative modes of assessment, and endeavour to make some recommendations for the future.

Across the world changes are being advocated in the way that mathematics is to be taught and learned at the school level. These changes are based on sound research findings in the field of mathematics education. The types of mathematical skills which are being encouraged are very different from the narrow objectives which have typified the teaching and learning of the subject in the past. Problem solving and its applications in mathematics is now being emphasised along with the advocacy of co-operative group processes including the use of language. Computers and calculators have changed permanently our vision of what constitutes an adequate mathematical education. But when it comes down to the crunch, our assessment procedures utilised in internal and external school mathematics examinations always dictate in the mind of the learner, what in their mathematics education is of greatest importance.

It is quite clear that secondary mathematics teachers continue to predominantly assess their students' understanding of the subject via the timed pencil and paper test. Despite many calls for change, both overseas and in Australia, (e.g. U.S.A. - N.C.T.M., 1989; United Kingdom - Department of Education and Science, 1990; Australia - National Mathematics Statement, 1990), resistance by teachers is very common and although some change is occurring, it is patchy and fairly slow. As outlined at MERGA - 15, the situation in New South Wales secondary schools is particularly resistant (Grimison, 1992). The mathematics curriculum is characterised by a very solid syllabus in mathematics for Years 7 - 12 which is almost universally assessed by the pencil and paper test (both internally and externally). This is especially true once students enter Year 9 and beyond.

Some experimentation with alternate forms of assessment was reported to be occurring in the largely mixed ability classes in Years 7 and 8, and ofcourse in the primary grades within this state (Grimison, 1992). Some progress has also been made in the internal assessment procedures in the recently developed (1989) Year 11 and 12 course, Mathematics in Practice . But this course is only currently attracting about 2000 out of the 60000 Higher School Certificate candidates. For the vast majority of students of mathematics in N.S.W., the method of assessment remains very traditional. Overseas and in a number of other Australian states, advances are occurring in the widening of secondary mathematics assessment procedures as reported by Leder (1992) and Stephens and Izard (1992).

In N.S.W. secondary schools, mathematics teachers are communicating to their classes the value of the product rather than the process of mathematics via the assessment techniques employed. Recent literature is filled with the repercussions of always using the test as the sole method of assessment in mathematics (e.g.

Clarke, 1987, Clarke and Lovitt, 1989, Mousley, 1991, etc.,). This research literature makes it quite plain that if we desire to communicate process as being important in mathematics learning as well as product, then this necessitates a broadening in the range of assessment tasks utilised. Timed pencil and paper tests rarely assess understanding in the higher cognitive levels, including comprehension, application, analysis, synthesis and evaluation. To function in these desirable higher cognitive levels, skills need to developed through a broader range of activities other than pen and paper knowledge or product type tests.

Some of the assessment alternatives which have been suggested in the literature include student journals, work profiles, interviews, observations, oral tests, parental and student self assessment, practical activities and investigational tasks. Historically, secondary mathematics teachers have been opposed to these alternative forms of assessment and have always relied on the tried and tested traditional form of the formal hand-written test. This is what they themselves experienced in their own schooling and they consider these practices to be fair and equable. Yet as Clarke claims "it is through our assessment that we communicate most clearly to our students which learning activities we value" (Clarke, 1987).

Secondly, mathematics is seen to be objective in comparison to subjects such as the humanities and social sciences. This very objectivity of mathematics is seen to be one of the subject's great advantages as there is much less of a problem encountered in its marking, as occurs in the subjectivity of marking essays in the humanities or social sciences. There is thus a very firm belief that using tests in mathematics will be fairer on all the students.

The current assessment techniques employed in N.S.W. secondary mathematics classrooms are suited most closely to the few students at the top of the class, who perform well in traditional formal tests. The remainder of the class are left to contemplate failure which naturally lowers their confidence and self esteem. With this in mind, it is important to research the current attitudes of secondary mathematics teachers towards alternative forms of assessment.

THE STUDY:

The purpose of our research study was to examine the attitudes of some N.S.W. secondary mathematics teachers towards assessment alternatives. Firstly, we tested the hypothesis that those teachers with more years of experience of teaching would report attitudes revealing that the current traditional assessment techniques are able to assess the students adequately. This would mean that the level of satisfaction with pen and paper tests would depend on the years of experience in teaching. Secondly, we hypothesised that the more experienced teachers would report negative attitudes towards alternative assessment techniques in mathematics and those teachers who have less teaching experience would report more positive ones. Thirdly, the teachers were asked to respond in detail to a number of forms of alternative assessment and an endeavour was made to determine their objections to them.

The subjects in this study conducted in 1992 were practising secondary mathematics teachers in eleven Sydney schools. There were three private and eight Government schools from varying socio-economic areas of Sydney. A total of 60 teachers participated in the study. Schools were randomly selected. Seven university students enrolled in their third year of a four year professional honours Bachelor of Education Degree at the University of Sydney, together with the author, conducted this small piece of research. A questionnaire was developed by the group which was to distributed to practising mathematics teachers in these selected schools. Both closed and open questions were included. A number of follow-up interviews with some of the subjects were intended to be carried out but this did bot happen because of the time factor.

The questionnaire was administered to teachers in the 11 schools, with 60 completed returns obtained, having each been given about a week to complete. The 60 subjects were grouped into years of teaching experience by the following categories: 0 - 9, 10 - 19, 20 - 29 and 30 - 39. Their profile is shown in Table 1.

Years of Teaching Experience	Number of Teachers
0 - 9	14
10 - 19	24
20 - 29	16
30 -39	6

Table 1

There were four questions in the survey instrument. Question 1 was included, as reported above, to enable the grouping of responses into length of time teaching. This then enabled the completed questionnaires to be sorted into these four groups.

Responses to Question 2 were recorded on a 5-point Likert Scale, which endeavoured to ascertain the respondents' attitudes to the usefulness, or otherwise, of paper and pencil tests for assessing students' ability in mathematics. These results were added together and averaged, counting the 5 point scale as ranging from 0/4 to 4/4, then converting these averages to percentages. This was done for Years 7-8, 9-10 and 11-12 and compared across each of the four groups.

Question 3 asked for a ticked response as to whether six methods of alternative assessment were used at all, or not, by the teacher. The alternative forms of assessment recorded in the question were oral, practical, observation, student journals, student analysis and parental assessment. These results were recorded and averaged and converted to a percentage for each of the four groups.

Question 4 endeavoured to ascertain the teachers' reasons for not utilising the six nominated forms of alternative assessment. These reasons tended to fall into five factors and they were then recorded as percentages for each form.

RESULTS:

Question 2

This question required the subjects to indicate on a scale how well paper and pencil tests assess student ability in Years 7-8, 9-10 and 11-12. These levels of satisfaction were converted into a percentage as explained above. The results are shown below in Table 2.

School Years	0 - 9 yrs teaching	10 - 19 yrs teaching	20 - 29 yrs teaching	30 - 39 yrs teaching
Yrs 7 - 8	67%	67%	74%	54%
Yrs 9 - 10	66%	72%	85%	71%
Yrs 11 - 12	77%	83%	94%	95%

Table 2

Question 3

This question required subjects to indicate alternate assessment techniques they used at all, from a given list. The results are shown in Table 3 below. The percentages indicate those in each category who used these alternate forms at all.

Type of Assessment	0 - 9 yrs teaching	10 - 19 yrs teaching	20 - 29 yrs teaching	30 - 39 yrs teaching
Oral	64%	83%	75%	83%
Practical	64%	67%	63%	67%
Observation	71%	79%	56%	83%

Student Journals	29%	17%	25%	0%
Self-assessment	29%	29%	38%	50%
Parental assessment	7%	13%	6%	17%

Table 3

Question 4

If alternate forms of assessment were not used, this question asked teachers to give reasons why they thought the six alternative forms of assessment offered in Question 3 would be unusable for them. These reasons fell into five factors .

1. Insufficient time for implementation / hard to organise
2. Unstructured nature
3. Unsuitable
4. Unreliable/subjective
5. Insufficient resources on hand to permit implementation

The percentages of respondents in each of the four categories of teaching experience giving the five factors as negative reasons for using each of the six forms of alternative assessment are shown in the Table 4 below:

Oral	0 - 9 Yrs	10 - 19 Yrs	20 - 29Yrs	30 - 39 Yrs	Pract.	0 -9 Yrs	10 - 19 Yrs	20 - 29 Yrs	30 - 39 Yrs
Fact 1	7%	0%	25%	0%		0%	5%	19%	0%
Fact 2	7%	0%	0%	17%		7%	0%	6%	0%
Fact 3	7%	5%	6%	0%		0%	5%	6%	0%
Fact 4	0%	11%	19%	0%		0%	0%	6%	0%
Fact 5	0%	0%	0%	0%		0%	16%	0%	0%
Observation					Journals				
Fact 1	0%	0%	6%	0%	Fact 1	0%	0%	13%	17%
Fact 2	14%	0%	6%	0%	Fact 2	7%	17%	0%	0%
Fact 3	0%	0%	0%	0%	Fact 3	12%	38%	13%	0%
Fact 4	0%	17%	19%	0%	Fact 4	0%	0%	6%	17%
Fact 5	0%	0%	0%	0%	Fact 5	0%	0%	0%	0%
Self-Asses					Parental				
Fact 1	0%	0%	0%	17%	Fact 1	0%	8%	0%	0%
Fact 2	0%	0%	0%	0%	Fact 2	7%	0%	0%	0%
Fact 3	0%	33%	0%	0%	Fact 3	0%	0%	6%	0%
Fact 4	43%	4%	13%	17%	Fact 4	64%	54%	50%	83%
Fact 5	0%	0%	0%	0%	Fact 5	0%	0%	0%	0%

Table 4

DISCUSSION:

In Years 7/8, the most experienced teachers in our sample reported lower levels of satisfaction with timed pencil and paper tests in mathematics assessment than teachers with less experience. This result did not support our first hypothesis for the junior years of secondary education. However with increases in grade levels from Years 9/10 to 11/12, the more experienced teachers reported much higher levels of satisfaction than do less experienced ones. General levels of satisfaction do increase as grade level increases, till over 90% of experienced teachers expressed satisfaction with solely timed pencil and paper tests in Years 11 and 12, and less experienced ones recorded about 80%. The belief that written tests are the most appropriate way of assessing "real" mathematics seems very strong. Except for Years 7/8, our first hypothesis is supported.

The results in the third question show that the most common forms of alternative assessment are oral, practical and observation. Student journals together with student and parental assessment were uncommon. The most experienced teachers expressed a universal opposition to the use of student journals as a mode of assessment. A problem with the phrasing of this question was that many teachers were confused as to whether the assessment was formal or informal, and this was often written on the questionnaire (eg. Do you mean formal or informal? I am assuming formal.) Many teachers indicated that many of the forms were used in their assessment, but only as informal assessment which did not contribute to the final mark. Clearly, in addition, a whole lot of confusion occurred as to the exact meaning of some of these alternative forms of assessment, many of which may have appeared foreign to Sydney secondary mathematics teachers.

The last question endeavoured to explore reasons why teachers were opposed to using specified techniques of alternative assessment. These results lend partial support to the second hypothesis, as the less experienced teachers reported fewer dissatisfied views about why these alternate forms were unsuitable. This could have some relationship to the recency of their pre-service teacher training which presumably, included an introduction into a range of alternative forms of assessment. The range of reasons for this conflict is varied. However, the most common opposition relates to the perceived subjectivity of alternatives and their perceived "unsuitability" to mathematics. These reasons for this resistance have many implications for the future manner in which some of these alternative techniques are introduced into the N.S.W. secondary mathematics curriculum and assessment system.

It is clear that the traditional written test dominates any other form of assessment form in the schools surveyed. Attitudes at present to using alternative methods of assessment in the classroom are fairly negative. Clearly, the way forward is to change the way that mathematics is assessed in the external School and Higher School Certificates, and to place much greater emphasis upon less traditional and formal mathematical assessment practices.

REFERENCES:

Australian Education Council, 1990, A National Statement on Mathematics for Australian Schools. Curriculum Corporation.

Clarke, D and Lovitt, C., 1989, Assessment Alternatives in Mathematics, Mathematics Curriculum and Teaching Program. Canberra: Curriculum Development Centre.

Clarke, D., 1987, "A Rationale for Assessment Alternatives in Mathematics", in The Australian Mathematics Teacher, Vol 43 No 3.

Department of Education and Science, 1990, A National Mathematics Curriculum. London: H.M.S.O.

Grimison, L., 1992, "Assessment in Mathematics - Some Alternatives" in Southwell, B., et al., Proceedings of the Fifteenth Conference of Mathematics β Education Research Group of Australasia, University of Western Sydney - Richmond.

Leder, G., 1992, Assessment and Learning of Mathematics. Melbourne: A.C.E.R.

Mousley, J., 1991, "Years 7 - 10: Preparing for the VCE" in Reilly, J and Wettenhall, S., Mathematics: Inclusive, Dynamic, Exciting, Active, Stimulating. Mathematics Association of Victoria.

National Council of Teachers of Mathematics, 1989, Curriculum and Evaluation Standards for School Mathematics. Reston, Virginia: N.C.T.M.

Stephens, M and Izard, 1992, Reshaping Assessment Practices: Assessment in the Mathematical Sciences Under Challenge. Melbourne: A.C.E.R.

YOUNG CHILDREN'S NUMBER CONCEPTS - The effect of calculator use on teacher expectations

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This paper examines the extent to which seven teachers changed their expectations of children's mathematical performance during their first two years of teaching at the preparatory level in the Calculators in Primary Mathematics project – a long-term investigation into the effects of calculator use on the learning and teaching of primary mathematics. Questionnaire results show a slight increase in expectations for most of the 21 items dealing with counting and "large numbers", with a greater increase for the 5 items dealing with negative numbers. In general, teachers' increased expectations reflected their observations of children's performance during the previous year. Nevertheless, teachers' predictions remained conservative compared to actual levels of performance. However, data obtained from interviews and teachers' comments suggest that teachers are adopting a more open-ended approach to their mathematics teaching in order to cater for the increased level of understanding revealed by the presence of the calculator.

The *Calculators in Primary Mathematics* * project is a long-term investigation into the effects of the introduction of calculators on the learning and teaching of primary mathematics. It is based on the premise that calculators have the potential to significantly change mathematics curriculum and teaching (Corbitt, 1985, p.14; Cockcroft, 1982, p.109). Apart from the *Calculator-Aware Number* (CAN) project (Shuard, Walsh, Goodwin & Worcester, 1991), there is little evidence that such changes are commonly occurring (Curriculum Development Centre, 1986, p.18; Hembree & Dessart, 1986, p.83; Reys, 1989, p.173).

The project commenced at prep and grade 1 level in six schools in 1990, with 45 prep to grade 3 classes participating in 1992. After three years of involvement, prep and grade 1 are no longer part of the project in 1993, leaving 33 grades 2 to 4 classes participating. All children are given their own calculator to use whenever they wish, while teachers are provided with systematic professional support to assist them in using calculators to create a rich mathematical environment for children to explore.

Changes in teachers' expectations of children's mathematical performance and consequent changes in the mathematics curriculum provide one of the major foci for the project.

Previous studies have found that young children's mathematical skills are consistently underestimated by teachers, with the consequence that a high percentage of tasks set by teachers fail to challenge many children (Young-Loveridge, 1989, p.60; Desforges & Cockburn, 1987, pp.90-94; Wright, 1991, p.13).

Young children in the CAN project used their calculators to explore numbers and developed mathematical concepts which were unexpected for their age (Shuard, Walsh, Goodwin, & Worcester, 1991, p.6). Furthermore, teachers began to listen more to children and encourage children to talk more about mathematics (p.45).

Yackel, Cobb and Wood (to appear) regard the classroom as the primary site for teacher learning as teachers interact with their students, with class discussions providing the opportunity for teachers to develop expectations and anticipate children's solutions.

The *Calculators in Primary Mathematics* project is based on a model of teacher change which assumes that the major motivation for teachers to change is the desire for improvement in student

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* This research has been funded by the Australian Research Council, Deakin University and the University of Melbourne. The *Calculators in Primary Mathematics* project team consists of Susie Groves, Jill Cheeseman, Terry Beeby, Graham Ferres (Deakin University); Ron Welsh, Kaye Stacey (Melbourne University); and Paul Carlin (Catholic Education Office).

learning outcomes and that changes in teachers' classroom practice need to precede changes in teachers' beliefs and attitudes (Guskey, 1986, pp.7-10).

Considerable anecdotal evidence exists to suggest that young children in the project are dealing with much larger numbers than would normally be expected, as well as negative numbers, and, to a lesser extent, decimals (see, for example, Groves, Cheeseman, Clarke & Hawkes, 1991; Groves & Cheeseman, 1992, pp.3-4). If children exceed teachers' expectations in certain areas, then it might be anticipated that teachers will change their expectations, which will then lead to curriculum change. In order to investigate changes in teachers' expectations, data is being collected each year by means of an extensive written questionnaire and teacher interviews.

This paper examines the extent to which the seven teachers, who participated in the project at preparatory level for two or more years, changed their expectations of children's mathematical performance during their first two years. It focusses on three areas where we hypothesised that children in the calculator project would exceed teachers' previous expectations: counting, "large numbers" and negative numbers. Decimals are not dealt with here, as evidence suggests that gains in this area are not evident at the preparatory level.

QUESTIONNAIRE RESULTS

The written questionnaire contains 121 specific attainment targets, relating to number, at levels of difficulty ranging from what would be usually accepted as relevant to prep children (e.g. *count aloud to 10*) to ones which would normally not be seen as part of the primary mathematics curriculum (e.g. *correctly approximate to 3 significant figures a number on a calculator display*). The choice of items was based on a selection of attainment targets from the *Number* strand of the UK National Curriculum (National Curriculum Council, 1988). The items have been arranged at random, to avoid preconceptions about levels of difficulty. A total of 26 items deal with counting, "large numbers" and negative numbers.

At the beginning of each year of their involvement in the project, teachers are asked to indicate, for each of the items, their expectations of children at the beginning and end of that year. They do this by indicating whether they expect all, most, some or none of the children to be successful on the item. Their own completed questionnaires are then returned to the teachers at the end of the year for them to indicate, in the same manner, children's actual performance by the end of the year, and to make free comments on how they think their expectations of children's mathematical performance have changed over the period of their involvement in the project.

For the purpose of reporting the data here, the responses relating to the number of children expected or actually able to perform a task was mapped onto a linear numerical scale as follows:
None = 0 Some = 1 Most = 2 All = 3 .

Table 1 **Teachers' expectations of children at beginning and end of prep year and children's actual performance for items related to counting**

Item number	Description of task	Response*				
		Yr 1 Beg	Yr 2 Beg	Yr 1 End	Yr 2 End	Yr 1 Act
7	Count aloud to 120 .	3	5	8	10	11
22	Know the number 1 greater than 17 .	5	7	14	13	16
23	Count backwards from 10 to 1 .	7	10	18	20	20

40	Count by 2's from 1 to 17 .	1	3	7	9	7
42	Count by 10's from 7 to 77 .	1	1	6	5	9
47	Count by 10's from 10 to 100 .	4	4	14	12	15
88	Count by 2's from 2 to 28 .	1	3	7	8	10
92	Continue 5 , 10 , 15 , 20 , ...	2	5	11	11	17
100	Count backwards in 10's from 100 to 0 .	1	2	5	8	8
107	Count by 5's from 5 to 50 .	0	2	11	10	16
111	Know that 10 more than 42 is 52 .	0	1	4	6	7

* sum of responses of seven prep teachers in their first and second years for expectations of children at beginning and end of year and children's actual performance, using the scale of None = 0 ; Some = 1 ; Most = 2 ; All = 3 .

Tables 1 to 3 display, in summary form, the data obtained for the 26 items dealing with counting, "large numbers" and negative numbers. For each item, the numerical scale referred to above was used to sum the responses of the seven teachers for their expectations at the beginning and end of the two years and children's actual performance in the first year. So, for example, for Items 7 and 107 in Table 1, the entries of 10 for expectations at the end of year 2 were obtained since three teachers expected most children to be able to do the task and four teachers expected only some children to do so (10 = 3 x 2 + 4 x 1).

Table 2 **Teachers' expectations of children at beginning and end of prep year and children's actual performance for items related to large numbers**

Item number	Description of task	Response*				
		Yr 1 Beg	Yr 2 Beg	Yr 1 End	Yr 2 End	Yr 1 Act
13	Know that "four hundred and two" is written as 402 and know why neither 42 nor 4002 is correct.	1	0	1	5	5
16	Count from 389 to 407.	0	1	3	5	6
19	Correctly read 14 560.	0	0	1	1	3
20	Know the value of the 2 in 521 400 .	0	0	0	0	2

32	Know that 78 is 7 tens and 8 ones.	0	1	4	6	9
36	Write in figures four thousand and seventy three.	0	0	0	1	2
44	Correctly read 1810 .	0	1	1	4	3
55	Correctly read numerals to 1000 .	1	2	4	5	4
83	Count in 10's from 960 to 1050 .	0	0	1	3	5
98	Know the next number to appear on the trip meter of a car after 06399.	0	0	0	0	1

* sum of responses of seven prep teachers in their first and second years for expectations of children at beginning and end of year and children's actual performance, using the scale of None = 0 ; Some = 1 ; Most = 2 ; All = 3 .

Tables 1 and 2 show that for most of the 21 items related to counting and large numbers there was a slight increase or no change in teachers' expectations for children at the beginning and end of the year.

Four items (Items 22, 42 , 47 and 107) showed a slight decrease in expectations at the end of the year, mostly corresponding to just one teacher lowering expectations by one category. All of these items related to counting, with three of the four items relating to counting by 5's or 10's. It is difficult to explain these decreases in expectations as, for each of the four items, teachers had rated children's actual performance at the end of the first year not only higher than their predictions for the second year, but also higher than their expectations for the first year. This tendency to make conservative predictions in comparison with perceived actual performance from the previous year was consistent with data for all 21 items, with only three items (Items 40, 44 and 55) showing teachers predicting a higher end of year performance than actually observed in the previous year, while 15 items produced lower predictions.

Only one item (Item 13) showed a decrease in expectations for the beginning of the year. This item required considerable knowledge of place value, as did five of the six items which showed no change in expectations at the beginning of the year. This result is not surprising as these place value tasks would normally be expected to be much too difficult for children just beginning school. However, Item 13 resulted in the greatest positive change in teachers' expectations for the end of the year, with 5 teachers expecting some children to know how to write 402 in their second year, compared to only one teacher in the first year.

The other items to show significant positive changes in teachers' expectations were Item 23 (beginning of year) and Items 100 and 44 (end of year). Two of these items relate to counting backwards – a popular calculator activity which leads many children to "discover" negative numbers.

Table 3 **Teachers' expectations of children at beginning and end of prep year and children's actual performance for items related to negative numbers**

Item number	Description of task	Response*				
		Yr 1 Beg	Yr 2 Beg	Yr 1 End	Yr 2 End	Yr 1 Act

9	Know the answer for 2 − 6 .	0	0	0	3	3
59	Know that − 3 is the number immediately to the left of − 2 on a number line marked off in 1's.	0	1	0	3	2
60	Understand what is meant by −7 on a calculator display.	0	0	1	7	6
94	Count backwards from 9 to − 9 .	0	1	1	3	5
118	Count backwards in 10's from 50 to − 70 .	0	0	0	0	2

* sum of responses of seven prep teachers in their first and second years for expectations of children at beginning and end of year and children's actual performance, using the scale of None = 0 ; Some = 1 ; Most = 2 ; All = 3 .

Table 3 shows that, for four of the five items related to negative numbers, there was a considerable shift in teachers' expectations of children's performance at the end of the year, with approximately half to all of the teachers moving to expecting some children to be able to perform the task. The biggest change occurred for Item 60, understanding what is meant by −7 on a calculator display, where all teachers in their second year of involvement expected some children to understand this, compared to no teachers in their first year. Teachers' expectations for the end of their second year were closer to their perceptions of actual performance than for the earlier items. The only exception to this was Item 118, counting backwards in 10's from 50 to − 70, for which no teacher expected any children to be successful either at the beginning or end of year, despite two teachers reporting that some children could do this in the first year.

TEACHERS' COMMENTS

Data on changing expectations obtained from interviews with the seven teachers and their free comments at the end of the written questionnaire falls into three major categories.

All seven of the teachers specifically refer to at least some children working with and understanding larger numbers than previously expected - e.g. "They can be counting on their calculator with long strings of numbers, counting from zero by fives. They can get right up into the hundreds and anticipate the next numbers. Once they stop keying in the numbers, they get up to 505 and they will know the next number is 510". Teachers sometimes directly attribute the understanding to the presence of the calculator - e.g. "I think it enables them to do more difficult things. They will make up counting grids and count up into the hundreds and even into thousands and they wouldn't be doing that if they didn't have the calculator". Teachers frequently comment that they are no longer able to "limit" children to numbers less than 10 (or 50 or 100) - e.g. "I'm not conscious of staying within a given number range. Although I might start that way, I then leave it up to the children".

This last comment also suggests a change in the way in which teachers are approaching their mathematics teaching. Five of the seven teachers reported changes in their teaching of mathematics, while the remaining two denied any such changes. The most common changes were in the extent of open-endedness of the tasks and the extent to which children are able to lead the learning process. For example, one teacher reports that "I have changed the way I plan the sorts of activities I do and allow the kids to extend themselves. I think that's what the calculator has allowed." Several teachers compare the changes to earlier changes in their language teaching - e.g. "I am asking a lot more questions and allowing children to arrive at their own answers. I am giving them a lot more freedom to experiment with number and problem solving. I am not giving approval or disapproval, I am just allowing children to share what they find out. So it's much more child-centred, as language has become".

There is considerable agreement that the calculator enables teachers to see what the children can do and hence affects their expectations. One teacher comments "Some things children are actually able to do might have been the same in previous years - but I may not have given them the opportunity to show me. Consequently past experiences and style of teaching have led to my end of year predictions". When asked whether they believed the calculator was causing changes in children's level of understanding or merely revealing what they already knew, three teachers believed that it was merely revealing what children knew, three believed it was both, while the remaining teacher answered by saying "I think we actually stopped children from going too far years ago - we [went] to 10 in prep and 20 in grade 1 ... I really believe it's causing a change in teaching rather than in children's levels of learning. I think teachers are changing because we are using the calculators now".

CONCLUSION

Teachers' expectations, as measured by the questionnaire, show a slight increase for most of the 21 items dealing with counting and "large numbers", with a greater increase for the 5 items dealing with negative numbers. In general, teachers' increased expectations reflected their observations of children's performance during the previous year. Nevertheless, teachers' predictions remained conservative compared to actual levels of performance.

One possible reason for these conservative expectations may be teachers' beliefs that the few children they observed being able to perform these tasks in their first year of involvement in the project were truly exceptional and that such children were unlikely to be present again in other years.

The items relating to counting which showed the greatest increase in expectations both involved counting backwards. Counting backwards on calculators has been cited by project teachers as leading many children to "discover" negative numbers – the area in which the greatest positive change in teachers' expectations was observed.

Previous studies have found that the unrestricted mathematical environment provided by calculators encourages children to explore concepts from which they may have previously been "shielded" and that, as a result, teachers reconsider their approach to teaching and begin to develop exploratory and investigative styles of teaching which allow children to take responsibility for their own learning (Open University, 1982, pp.49-51; Shuard, Walsh, Goodwin, & Worcester, 1991, p.44).

Classroom observations and other data as reported elsewhere (see, for example, Groves, 1991) suggest that teachers involved in the *Calculators in Primary Mathematics* project are changing their teaching practice in mathematics. These observations are now supported by data obtained from interviews and the free comments on the questionnaire, which suggest that teachers, at least at the preparatory level, are adopting a more open-ended approach to their mathematics teaching in order to cater for the increased level of understanding revealed by the presence of the calculator. In particular, teachers are allowing children to lead the learning process by presenting them with starting points which allow for extensions in many different ways.

Future reporting of research on project teachers' expectations will focus on other aspects of number, such as operations, and other grade levels. If teachers' conservative predictions compared to children's actual levels of performance persists, further data will be collected through interviews.

REFERENCES

Corbitt, M. K. (1985). The impact of computing technology on school mathematics: Report of an NCTM conference. *Arithmetic Teacher, 32* (8), 14-18.

Cockcroft, W.H.(1982). *Mathematics Counts* . London: Her Majesty's Stationery Office.

Curriculum Development Centre and The Australian Association of Mathematics Teachers (1987). *A National Statement on the Use of Calculators for Mathematics in Australian Schools.* Canberra: CDC.

Desforges, C. & Cockburn, A. (1987). *Understanding the mathematics teacher: A study of practice in first schools.* London: Falmer Press.

Groves, S. (1991, November). *Calculators as an agent for change in the teaching of primary mathematics: The Victoria College Calculator Project.* Unpublished paper presented at the Annual Conference of the Australian Association for Research in Education, Surfers Paradise.

Groves, S., Cheeseman, J., Clarke, C. & Hawkes, J. (1991). Using calculators with young children. In O'Reilly, J. & Wettenhall, S. (Eds.). *Whither mathematics?* (Proceedings of the 28th Annual Conference of the Mathematical Association of Victoria, pp. 244-250). Melbourne: Mathematical Association of Victoria

Groves, S. & Cheeseman, J. (1992, November). *Calculators in primary mathematics: Changing expectations and curriculum issues.* Unpublished paper presented at the Joint Conference of the Australian Association for Research in Education and the New Zealand Association for Research in Education, Geelong.

Guskey, T. R. (1986). Staff development and the process of teacher change. *Educational Researcher, 15* (5), 5-13.

Hembree, A. & Dessart, D. (1986). Effects of hand-held calculators in pre-college mathematics education: A meta-analysis. *Journal for Research in Mathematics Education, 17,* 83-99.

Ministry of Education, Victoria (1988). *The mathematics framework: P-10.* Melbourne: Ministry of Education (Schools Division), Victoria.

Open University (1982). *Calculators in the primary school.* Milton Keynes: The Open University Press.

Reys, B. J. (1989). The calculator as a tool for instruction and learning. In P. R. Trafton & A. P. Shulte (Eds.). *New directions for elementary school mathematics* (1989 yearbook of the National Council of Teachers of Mathematics, pp. 168-173). Reston, VA: NCTM.

Shuard, H., Walsh, A., Goodwin, J. & Worcester, V. (1991). *Calculators, children and mathematics.* London: Simon & Schuster.

Wright, R.J. (1991). What number knowledge is possessed by children beginning the kindergarten year of school? *Mathematics Education Research Journal, 3* (1), 1-16.

Yackel, E., Cobb, P. & Wood, T. (to appear). The interactive constitution of mathematical meaning in one second grade classroom: An illustrative example. *Journal of Mathematical Behaviour.*

Young-Loveridge, J. (1989). The development of children's number concepts: The first year of school. *New Zealand Journal of Educational Studies, 24* (1), 47-64.

TEACHING MATHEMATICS USING THE PROCEDURAL ANALOGY THEORY

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The role and value of concrete materials in teaching and learning mathematics is uncertain, yet mathematics educators tend to assume their use is essential. Is this an act of faith? This paper describes a Procedural Analogy Theory which attempts to explain the value of concrete materials in the teaching of mathematics. Given the range of teaching possibilities for using concrete materials to help the learning of a particular concept or skill, this theory claims to be able to help teachers develop a teaching approach which will be superior to others. Aspects of both cognitive science and mathematics education are discussed in relation to this theory.

The paper reports on findings when the procedural analogy theory was applied in a number of Year 4 classrooms where Multibased Arithmetic Blocks were used to support the teaching and learning of subtraction algorithms.

An important goal of mathematics education is for students to develop understanding of what they learn in school mathematics, to move beyond the answer to a problem. A pedagogical approach widely assumed to support this goal involves the use of concrete materials in teaching and learning. The use of such materials has been touted in teacher education programs, mathematics education textbooks, school curriculums and academic papers with an intensity of purpose that sometimes approaches missionary zeal. Unfortunately anecdotal accounts, systematic observation of classroom practices, and research in general have been unable to explain the value and role of concrete materials with any degree of certainty. Mathematics educators appear to have set aside this question, perhaps in the belief that it has been answered, or perhaps because there are more fundamental questions that need to be addressed. Lamentably, the answer seems to lie more in human frailty of pursuing interesting ideas, endeavouring to keep current and to find more intriguing questions rather than attempting to find answers to pragmatic problems. Given the prosperous days of mathematics education in the 1960s and 70s, and given the quality of much research in mathematics education since that time, it is disappointing to find ourselves in such a position.

This paper analyses aspects of cognitive science and mathematics education to explain the rationale and the detail of the Procedural Analogy Theory. This theory is intended to answer a range of questions about the effectiveness of concrete materials in teaching and learning. In particular, the theory is a theory of instruction, so this paper reports on an application of the theory, in this case the teaching of subtraction algorithms to Year 4 students, through the use of Multibased Arithmetic Blocks (MAB).

THE VIEW FROM COGNITIVE SCIENCE

Intelligent Tutoring Systems, typically costing hundreds of thousands of dollars to develop, have been shown to be an effective teaching medium in a range of areas (Polson and Richardson, 1988; Self, 1988). So why were students using an Intelligent Tutoring System to learn aspects of arithmetic not as successful as the developers of the system had hoped (Ohlsson, Bee and Zeller, 1990; Ohlsson, Nickolas and Bee, 1987)? Answers to this question include the need for the software to effectively represent or model mathematical concepts and skills, and for it to provide the opportunity for effective students interactions with the system.

Intelligent tutoring systems are part of the field of artificial intelligence, and their specification and development requires multidisciplinary teams to cover aspects of computer science and cognitive psychology. The combination of computer science, cognitive psychology and artificial intelligence forms the basis of the field of

Cognitive Science. In part cognitive science is concerned with declarative and procedural knowledge, with the use of analogy in learning, with machine learning, with planning nets and production rules that show the interconnectedness of learning, and with representations of the knowledge structure of the domain in question (Anderson, 1985; Holyoak and Thagard, 1989; Michalski, Carbonell and Mitchell, 1986). And it is these areas of study that provide the cognitive science basis for the procedural analogy theory discussed here.

THE VIEW FROM MATHEMATICS EDUCATION - BELIEFS

Concrete representation of mathematical ideas is hardly a new idea. Bowen (1972) suggests that as early as 3000BC the Mesopotamians were using a one-to-one correspondence to perform counting, and we are all familiar with tally sticks and with the abacus, though few of us may have actually used these. But how did concrete representation come to take on the importance it has today?

The movement away from book learning to a more pragmatic approach to education, to learning through experience, gained momentum from the seventeenth century through Bacon's work on empiricism, together with Descartes' writings on rationalism, Newton's scientific empiricism and Locke's idea of Tabula Rasa. These were followed by Rousseau's *Emile* in 1762, and the writings of Pestalozzi, Kant and Froebel, forming a body of philosophy expressing the importance of the individual, of the innate goodness of humanity, of the need to respect these qualities in education, and a view of children as more than simply 'little adults' (Bowen 1981; Boyd and King, 1966; Good and Teller, 1969; Mayer, 1973). These philosophies saw education as being child centred, and requiring the child to be an active learner. In the present century the work of Dewey, Montessori and the Progressive Education Movement have provided this child centred and active learning approach with further ideological support. Taking all these various components together, this body of knowledge provides a strong philosophical foundation for the value of concrete materials in teaching mathematics as a means of providing child centred, active learning.

But is this philosophical foundation sufficient reason for using concrete materials in contemporary mathematics education? Where else can support be found? In 1929 Durell suggested "the use of splits tied up in bundles of ten, the use of diagrams to illustrate fractions and the use of graphs" encouraged the saving of time, better retention and transfer of learning. Breslich (1933), Christofferson (1937) and Taylor (1938) also published articles supportive of the use of concrete materials with some of their ideas pre-empting contemporary approaches. So the use of concrete materials in teaching mathematics has been supported for decades in mathematics education literature. We can also cite the work of Piaget on concrete operational thinking and Dienes' writings on the structure of mathematics to support the use of concrete materials, and we can draw comfort from the position taken by the Nuffield Mathematics Project's slogan

> ***I hear, and I forget***
> ***I see, and I remember***
> ***I do, and I understand.***
> (1967a,b)

But what are we expecting from the use of concrete materials? Exactly what kind of mathematical ideas are these materials supposed to represent, and what kind of learning is envisaged? What research indicates that concrete representations of mathematical concepts are actually valuable, that they do represent the concepts we intend, and that they do have a real and measurable impact on students' learning? And where is the data indicating exactly how concrete representations allow the learner to better arrange his or her cognitive structure so that learning is more effective?

What there are underlying ideologies and values about the roles of concrete materials, and that such beliefs may be as worthy of merit as empirical findings. It is possible to debate such values and beliefs, but data cannot be collected to prove that one view is more correct than another. These are issues fundamental to individual educator's beliefs about humanity, about students in particular, and what the individual sees as constituting teaching, learning and mathematics. That is, we use certain teaching approaches and materials in mathematics education because we believe in them. We may seek support for our opinions, but there is unlikely to be any logical necessity to accept or reject a particular point of view. Much of the pedagogy adopted by mathematics educators, and what mathematics educators tell other mathematics educators about effective pedagogies in research publications, curriculum development activities and teacher education courses is based on beliefs, with the addition of selected philosophical, anecdotal and empirical evidence.

THE VIEW FROM MATHEMATICS EDUCATION - RESEARCH

Resnick and Omanson (1987) sought to establish the relationship between performing arithmetic and understanding it, especially by illustrating procedural learning with "well-grounded mathematical principles". They developed a *mapping instruction* in which they maintained "a step-by-step correspondence between the blocks and written symbols throughout the problem". They had 80 fourth, fifth and sixth grade students perform tasks, both written and using MAB materials, where representations of numbers were constructed and decomposed, and where activities involved addition with carrying, and subtraction with decomposition. After a period of instruction, posttest scores showed children taught with the mapping instruction did not differ significantly from children in the comparison group, but in a delayed posttest the mapping instruction group gained higher scores. All the same, the researchers expressed disappointment at children's levels of achievement, and concluded that the mapping instruction was not effective in curing subtraction bugs. These findings suggest at least some of our beliefs about the value of concrete materials are questionable. Mathematics educators need to be concerned with these outcomes, especially since Resnick and Omanson's research was well designed, is frequently cited, was conducted by well known researchers and employed what appeared to be a detailed and sensible pedagogy - yet the use of concrete materials does not appear to have led to much in the way of positive outcomes.

In another important paper, Sowell (1989) reported a meta-analysis of 60 studies designed to assess the value of manipulative materials in mathematics instruction. The studies ranged from those involving kindergarten children to those in which college students participated, and employed a wide range of manipulatives and mathematics topics. Sowell found that treatment lasting a school year or longer favoured the manipulative groups but only for the use of concrete materials and not for pictorial representations. Treatments for shorter periods showed no difference between the manipulative and nonmanipulative groups on either posttest or delayed posttest scores.

Reference to a wider range of research literature simply confirms the uncertain value of concrete materials (for example, Hart, 1989; Hiebert and Carpenter, 1992). Treatment time is an element in teaching and learning, and so is pedagogy, but one of the difficulties in analysing the literature on concrete materials is the lack of detail given about the actual teaching methods employed. Statements about an experimental teaching approach contrasted with a traditional approach give insufficient detail as to the intricacies and nuances of the learners' experiences. I believe it likely that the pedagogy used in some studies could be improved simply through more attention to detail in the teaching learning process. For example, the procedural analogy theory outlined below provides one set of guidelines for improving instruction.

THE PROCEDURAL ANALOGY THEORY

How then can we show that the construction of meaning, through the internalisation of mathematical skills and concepts into a richly connected cognitive network, will be assisted by the manipulation of concrete materials? If teaching is to lead to understanding and to the learning of those standardised written procedures that continue to be an important goal of school mathematics such manipulation will need to be complemented by a pedagogy that encourages cognitive re-construction.

The procedural analogy theory describes how concrete materials assist the learning of declarative and procedural knowledge, and movement to the required target behaviour. Simplification, procedural analogy and symbolism, together with practice, lead finally to automatic responses. The procedural analogy theory is a theory of instruction, and has its basis in both cognitive science and mathematics education. In addition to the original publication concerning this theory (Ohlsson and Hall, 1990), aspects of the theory have been presented elsewhere (Hall, 1990, 1991, 1992a, 1992b). In solving a particular problem the theory relies heavily on the analogy between the process of acting upon concrete materials representing mathematical concepts and skills, and the written algorithm that corresponds to that process. The theory asserts that while concrete materials may be used in a wide range of ways to achieve a correct answer, there are some ways that will be more effective than others because they more closely mirror the desired target behaviour, and these latter ways provide the more effective teaching approach.

Table 1 shows one use of MAB materials and the target algorithm that is developed from this material. The steps emphasised both in the use of MAB materials and in the target algorithm are not unique, and must be

Table 1: Procedural analogy: MAB and target procedures

MAB procedure	Target procedure
0.0 428 - 169	0.0 428 - 169
0.1 Subtract 169 from 4H, 2T, 8U	
1.0 Process units	1.0 Process units
1.1 Take 9U from 8U (cannot)	1.1 Take 9 from 8 (cannot)
1.1.1 Trade for more units	1.1.1 Trade for more units
1.1.2 Move 1T from 2T to bank, bring back 10U	1.1.2 Recall 2 - 1 = 1
	1.1.3 Cross out 2, write 1
1.1.3 Join 10U and 8U	1.1.4 Write 1 next to 8
1.1.4 Recall 10U + 8U = 18U	1.1.5 Recall this is 18
1.2 Take 9U from 18U	1.2 Take 9 from 18
1.3 Recall 18U - 9U = 9U	1.3 Recall 18 - 9 = 9
1.4 Record answer, 9U in answer space	1.4 Record 9 in answer space
2.0 Process tens	2.0 Process tens
2.1 Take 6T from 1T (cannot)	2.1 Take 6 from 1 (cannot)
2.1.1 Trade for more tens	2.1.1 Trade for more tens
2.1.2 Move 1H from 4H to bank, bring back 10T	2.1.2 Recall 4 - 1 = 3
	2.1.3 Cross out 4, write 3
2.1.3 Join 10T and 1T	2.1.4 Write 1 next to 1
2.1.4 Recall 10T + 1T = 11T	2.1.5 Recall this is 11
2.2 Take 6T from 11T	2.2 Take 6 from 11
2.3 Recall 11T - 6T = 5T	2.3 Recall 11 - 6 = 5
2.4 Record answer, 5T in answer space	2.4 Record 5 in answer space
3.0 Process hundreds	3.0 Process hundreds
3.1 Take 1H from 3H	3.1 Take 1 from 3
3.2 Recall 3H - 1H = 2H	3.2 Recall 3 - 1 = 2
3.3 Record answer, 2H in answer space	3.3 Record 2 in answer space
4.0 Read answer (2H 5T 9U)	4.0 Read answer (259)

developed by the teacher. Once the teacher has decided on the target behaviour, a teaching sequence can be developed for the concrete materials that increases the likelihood that learners will structure their own knowledge in a similar manner.

The procedural analogy theory uses an isomorphism index ($I_{1,2}$) as a measure of analogy between the two procedures. The index is given by the formula

$$I_{1,2} = \frac{(N_1 + N_2 - 2) - (D_1 + D_2)}{N_1 + N_2 - 2}$$

where N_1 is the number of steps in the first procedure, N_2 the number of steps in the second procedure, D_1 the number of steps in the first procedure but not in the second, and D_2 the number in the second procedure but not in the first. In Table 1, $N_1 = 25$, $N_2 = 26$, $D_1 = 3$ and $D_2 = 4$ giving a high isomorphism index of 0.86. Slight variations in the steps will lead to a lower isomorphism index. The theory argues that the closer the relationship between the procedure involving the use of concrete materials and the target procedure, the higher the $I_{1,2}$ value, so the more effective will be the value of the concrete materials, and the greater the level of learning outcomes. That is, the procedural analogy theory allows an analysis of teaching steps prior to teaching and provides a method of measuring likely pedagogical success.

METHODOLOGY

The research reported here involved 110 students, two Year 4 classes in each of two schools, where students were randomly assigned to one of three groups to learn subtraction algorithms through the use of MAB materials. For each school the regular classroom teachers taught one group during the period of the research, the researcher taught the third group. Two of the three groups used a teaching method where there was a high isomorphism index (High I), one group using expanded numerals in the movement from concrete materials to written algorithm, the other moving directly from concrete materials to the target procedure. The third group used a method with a lower value isomorphism index (Low I), where the teaching approach was acceptable to all teachers in the research, was a typical teaching approach and one supported by curriculum statements and textbooks. These details are summarised in Table 2.

Table 2: **Teaching groups**

School	Group/Teaching Approach	Teacher
A	High I, expanded	1
	High I	2
	Low I	3
B	High I, expanded	4
	High I	5
	Low I	2

The teaching approaches of the two experimental groups differed from the teaching approach of the comparison group in the detail of the correspondence between the actions on concrete materials and the written algorithm, and in the detail of the guidance and description given by the teacher.

RESULTS AND ANALYSIS

All students were given a pretest on subtraction algorithms, and parallel posttest and delayed posttest. An analysis of variance showed no significant differences in pretest scores between students in the three teaching approaches. An analysis of variance on posttest scores showed there was a trend for students in the High I teaching approaches to have higher scores than for students in the Low I approach, but the differences were not statistically significant ($p < .10$). An analysis of variance on delayed posttest scores showed a significant difference ($p < .05$) in favour of the High I teaching approaches over the Low I teaching approach. There was no statistical difference in scores between members of the two groups with High I teaching approaches.

The trend for students exposed to teaching approaches with high isomorphism indices to show greater gains on posttests scores than students where the teaching approach had a low isomorphism index is a positive finding in terms of the procedural analogy theory, but clearly needs further investigation. It may be that for a range of teaching approaches students will not score significantly differently on posttests if the test is administered immediately after completion of the topic, and that differences in test scores resulting from different teaching approaches become evident only over time after instruction has finished. The topic chosen may also explain the sameness of the posttest scores. That is, subtraction was not a new topic for any of the students involved in this study, so posttest results may have differed had the topic been new to all students.

The finding that students exposed to teaching approaches with high isomorphism indices showed greater gains on delayed posttest scores than those students where the teaching approach had a low isomorphism index gives some support to the procedural analogy theory and to the longer term value of using concrete materials. These findings also suggest that the use of concrete materials together with specific pedagogies assist the development of an effective cognitive structure, one in which concepts and skills are stored in a meaningful and efficient manner, where they can be remembered, recalled and reconstructed as necessary.

DISCUSSION

How generalisable is the procedural analogy theory in terms of mathematics topics and ages of learners? Does application of this procedural analogy theory encourage learners to develop a richly connected network of cognitive structures? Is the cognitive development taking place through the application of this procedural analogy theory superior to developments using other teaching approaches?

This research has generated many questions, there is clearly a need for the research to be replicated, and further investigation is necessary to assess the relevance of the procedural analogy theory in other school mathematics topics. At the same time the research reported here appears to have some potential in facilitating the design of teaching approaches involving the use of concrete materials, and in helping teachers guide learners in the construction of knowledge.

REFERENCES

Anderson, J. R. (1985). *Cognitive psychology and its implications*, 2nd edition. New York: Freeman.

Bowen, J. (1972, 1975, 1981). *A history of western education*, three volumes. London: Methuen.

Boyd, W. and King, E.J. (1966). *The history of western education*, 8th edition. New York: Barnes and Noble.

Breslich, E.R. (1933). Secondary school mathematics and changing curriculum. *Math T*, 26, 327-9.

Christofferson, H.C. (1937). Trends for improving instruction in mathematics. *Math T*, 30, 15-20.

Durell, F. (1929). Making the concrete and abstract help each other in mathematics. *Sch Sci Math*, 29, 702-713.

Good, H.G. and Teller, J.D. (1969). *A history of western education*, 3rd edition. London: Macmillan.

Hall, N. (1992a). *Constructivism, concept maps and procedures: making meaning in school mathematics*. Paper presented to the AARE conference, the annual conference of the Australian Association for Research in Education, Geelong.

Hall, N. (1992b). *From procedural analogy to understanding*. In B. Southwell, B. Perry and K. Owens (Eds.), *Space -The First and Final Frontier*, proceedings of the fifteenth annual conference of the Mathematics Education Research Group of Australasia, Sydney.

Hall, N. (1991). *A procedural analogy theory: the role of concrete embodiments in teaching mathematics*. In F. Furinghetti (Ed.), Proceedings of the fifteenth annual conference of the International Group for the Psychology of Mathematics Education, Assisi, Italy.

Hall, N. (1990). *Developing intelligent tutoring systems.* In J.G. Hedburg, J. Steele and M. Mooney (Eds.), *Converging Technologies.* Papers from EdTech'90, the 1990 Conference of the Australian Society for Educational Technology. Sydney.

Hart, K. (1989). *There is little connection.* In P. Ernest (Ed.), *Mathematics teaching: the sate of the art.* (pp. 138-142). London: Falmer Press.

Hiebert, J. and Carpenter, T.P. (1992). Learning and teaching with understanding. In D.A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 65-97). New York: Macmillan.

Holyoak, K.J. and Thagard, P. (1989). Analogical mapping by constraint satisfaction. *Cognitive Science* , 13, 295-355.

Mayer, F. (1973). *A history of western education*, 3rd edition. Columbus, Oh: Merrill.

Michalski, R. S., Carbonell, J. G. and Mitchell, T. M. (Eds.), (1986). *Machine learning. Vol. II.* Los Altos, CA: Kaufmann.

Nuffield Mathematics Project (1967a). *I do, and I understand.* New York: John Wiley.

Nuffield Mathematics Project (1967b). *Mathematics begins.* New York: John Wiley.

Ohlsson, S., Bee, N. and Zeller, (1990). *The fractions tutor.* Learning Research and Development Centre, University of Pittsburgh, Technical Report.

Ohlsson, S., Nickolas, S., and Bee, N. (1987). *Interactive illustrations for fractions: A progress report.* (Tech. Rep. No. KUL-87-03). Pittsburgh, PA: University of Pittsburgh, Learning Research & Development Center.

Ohlsson, S. and Hall, N. (1990).*The cognitive function of embodiments in mathematics instruction.* Learning Research and Development Centre, University of Pittsburgh, Technical Report No. KUL-90-02.

Polson, M.C. and Richardson, J.J. (Eds.) (1988). *Foundations of intelligent tutoring systems.* Hillsdale, N.J.: Erlbaum.

Resnick, L.B. and Omanson, S. (1987). Learning to understand arithmetic. In R. Glaser (Ed.), *Advances in Instructional Psychology*, (Vol. 3, pp. 41-95). Hillsdale, N.J.:Erlbaum.

Self, J. (Ed.) (1988). *Artificial intelligence and human learning.* London: Chapman and Hall.

Sowell, E.J. (1989). Effects of manipulative materials in mathematics instruction. *J. Res. Math. Educ.*, 20 (5), 498-505.

Taylor, E.H. (1938). Mathematics for a four-year course for teachers in the elementary school. *Sch Sci Math*, 38, 499-503.

A METACOGNITIVE APPROACH TO TEACHING MATHEMATICS THROUGH COMPUTER SUPPORTED ENVIRONMENTS

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The research reported here focussed on the mathematical learning of "at risk" preschoolers in computer-supported interactive environments. Specifically, our investigations have explored the effectiveness of metacognitively-rich computer-supported contexts on early mathematical learning. Results indicated that children who participated in mathematical learning activities within these contexts performed significantly better in tests of mathematics competence than did children engaged in more regular instructional approaches, including those situated within computer-based settings.

This paper presents some of the theoretical background to our research, some findings, and considers a range of implications for teaching mathematics in early childhood contexts.

In the research outlined in this paper we investigated the mathematical learning of "at risk" preschoolers in computer-supported interactive environments. Specifically, there are three theoretical perspectives that inform the research. The first is concerned with Vygotskian notions of the role of social mediation in scaffolding cognitive activity and the importance of self-regulatory strategies in internalising thinking. Secondly, we are concerned with aspects of cognitive science, in particular the processing of information, for example cognitive structure and how metacognitive activity improves storage and retrieval of data. Finally, there is a mathematics education perspective in which we consider optimal instructional approaches for early mathematical learning.

According to Vygotskian perspectives on cognitive development the interpersonal nature of cognitive activity is critical to learning. In this sense children's interactions with competent others serve to mediate thinking and problem-solving in the cognitive space between what can be accomplished alone and in collaboration with more capable others, that is, in the zone of proximal development. It is suggested that this "scaffolding" may provide ongoing stimulation and motivation for learning, as well as support of a more metacognitive, and particularly, self-regulatory nature. Subsequently, the self-regulatory strategies are internalised to become part of the learner's independent repertoire of competencies for application in similar and novel contexts (Rogoff, 1990; Vygotsky, 1978).

The role of the teacher in this scaffolding process is to promote an awareness of the cognitive demands of the task and to guide activity within a purposeful and goal directed framework focussing on, for example, overall task orientation, and planning, monitoring and evaluating cognitive activity, rather than on the management and organisation of the learning experience itself. Complementary peer interaction, as well as the individual's own structuring of the activity, generate the cognitive and metacognitive dialogues that serve to support and mirror knowledge and thinking. In this sense the interrelatedness of the contextual supports lend themselves to framing aspects of self-regulated learning.

While this interactivity is usually envisaged as mediated by adults and/or peers, computer environments may afford additional and complementary scaffolds through provision of prestructured content and in-built cognitive supports, for example, feedback, sequencing, and the predictable flow of a the activity. It is contended that intertwining these cognitive supports with those afforded by co-participants (teachers and peers), as well as the individual's own structuring of activity, serves to foster thinking and problem-solving (Salomon, 1990).

A key function of self-regulatory behaviour is its role in helping children increase their control over learning and problem-solving activity and the application of knowledge about cognitive states and processes to the learning process (Wellman, 1985; Zimmerman, 1990). For preschoolers, this cognitive activity, generally manifested at an external level and evidenced largely through language, results in the gradual internalisation of thinking skills (Vygotsky, 1978).

Our research is concerned also with how data are processed in the brain. In particular, we are interested in declarative and procedural knowledge and the roles of representations and analogies in assisting the movement from declarative to procedural knowledge. Specifically, we are concerned with ways in which learners can be encouraged to develop a richly connected cognitive network of skills and understandings. For example, teacher generated questioning designed to extend new learning and link it with present knowledge facilitates proceduralisation of the declarative knowledge initiated by the computer. Subsequent practice makes a learner's response time quicker, and the metacognitive scaffolds encourage chunking of related concepts or procedures. Through this process recall becomes faster and the efficiency of short term memory processing is increased. Central to this information processing are the self-regulatory strategies that enable learners to consciously control the processing of data in short term memory, and its flow in and out of long term memory. An individual's awareness and use of metacognitive strategies is increasingly accepted to be associated with high levels of performance on a range of cognitive tasks (Paris and Winograd, 1990).

Finally, we're interested in conceptualising our investigations within a mathematics education framework. For example, what does mathematics education tell us about young children and their mathematical learning? We know that it is unusual for preschools to conduct formal lessons in mathematics, though one hopes there will be a myriad of practical experiences in which mathematical ideas are fostered. Generally, experiences of this kind are categorised as pre-mathematical in nature, and include measuring, counting and naming, and assisting children gain conventional meanings for a range of ideas and terms essential to later mathematics learning.

We know that by the end of preschool children typically have some number concepts and skills and that these range across a variety of levels. For example, they may see counting as a word game, a number sequence, or sets of numbers, so "(t)hey need a great deal of experience of acting with objects... to reach counting and structuring levels of number" (Leino, 1990, p. 44). Charlesworth (1984) notes that many children start school knowing which of two small groups contains "more", knowing that two objects and three objects make five, can count objects and use the correct number name to ten. These views are consistent with other findings concerning the level of children's counting ability prior to the start of formal schooling (Fuson and Hall, 1983; Gelman and Gallistel, 1978; Ginsburg, 1983; Hiebert, 1986).

One of the problems in early childhood mathematics is that adults often see little difficulty in the learning of pre-mathematical ideas and elementary number work. Baroody (1985), however, emphasises that learning basic number combinations is not "a straightforward, rote memory task". Indeed, learning number combinations is a slow process requiring considerable experience with counting. Importantly, the process is dependent on development of procedures or invented rules as learners gradually replace slow counting procedures with rapid fact retrieval. What these mental representations look like is presently uncertain. But, it appears that learners use more than simple associative models. Indeed, the complexity of human information processing suggests that there may be a series of rules and that they may differ significantly from one learner to the next.

In dealing with the mathematical education of young children it is unfortunate that the majority of early childhood teachers have narrow conceptualisation of mathematics, lack knowledge and experience of mathematics, and know little about processes of mathematics teaching and learning. Burton (1990) believes that in teaching mathematics to young children, and in training early childhood teachers, there is an overly simplistic approach. For example, mathematics is so simplified that it is removed from its relationship to other mathematical ideas and environments. There is also a widespread view that simple mathematics ought to be easy to learn so there is a failure to recognise the highly complex nature of mathematical learning. In a climate where mathematics is viewed largely as a body of concepts and skills for learners to codify and digest there is little recognition that it can or should be a rewarding experience.

The two studies reported here have been conducted with preschool-aged children identified as "at risk" (Elliott, 1991; Elliott and Hall, 1985, 1990 a,b,c; Hall, 1992; Hall and Elliott, 1992). That is, the children are considered "at risk" of early school failure and enrolled in preschool settings that provide early intervention programs aimed at strengthening cognitive and social competence. Given increasing recognition that skilled mathematical problem-solving is dependent on a range of metacognitive activities that are amenable to classroom instruction, our research is concerned with explicating and facilitating metacognitively-rich instructional approaches and examining their effectiveness. Such approaches may be especially important for children experiencing early difficulties with learning (Borkowski, Carr, Rellinger and Pressley, 1990).

STUDY A

The source group for this study was 50 "at risk" preschool children enrolled in early intervention programs. Children were assigned to one of two experimental groups or to a control group, based on assessment scores using the *Test of Early Mathematical Ability* (TEMA 2). Children in the two experimental groups participated in fifteen twenty-minute computer-supported mathematics teaching sessions where they worked in pairs with teacher guidance. Children in Experimental Group 1 participated in a metacognitively-rich mathematical environment based on popular commercial software such as *Number Farm*. Teacher guidance aimed to scaffold cognitive activity by emphasising self-regulatory strategies through questioning and cuing, and modelling and demonstrating aspects of material to be learned. Teaching focus was on extending children's knowledge through broadening and elaborating computer-generated questions and cues, by rectifying discrepancies between the child's response and the ideal situation, through minimising frustration and encouraging risk in problem solving, and by demonstrating an idealised version of the problem's solution. In Experimental Group 2 the teaching emphasis was on informal, caring and supportive teaching much like that normally used in the preschool. Children in the third group, the control group, did not participate in any computer-supported mathematics activities. All children continued to engage in the usual range of informal preschool experiences designed to engender mathematical activity, such as sand, water and block play. The TEMA 2 instrument was used to measure mathematical achievement after the completion of the intervention program.

An example of the interaction generated in the metacognitive teaching session is shown in Table 1 below where a teacher is talking to two children (C_1 and C_2).

In the metacognitive teaching approach adult mediation serves to focus the child's activity on the task at hand, to encourage planning, regulation and evaluation of performance in each task, and to link new information with present knowledge. The cognitive consequence of this learning is that children become enable to select and attack problems strategically (Paris and Winograd, 1990) and to develop rich conceptual networks. Engagement in intellectual activity that exercises and stretches thinking processes is dependent on the interactivity generated by the learners and adults, as well as the computer and software. As we all well know computers do not in themselves generate cognitive activity that results in learning (de Corte, 1990; Elliott and Hall, 1985; Pea, 1987; Salomon, 1990).

Teaching activities selected by either the computer or the teacher included counting a variety of objects, either right to left or left to right, but discouraging random selection because of errors that would follow, drawing objects then counting them, finding then typing the numeral on the computer keyboard, representing the set, and writing numerals. These activities are consistent with, though not as comprehensive as the approaches reported by Saxe, Becker, Sadeghpour and Sicilian (1989) where the researchers gave emphasis not only to one-to-one correspondence, to the last element as the number representing that set, and to counting, but to the arbitrary nature of the conventional names and symbols used in counting. Our teaching activities are also consistent with Steffe, von Glasersfeld, Richards and Cobb's (1983) perceptual, figural, motor and verbal conceptual structures of number development. That is, in Steffe et al's terms, the activities encouraged using representations to count in order to begin counting at one, to re-present a collection of objects, and to re-present a counting activity. And our approach is consistent too, with Greeno, Riley and Gelman (1984) who had children create sets of objects, count them and find the correct numeral to represent them, so as to increase the likelihood that learners would associate the number of objects counted with its numeral representation. They claimed that young children's counting was far more than rote recitation. Rather, it reflected understanding of counting principles including cardinality, one-to-one correspondence and order.

Table 1: **An extract from dialogue during the metacognitive teaching approach**

Four objects on the computer screen.
T: How many have we got here?
C_1: (Points with finger on screen, and counts)
One, two, three, four.
T: (Asks second child)
How many do you think there are?
C_2: Four.
T: Well, can you show me four fingers?
C_2: (Holds up four fingers, counts)
One, two, three, four.
T: How old are you C_1?
(C_1 shows four fingers)
How many is that?
C_1: Four.
T: How many candles did C_1 have on her tray?
C_2: Four
T: How many eyes have you and C_2 got together?
C_1: Four.
T: How many hands have you and C_1 got together?
C_2: Four.
T: How many ears have you and C_1 got together?
C_2: Four.

RESULTS A

Analysis of variance procedures showed significant differences in mathematical gain scores between the three groups ($p < .001$), with the control group performing least well. Further analyses showed that the gains made by the metacognitive teaching group were significantly higher than those of the non-metacognitive teaching group ($p < .01$), and that the comparison group made no significant gains in mathematics scores during the period of the research.

We suggest, then, that computer mediated learning environments are successful in helping "at risk" preschoolers learn pre-mathematical and arithmetic concepts and skills. Further, we contend that our metacognitive teaching approach encourages greater mathematical learning than a typical caring, concerned teaching approach even when that approach was embedded in an interactive computer-supported setting. That children in the control group did not improve their mathematics achievement scores over the research period is cause for concern. It would seem there are too few informal mathematical experiences provided by the preschools. Given that the children in the study were already exhibiting signs that early mathematical learning was not proceeding as well as might be expected, it is important that opportunities be provided to strengthen these children's mathematics learning.

STUDY B

The success of the computer-supported metacognitive approach and concern for the overall quality of mathematics curriculums for "at risk" preschoolers led to further research and program development in one of the preschools. During the period of this second study the preschool revised its mathematics program, so that it went from exclusively unstructured mathematical play experiences, to one in which mathematical activities were more regularly included in the curriculum, supported by a mathematics corner and by newly purchased mathematical materials. There was also a more conscious effort to intervene so as to draw out the mathematical aspects of all free-play and guided learning settings. Further, the teachers adopted the metacognitive teaching approach described above, both in the computer context as well as in the broader activities of their teaching program.

Each child's concepts and skills in mathematics were assessed using the TEMA 2 test at the start of the research period, and were reassessed six months later. All 24 children in the class, most of whom were "at risk", were assigned to one of two groups based on their pretest scores. Both groups of children experienced computer-mediated learning for fifteen twenty-minute sessions during the six months. Teaching in the experimental group emphasised mathematical learning experiences of the kind described as the metacognitive teaching approach in Study A. The comparison group's computer experiences were non-arithmetical, generally language and drawing programs. This experimental design allowed us to see if changes to the preschool's mathematics program had led to improvements in children's mathematical learning, to contrast gains of the experimental group with gains of the comparison group, and to see if a less intensive computer experience, than implemented in Study A, would still lead to gains in mathematics scores.

RESULTS B

An analysis of variance of the gain scores for the two groups showed a significant difference ($p < .01$) favouring the group with additional mathematical experiences. Further analyses showed the comparison group's gains to be significant at the .05 level, thus indicating that the preschool's revised mathematics program was having an impact on children's mathematical learnings.

These results indicate the cognitive benefits of a more focussed and metacognitively-rich pre-mathematical and mathematical program. Additionally, they suggest that even when a preschool's mathematics program is improved, it is likely that children can gain further through experiences within computer-mediated contexts. We know, too, that these gains can be achieved through a modest investment in computer and human resource time; in this research children had computer access for less than twenty minutes per week.

CONCLUSIONS

Findings from the studies reported above suggest the effectiveness of interactive, metacognitively-rich computer environments for enhancing "at risk" preschoolers' mathematical learning. That is, children's gains in mathematical achievement are greater in computer-mediated environments, even when more traditional environments are metacognitively enriched. The validity of teaching those concepts and skills that formed the subject matter of our research, at the preschool level is, however, debatable, and our answer is very pragmatic. We accept Cobb's point concerning emergent mathematical meanings and institutionalised mathematical knowledge (1990), and note high teacher, parent, pupil and broader community expectations of children in the first years of school, especially as they concern learning arithmetic. In our opinion, these expectations mean that for many "at risk" children, early mathematical disadvantages are magnified so that they fall further and further behind their peers, and so move further and further away from the goal of institutionalised mathematical knowledge. So the gap between mathematically competent children and those struggling to learn concepts and skills grows from the very beginning of school.

The metacognitive teaching approach adopted in these studies, especially when embedded in an interactive computer context seemed to provide a valuable scaffolding for the development of cognitive competence in mathematics. Importantly, it encouraged planning, reflection and evaluation as children constructed their own knowledge (Steffe and Wood, 1990; von Glasersfeld, 1991). The interactivity within the computer environments seemed to allow for more cognitive conflict and re-construction than was the case without computer intervention

In our continuing research in this area we are focussing on refining the metacognitive model, particularly in respect of those contextual supports most effective in maximising metacognitive activity. First though, we need to ascertain exactly which contextual supports are most effective in maximising academic competence, and then determine the most effective ways of designing environments to enhance the benefits of these supports. Further research is also required to examine the impact of enriched mathematical preschool experiences on children's first years of school. Additionally, we need to consider the professional development of teachers, the appropriate content of preschool mathematics programs, and the specifications of computer software.

REFERENCES

Baroody, A.J. (1985). Mastery of basic number combinations: internalisation of relationships or facts? *Journal for Research in Mathematics Education*, 16 (2), 83-98.

Berk, L.E. (1991). *Child Development*, 2nd edition. Boston, MA: Allyn and Bacon.

Borkowski, J.G., Carr, M., Rellinger, E. and Pressley, M. (1990). Metacognition, attributions and cognition. In B.F. Jones and L. Idol (Eds.). *Dimensions of thinking and cognitive instruction* (pp.52-92). Hillsdale, N.J.: Erlbaum.

Burton, L. (1990). What could teacher education be like for prospective teachers of early childhood mathematics - with particular reference to the environment. In L.P. Steefe and T. Wood (Eds.), *Transforming Children's Mathematics Education* (pp. 334-344). Hillsdale, N.J.: Erlbaum.

Charlesworth, R. (1984). *Kindergarten mathematics: step-by-step from concrete materials to paper and pencil.* Paper presented to the Annual Conference of the National Association for the Education of Young Children, Los Angeles.

Cobb, P. (1990). Multiple perspectives. In L.P. Steefe and T. Wood (Eds.), *Transforming children's mathematics education* (pp. 200-215). Hillsdale, N.J.: Erlbaum.

de Corte, E. (1990). Learning with new information technologies in schools: perspectives from the psychology of learning and instruction. *Journal of Computer Assisted Learning*, 6, 69-87.

Elliott, A. (1991). The role of contextual supports in computer-based mathematical experiences. In *Crossing Boundaries: present realities and future possibilities* (pp. 107-112). Proceedings of the 15th National Australian Association for Special Education conference, University of Queensland.

Elliott, A. and Hall, N. (1990a). Computer based mathematical experiences in an early intervention program. *Australian Journal of Early Childhood*, 15(3), 37-45.

Elliott, A. and Hall, N. (1990b). *Computer Applications for Young Children with Special Learning Needs: a Guide for Educators.* Report to the Early Special Education Program, Department of Education, Employment and Training, Sydney.

Elliott, A. and Hall, N. (1990c). *Evaluating the use of computer based learning activities with young children exhibiting aspects of developmental delay.* Report to the Early Special Education Program, Department of Education, Employment and Training, Sydney.

Elliott, A. and Hall, N. (1985). Computers in the preschool: Enhancing curriculum goals. In *Early childhood: Ideals and realities.* (Vol 2, pp. 372-386). Brisbane: AECA.

Fuson, K.C. and Hall, J.W. (1983). The acquisition of early number word meanings: a conceptual analysis and review. In H.P. Ginsburg (Ed.), *The development of mathematical thinking*. New York: Academic Press.

Gelman, R. and Gallistel, C.R. (1978). *The child's understanding of numbers*. Cambridge, Mass: Harvard University Press.

Ginsburg, H.P. (Ed.) (1983). *The development of mathematical thinking*. New York: Academic Press.

Greeno, J.G., Riley, M.S. and Gelman, R. (1984). Conceptual competence and children's counting. *Cognitive Psychology*, 16, 94-143.

Hall, N. (1992). *Computer mediated learning in preschools: Why bother?* Paper presented to the AARE conference, the Annual Conference of the Australian Association for Research in Education, Geelong.

Hall, N. and Elliott, A. (1992). *Mathematics, computers and "at risk" preschoolers*. In B. Southwell, B. Perry and K. Owens (Eds.) *Space -The first and final frontier*. Proceedings of the Fifteenth Annual Conference of the Mathematics Education Research Group of Australasia, Sydney.

Hiebert, J. (Ed.) (1986). *Conceptual and procedural knowledge: the case of mathematics*. Hillsdale, N.J.: Erlbaum.

Leino, J. (1990). Knowledge and learning mathematics. In L.P. Steefe and T. Wood (Eds.), *Transforming children's mathematics education* (pp. 41-46.) Hillsdale, N.J.: Erlbaum.

Paris, S.G. and Winograd, P. (1990). Metacognition can promote learning and instruction. In B.F. Jones and L. Idol (Eds.). *Dimensions of thinking and cognitive instruction* (pp.15-51). Hillsdale, N.J.: Erlbaum.

Pea, R. (1987). Integrating human and computer intelligence. In R.D. Pea and K. Sheingold (Eds.), *Mirrors of the mind. Patterns of experience in educational computing* (pp. 128-146). Norwood, N.J.: Ablex.

Rogoff, B. (1990). *Apprenticeships in thinking. Cognitive development in a social context.* NY: Oxford University Press.

Salomon, G. (1990). Cognitive effects with and of computer technology. *Communication Research*, 17(1), 26-41.

Saxe, G.B., Becker, J., Sadeghpour, M. and Sicilian, S. (1989). Developmental differences in children's understanding of number word conventions. *Journal for Research in Mathematics Education*, 20 (5), 468-488.

Steffe, L.P., von Glasersfeld, E., Richards, J. and Cobb, P. (1983). *Children's counting types: Philosophy, theory, and application*. New York: Praeger Scientific.

Steefe, L.P. and Wood, T. (Eds.), (1990). *Transforming children's mathematics education*. Hillsdale, N.J.: Erlbaum.

Wellman, H. (1985). The origins of metacognition. In D.L. Forrest-Pressley, G.E. MacKinnon, T.G. Waller, (Eds.), *Metacognition, cognition and human performance* (pp. 1-32). Orlando: Academic Press.

von Glasersfeld, E. (Ed.) (1991). *Radical constructivism in mathematics education*. Dordrecht, The Netherlands: Kluwer.

Vygotsky, L.S. (1978). *Mind in society. The development of higher psychological processes.* Cambridge, MA: Harvard University Press.

Zimmerman, B.J. (1990). Self-regulated learning and academic achievement. An overview. *Educational Psychologist*, 25(1), 3-17.

ON THE ROLE OF DIDACTIC MATERIALS IN TEACHING AND LEARNING MATHEMATICS

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Abstract[*]

We begin with a discussion about how we come to know the concept of a square. The problem of how best to teach fractions, and by what means we approach this daunting task follows. The example of the CopyCat, a prototype computer-based learning environment is used to illustrate strengths and weaknesses. Other didactic materials such as Steffe and Olive's Sticks, and Kieren and Pirie's paper folding are discussed, and we conclude that there is no possibility of finding physical material with the properties that will mirror the precise essence of the mathematical concept we have in mind to teach. A view of the role of physical materials in an enlarged pedagogy involving what Cobb called the interactive negotiation of mathematical meaning is suggested as a solution. We are reminded also that students operate on material of the mind including re-presentations of past actions, memories, and visual images. Remaining unanswered questions are posed.

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[*] Paper not available for publication

HOW MUCH IS ENOUGH IN DESIGNING CLINICAL TASKS FOR MATHEMATICS ASSESSMENT?

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Abstract[*]

An ARC funded project of the Institute of Mathematics Education is to develop and validate a clinical instrument suitable for an initial assessment of a student. Priorities need to be set about what constitutes a "critical task", since there is simply not enough time to systematically sample every content cell from a traditional scope and sequence analysis of the curriculum. We provide background to the project, including phases in the validation process. Issues raised by consultant experts, practitioners, and the development team in making decisions about revision, inclusion and exclusion of tasks will be discussed.

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[*] Paper not available for publication

DETERMINING THE EDUCATIONAL POTENTIAL OF COMPUTER BASED STRATEGIES FOR DEVELOPING AN UNDERSTANDING OF SAMPLING DISTRIBUTIONS

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An analysis of the steps involved in forming the idea of an empirical sampling distribution and the nature of the methods and/or images used in most computer based strategies to teach this idea suggest that this way of using the computer adds little insight to the usual text based explanations that they are designed to complement. This analysis suggests reasons why a more recent approach which uses the computer to model and dynamically display the processes that underlie the idea is more likely to be successful.

A critical step in developing the theory of statistical inference is the idea of a sampling distribution – the recognition that the estimates of a population parameter will vary and that this variation will conform to a predictable pattern. Yet, for all its importance, experience and research have shown that the idea is generally poorly understood (Moore, 1992; Rubin, Bruce, & Tenney, 1990 for example). One reason for this might be the way in which the idea has been traditionally introduced in statistics courses using a deductive approach based on probability theory (Johnson & Bhattacharyya, 1987; Mendenhall, Wackerly, & Scheaffer, 1990 for example). Such explanations are usually expressed in a highly mathematical language which tends to make the argument inaccessible to all but the mathematically able, now a very small minority of the students taking introductory courses in inferential statistics. But perhaps more importantly, it is a theoretical development which is difficult to relate to the physical process of drawing a sample from a population. Statistics educators have come to recognise that there are deficiencies with a purely theory based explanation and now often accompany or replace this with an empirical argument. The alternative interpretation uses the long run relative frequency approach, where the sampling distribution is viewed as the result of taking repeated samples of a fixed size from a population and calculating the value of the sample statistic for each (Devore & Peck, 1986; Ott & Mendenhall, 1990 for example). The empirical approach has the advantages of being more readily related to the actual physical process of sampling and requiring minimal use of formal mathematical language.

Because the computer has an obvious role in the empirical development of the idea of a sampling distribution, by carrying out the repeated sampling and summarising the results, a number of instructional sequences have been developed built around these capabilities. Unfortunately these approaches, although widely promoted and now commonplace activities in introductory statistics courses, have been less successful than statistics educators might have hoped for, as noted by Hawkins (1990):

> *ICOTS 2 delegates were treated to "101 ways of prettying up the Central Limit Theorem on screen", but if the students are not helped to see the purpose of the CLT, and if the software does not take them beyond what is still, for them, an abstract representation, then the software fails. (p 28)*

In this paper we will look at why early attempts at computer based explanations have not been as successful as they might be, and suggest reasons why a more recent approach by Rubin and her colleagues (Rubin, 1991) is more likely to succeed.

THREE EMPIRICALLY BASED STRATEGIES FOR INTRODUCING THE IDEA OF A SAMPLING DISTRIBUTION

In this analysis we will consider three empirically based instructional strategies for introducing the idea of a sampling distribution. For convenience we will restrict ourselves to the distribution of a sample proportion. The first strategy is a typical text based explanation. The second strategy utilises the general purpose computer package Minitab. The third strategy involves a computer package which explicitly makes use of the increased graphics potential of the new desktop computers.

Strategy 1 (Text only)

Text based instructional strategies verbally describe the process of forming an empirical sampling distribution. These explanations are often accompanied by one or more relative frequency histograms showing the distribution of the sample proportion $\hat{p}$ for a large number of trials, typically several hundred. The reader is then asked to note the (generally) near normal shape of the distribution and that it is centred on or around the (known) population proportion p. Figure 1 is taken from such a development sequence in a typical introductory statistic text (Devore & Peck, 1986 p 255). The population under consideration here is the labour force in Ireland, and the population proportion p = 0.265 is the percentage of females in this population.

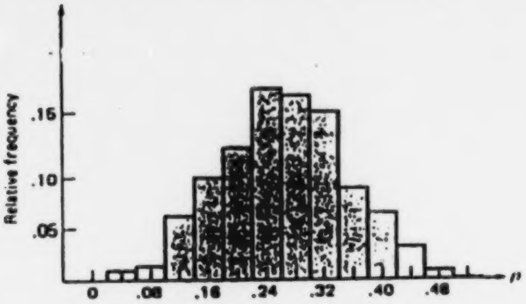


Figure 1: Histogram of 500 values of $\hat{p}$ based on a random sample of size n =25 (p=0.265)
(Devore & Peck, 1986 p 255)

Strategy 2 (Minitab)

In the early 1980's the more innovative statistics educators began using the computer as part of their teaching sequence (Bloom, Comber, & Cross, 1986; Thomas, 1984 for example). In the earliest attempts complicated programming was required, but now commonly available statistical computer packages such as Minitab may be used to produce empirical sampling distributions. Students are given the appropriate computer code to generate random samples, calculate corresponding values of the sample proportion $\hat{p}$, and display the distribution graphically (generally in the form of a histogram). For example, using Minitab, a histogram similar to that shown in Figure 1 can be reproduced using the following commands:

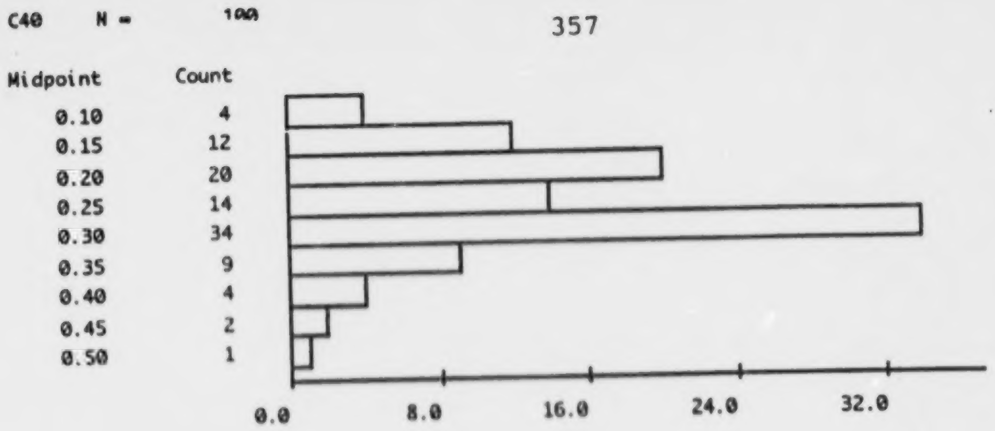
```
MTB > random 100 c1-c25;
SUBC> bernoulli 0.265.
MTB > rsum c1-c25 c30
MTB > let c40=c30/25
MTB > ghist c40
```

Using similar commands we can create any other histogramswe might like by varying the population proportion p, the sample size n, or both.

The resulting histogram is shown in Figure 2.

Strategy 3 (Sampling Laboratory)

More recent computer applications in mathematics and statistics have tended to de-emphasise the use of the computer as a computational tool and focus on our additional ability to use current technology to build a working model of the process under consideration and to display the results graphically, for example



```
C40     N =       100

Midpoint      Count
    0.10          4
    0.15         12
    0.20         20
    0.25         14
    0.30         34
    0.35          9
    0.40          4
    0.45          2
    0.50          1
```

Figure 2: Histogram of sampling distribution produced by Minitab

Geometric Supposer (Shwartz & Yerushalmy, 1985). Such an approach has been followed in Sampling Laboratory by Rubin et al. To use the package no programming is required by the user who simply makes the appropriate entries as requested. To replicate the example previously used the user enters the name the population (Labour Force Ireland), the attribute of interest (female, male) and the appropriate value of p (0.265). 'Experiments' are then invited, with the student requested to name the experiment (n=25), enter the sample size (25) and the number of samples to be drawn (100). The required number of samples are drawn sequentially, and the screen shows simultaneously the following three windows:

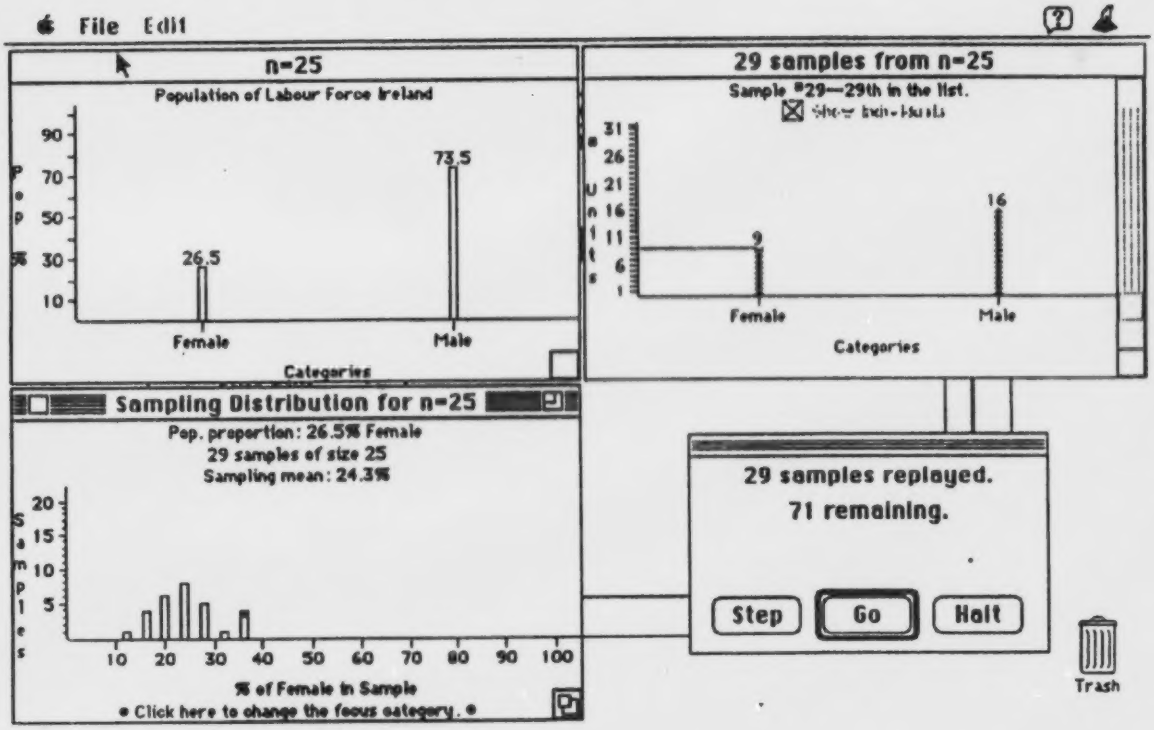
1. A probability distribution/bar chart of the population proportions.
2. A histogram which shows individual sample outcomes as well as the number of each.
3. An empirical sampling distribution of the values of the sample proportion $\hat{p}$ which builds as the sampling proceeds, with the value of the sample proportion $\hat{p}$ from the last sample shown explicitly in black, and also the overall sample proportion.

The 'Experiments' screen after 29 samples have been drawn is shown in Figure 3.

The same calculations which are performed in Minitab are also carried out here, but the emphasis is on the sampling process and the calculations remain very much in the background. In Sampling Laboratory the sampling process can be observed in real time and students see the sampling distribution form as more and more samples are taken. The process may be paused and restarted at any time, or may be conducted stepwise, one sample at a time.

Comparative educational gains through computerisation

In order to compare potential educational gains made by introducing the computer into the instructional strategies considered here it is useful to first identify the various steps involved in developing the idea. This has been done in the following table which also indicats the method or image primarily used to represent each step in the three instructional strategies. Note that we would not expect the computer based strategies to stand alone, but to be considered as complementary to the text based strategy. Thus, any representations available using the text based strategy would also be available when using the computer based strategies.


File Edit
n=25
Population of Labour Force Ireland
Pop %
90
70
50
30
10
26.5
73.5
Female
Male
Categories
29 samples from n=25
Sample #29—29th in the list.
☒ Show Individuals
Units
31
26
21
16
11
6
1
9
16
Female
Male
Categories
Sampling Distribution for n=25
Pop. proportion: 26.5% Female
29 samples of size 25
Sampling mean: 24.3%
Samples
20
15
10
5
10
20
30
40
50
60
70
80
90
100
% of Female in Sample
● Click here to change the focus category. ●
Step
Go
Halt
Trash

29 samples replayed.
71 remaining.

Figure 3: The 'Experiments' screen from Sampling Laboratory after 29 samples have been drawn showing the population proportion, the last sample and the empirical sampling distribution.

Development Sequence	Strategy 1 (Text)	Strategy 2 (Minitab)	Strategy 3 (Samp Lab)
1 A sample of individuals of given size is chosen from a *population*.	written explanation	implicit in code	population named, sample size specified
2 Those chosen in the *sample* may exhibit many attributes, one of which is to be recorded.	written explanation	implicit in code	attribute name specified
3 The proportion of individuals in the population which possess the attribute of interest is constant.	written explanation	implicit in code	the population proportion p is specified
4 The number of individuals in the sample which possess this attribute can be calculated.	written explanation	implicit in code	sample outcomes are depicted in a histogram
5 Many different samples can be selected from a population.	written explanation	samples can be listed and compared if requested	dynamic display of sample outcomes show variation from sample to sample

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6 Each sample gives rise to a value of the sample proportion $\hat{p}$.	written explanation	the value of $\hat{p}$ can be printed if requested	the value of $\hat{p}$ is shown on the histogram
7 These values of the sample proportion $\hat{p}$ will not all be the same.	written explanation	all values of $\hat{p}$ can be listed if requested	dynamic histogram which accumulates all values of $\hat{p}$
8 The values of the sample proportion $\hat{p}$ will form a distribution, called the (empirical) sampling distribution of $\hat{p}$.	written explanation and static histogram	static histogram	dynamically formed histogram of values of $\hat{p}$

Discussion and Conclusion

What then has the introduction of the computer contributed to the development of the idea of a sampling distribution? An analysis of the steps involved in developing the idea of a sampling distribution shows that it is a complex idea whose understanding cannot be dissociated from an understanding of the sampling process by which it is formed. The sampling process is dynamic and involves the linking of several elements: a parent population, the samples drawn from the population, the values of the test statistic extracted from each of the samples, and the sampling distribution they give rise to. In a purely text based explanation, the written word is used to describe the process outlined above and graphics are used to illustrate the end product of the process, the empirical distribution of the sample statistic. The bringing together of the sampling process and the resulting sampling distribution requires a high degree of mental processing and visualisation which seems to be difficult for many students because of the high cognitive load it potentially imposes.

However, an analysis of the second strategy shows that the introduction of the computer has done little to ameliorate the problem. Certainly, the computer gives the student an active role to play in generating the samples, extracting the values of the test statistic and forming the sampling distribution. It also gives them the opportunity to experiment with parameter values of their own to see how the form of the sampling distribution changes in response. But what has been done by the computer could have been achieved, in theory at least, with pen and paper. The product (the empirical sampling distribution) is certainly there but the process involved in generating the sampling distribution is hidden in the code (see previous section). As a result, it would appear that this computer based strategy would do little to reduce the high cognitive load that is inherent in a purely text based explanation. This contention would appear to be supported by classroom experience.

The third strategy utilising the computer package Sampling Laboratory creates a working model of the sampling process which has as its product a histogram displaying the resulting sampling distribution (similar to that produced by Minitab), but also displays the process by which the sampling distribution is obtained. This is done by simultaneously displaying the parent population, the current sample, the state of the sampling distribution with the addition of information from that sample, and finally the way in which each of the elements changes as the sampling process is set in motion. While graphical images similar to those that make up this display could have been generated by Minitab on an individual basis, the power of computer based representation of Sampling Laboratory is the simultaneous display and interlinking of these images so that changes in one element are immediately reflected, where appropriate, in the other elements. These dynamic, interactive displays have the potential to considerably reduce the cognitive load involved in understanding text based explanations of the idea of a sampling distribution and should lead to greater understanding. However, this has yet to be empirically tested.

The analysis so far has been concerned with a particular instructional problem, the understanding of the idea of an empirical sampling and specific technology, Minitab and Sampling Laboratory. However, as indicated by Kaput (1992), if we are to make progress in understanding the way in which technology can be integrated into the educative process and not just replicate what we have done in the past, we need to be able to step back from specific content areas and technologies and look at the general principles involved. Two of Kaput's key principles emerge from this analysis. Firstly, with computers we are now in the position to create new notations (ways of recording and /or displaying information) (Kaput, 1992 p523) that are more capable of conveying a complex idea than the traditional paper based notations used currently. This has been clearly illustrated in Rubin's Sampling Laboratory. While packages such as Minitab could actually generate similar graphical images to those used by Sampling Laboratory, even if the individual images produced could be arranged appropriately and displayed simultaneously, the end result would be no different than printing out the same images on a sheet of paper. To use Kaput's terminology, the notations used by Minitab, like those of most other statistical packages, have not been designed to do any more than replicate paper based notations which in turn have been designed to convey information in an inert medium. Whilst there may be an educational payoff of using the computer in this instance, by enabling the student to be more actively involved in the learning process, the images produced convey no additional information to the student.

The second general point made by Kaput and others (Lesh, Post, & Behr, 1987 for example) is the need to recognise that many concepts in quantitative disciplines like mathematics and statistics are multi-faceted and cannot be fully understood from a single representation. The sampling distribution is such a concept. Its definition as the distribution of a combination of random variables is not easily related to the physical process of sampling. Similarly, the empirical approach, which sees the sampling distribution as a long term frequency distribution, captures this aspect but does not lead to the precise relationships specifying the centre and spread of the sampling distribution that are the corner stones of almost all introductory statistical inference courses. Kaput suggests that to fully understand a concept it is necessary to understand it in all its manifestations. In terms of the sampling distribution, this would call into question any instructional sequence that presented only one approach, in this case purely theoretical or purely empirical. In this regard, neither of the computer based instructional sequences presented here come to terms with the issue.

REFERENCES

Bloom, L. M., Comber, G. A., & Cross, J. M. (1986). Using the microcomputer to simulate the binomial distribution and to illustrate the central limit theorem. International Journal of Mathematics Education in Science and Technology, 17(2), 229-237.

Devore, J., & Peck, R. (1986). Statistics. St Paul: West Publishing Company.

Hawkins, A. (1990). Success and Failure in Statistical Education - A UK Perspective. In D. Vere-Jones (Ed.), Proceedings of the Third International Conference on Teaching Statistics, 1 (pp. 24-32). Dunedin, NZ: ISI.

Johnson, R., & Bhattacharyya, G. (1987). Statistics Principles and Methods. New York: John Wiley & Sons.

Kaput, J. J. (1992). Technology and Mathematics Education. In D. A. Grouws (Eds.), Handbook of Research on Mathematics Teaching and Learning (pp. 515-555). MacMillan.

Lesh, R., Post, T., & Behr, M. (1987). Representations and Translations among Representations in Mathematics Learning and Problem Solving. In C. Janvier (Eds.), Problems of Representation in the Teaching and Learning of Mathematics (pp. 33-40).

Mendenhall, W., Wackerly, D. D., & Scheaffer, R. L. (1990). Mathematical Statistics with Applications (Fourth ed.). Boston: PWS–Kent Publishing Company.

Moore, D. S. (1992). Teaching Statistics as a Respectable Subject. In F. Gordon & S. Gordon (Eds.), Statistics for the Twenty-First Century (pp. 14-25). MAA Notes No 26, Mathematical Association of America.

Ott, L., & Mendenhall, W. (1990). Understanding Statistics (Fifth ed.). Boston: PWS–KENT Publishing Company.

Rubin, A. (1991). Using Computers in Teaching Statistical Analysis: A Double Edged Sword. Unpublished Manuscript.

Rubin, A., Bruce, B., & Tenney, Y. (1990). Learning about sampling: Trouble at the core of statistics. In Proceedings of the Third International Conference on Teaching Statistics, 2 (pp. 314-319). Dunedin, NZ: ISI.

Shwartz, J., & Yerushalmy, M. (1985). The Geometric Supposers. Pleasantville, NY: Sunburst Communications.

Thomas, D. A. (1984). Understanding the central limit theorem. Mathematics Teacher(October), 542 - 543.

LANGUAGE, SPEECH AND *SEMIOSIS*: APPROACHES TO POSTCONSTRUCTIVIST THEORIES OF LEARNING IN MATHEMATICS

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In the usual constructivist view of learning mathematics the student is engaged in the active process of constructing meaning for instructionally given target concepts. Cobb, Yackel, and Wood (1992) and others propound an alternative view, social constructivism, which treats mathematics both as an individual constructive activity, and as a social practice. On this view, learning operates as an individual cognitive action (construction of interpretations) made compatible by social interaction with the collective interpretations of mathematically acculturated practitioners. One argument used by Cobb et al *against standard forms of constructivism is that they commonly rely on an essentialist theory of knowledge. In this paper, however, it is argued that the notions of interpretation, meaning, and construction crucial to the position of Cobb* et al *also depend on essentialist theories of knowledge. Approaches to validating this claim are made in three lines of inquiry drawing upon notions of theory of practice, philosophy of language, and the science of signs or semiotics. Important parallels between these discussions, and between certain poststructuralist formulations are noted in the process of argument. An outline of elements of a new postconstructivist theory of learning emerges. A touchstone for these ideas is provided by Lave and Wenger's (1991) theory of learning as* legitimate peripheral participation*; allusions to this theory are made throughout the paper, although no direct study of it, or it's relation to the theory figured here is attempted.*

Viewed as a strategy to oppose the traditional "broadcast" view of education (Seeger and Steinbring, 1992) constructivism provides a valuable space in which teaching alternatives more favourable to the learner as an active meaning maker may be developed. Whether this strategy has been successful in reformulating classroom practice may be a moot point, however, we can say that as a name signifying a coherently organised set of beliefs about learning and teaching mathematics, constructivism has been singularly unsuccessful. Indeed, attempts to transcend an essentially negative, or mainly polemical formulation, have lead researchers to a wide variety of sometimes incompatible theoretical positions; Ernest (1991), for example, has observed a "panoply of theoretical positions" and that no single theory of constructivism holds general sway. One aim of this paper is to strengthen this observation by advancing the view that constructivist theories are obliged to covertly reinstate notions of knowledge characteristic of the "broadcast" model, and that therefore, no *positive* theory of constructivism is, in principle, possible. Three lines will be involved in establishing this argument, and these will draw on theories of practice, language, and signification and will be treated in separate sections below. The theory of 'social constructivism', as set out by Cobb, Yackel and Wood, in their article "A constructivist alternative to the representational view of mind in mathematics education" (1992), will be used to illustrate the main points.

Wittgenstein remarked that his work could be distinguished from his predecessors in that he found difference where they found identity, and identity where they saw only difference. Perhaps a clue for future directions in educational theory may be found in this remark; certainly, amongst recent authors, Lave and Wenger (1991) are exemplary in their treatment of theoretical issues in learning consistent with this dictum. At root, it may be observed that constructivism is beset by two levels of false identification: first, there is the forced insistence of an identity between terms such as "knowledge" "construction" "meaning" "interpretation" *etc*; and second, related to this, there has been too ready an acceptance of the view that there must necessarily hold an identity between a theory of learning, and a theory of learning *for teaching*. Such an identity carries with it the dubious assumption that teaching is principally a technical problem to be addressed by an appropriate learning theory in conjunction with the a correct method for transferring this knowledge to pedagogic sites.

On the other hand, attempts to develop a positive theory of constructivism are frustrated by what Lave and Wenger have call "discourses of duality". Dualities constitute and affirm the objects of theoretical knowledge, in doing so they afford the structures by which traditional theories of learning/teaching are proposed and validated, but in the process, it is crucial to recognise that they systematically distort the focus of their inquiry. This insight will be revisited frequently in the paper, although not explicitly discussed. Dualities to be addressed and questioned more or less openly in this paper will include: learner/teacher, {learner/teacher}/researcher, mind/body, language/world, structure/function, psychological/social.

In raising the above points an outline for a theory of learning alternative to the constructivist model will be provided. Such an alternative may be called *postconstructivist* since it would reject both transmissionism and the constructivist variations. Postconstructivist theories of learning transcend epistemological constraints, in particular, covert essentialism, and attempt to subvert and reorganise structures of identity and difference. Such theories take their cue from a remark of Lave and Wenger (1991, p 35 and p 93) who urge that learning be considered "an improvised practice". Approaches to such a theory, it will be suggested, emerge from theories of practice, Wittgenstein's later philosophy of language, and poststructuralist theories of signification.

SUMMARY OF A CASE FOR SOCIAL CONSTRUCTIVISM

Cobb *et al* (1992) argue that mathematics is "both a collective human activity and an individual constructive activity"(p 17). Individuals construct mathematical meanings (interpretations), and these are tacitly identified with "internal representations"(p 2). Cobb *et al* note, however, that the concept of representation typically functions in two ways: "internal representations are located in students' heads and external representations are located in the environment" (p7) and it is this separation that their analysis is designed to side step. One way to consider the efficacy of the individual construction of representations considered by the authors relies on a philosophical position known as the *representational view of mind* whose central tenet, Cobb *et al* report, may be summarised as follows

> To know is to represent accurately what is outside the mind; so to understand the possibility and nature of knowledge is to understand the way in which the mind is able to construct such [internal] representations. (Rorty, 1977, p 3 quoted in Cobb *et al*)

Cobb *et al* demonstrate that the application of this principle provides the basis for constructivist theories of learning in which students construct internal representations which match the mathematical relations inherent in external representations (p 17); and they note that such a formulation has "a certain explanatory power". However, the authors argue that since such theories lead to paradoxes in learning (Bereiter, 1985), have problems with an adequate theory of transfer, and ignore certain anthropological evidence they must, nevertheless, be rejected.

Thus the authors are lead to consider an alternative account in which the representational view of mind plays no part. A theory of social constructivism emerges in which both individual interpretation and the "taken-as-shared interpretations of mathematically acculturated members of the wider community" (p17) are taken into account. Teacher and student each construct individual interpretations which in principle, of course, they cannot share with each other; these, however, come to be taken as shared once a criterion of compatibility or fit for the purposes at hand is obtained. Note, finally, that the arguments of Cobb *et al* for the rejection of the representational view of mind amount only to a rejection of the hypothesis that mental representations can be known to be accurate; indeed, it would appear that individual cognitive representations play a tacit role in their theory, and that these enable the construction of individual interpretations. However, no account of the concept of representation, or of it's use by them in their theory, is provided by the authors. In the following section therefore, I will attempt to grapple with the question of the character and adequacy of their tacit formulation of this concept. I will also consider the efficacy of their application of teachers' and students' interpretations in order to construct mathematical relationships which, in their words, "have clout" (p25).

AN APPROACH THROUGH THEORIES OF PRACTICE AND REPRESENTATION

Theories of practice

In philosophical terms, the work of Cobb *et al* can be read as an attempt to counter a variant of an essentialist view of mathematics which holds that mathematical concepts exist in a reality separate from the minds and social space of the people who "discover" them and utilise them; and when concepts are to be utilised, so it is argued, they can be embodied in practical situations. An instance of this would be teaching mathematics itself, in which mathematical concepts are embodied by external representations in pedagogically transparent ways. In arguing against this position, known in philosophy as metaphysical realism, the authors embrace a positivist view of knowledge (*eg* privileging sense data as in the "experientially real scenario of the candy factory", p 22) and validate this by advocating a theory of truth in which statements are judged true in so far as they help us move toward satisfactory relations with general experience (*eg* formation of "increasingly sophisticated mathematical conceptions", p 22).

But was sense experience *actually* fundamental for the subjects' learning in the situations observed? Were concluding propositions *actually* taken-as-shared by the participants in the manner the authors described? Little evidence/argument is provided by the researchers in support of their crucial presupposition that variability amongst these factors was critical to the learning outcomes of the students. Nor, indeed, is it clear on what basis evidence for such claims could be provided. Cobb *et al* argue that learner and teacher develop meanings which they take to be shared between them. But how does the researcher know this? Three possibilities will be considered: (1) The researcher has a way of knowing which can see this truth directly (and this essentialism would need to be explained); and (2), the researcher enters the learner/teacher relationship as a third element. On this view, the truth is formed as a taken-as-shared meaning developed in the concrete relations which hold between these three agents. However, review of the proposals put forward by Cobb *et al* reveals no evidence that either of these alternatives is intended. The remaining alternative must then be accepted, and this is that the key relation from the researcher's point of view is between himself/herself (as observer) and a repertoire of learner/teacher duads (as actors). On this view, researchers work by formulating the most plausible story abstractable from each of the learner/teacher duads investigated. But by what criteria are these stories abstracted? With what prejudices and blindnesses? For what audience are they meant to be plausible? To what extent does the condition of their plausibility depend on what an audience will accept? Entailed within the operations of this alternative is an agreement (amongst the observers) to privilege abstracted relations between learners and teachers (the actors) over concrete relations between the particular learners, teachers, and researchers. What justifies this asymmetrical treatment? Why have the authors not provided a reflective account of their method and it's justification for obtaining abstracted relations? Why have the authors not critically reviewed the consequences of their choices? One response to the issues raised here is the suggestion that abstracted relations observed by the researchers exist in reality, and so are naturally privileged; moreover, on this view, abstracted relations are observed directly since they exist as separate elements within the system under review. Such an argument would, however, involve the paradoxical reinstatement of an essentialist epistemology, albeit at the level of the relation between the researcher and his/her subjects *ie* between observers and actors. A key question to be asked is: What are the epistemological implications of such a reinstatement for the theory which Cobb *et al* propose?

In conclusion, the arguments presented above demonstrate the need for Cobb *et al* to explicitly state their theory of practice (Bourdieu ,1977; Carr and Kemmis, 1986; Lave, 1991, pp 49-52). Such a theory would indicate by what processes and organisation researchers' interpretations of data are fashioned, and how their theoretical knowledge is formulated. A theory of practice formalises the distortions of theory, and provides research with a language in which to scrutinise, and assess the systematic (and necessary) failure of research to fully bring to light the character of what is being examined.

Theories of representation

I have already mentioned that Cobb *et al* do not explicitly analyse the notion of representation, although they extensively rely on it. In their hands it seems to be a simple duad: Two objects related in some way. Whether one or both of the objects are internal or external to the learner does not seem to matter - what is of importance is that the objects are similar (for/to the learner) in some important way, and that the tasks of construction ensure this. Indeed, in some instances Cobb *et al* seem prepared to go further and appear to have in mind some kind of equivalence between these elements (as in talk of "re-presentation", p 25, and the like). Thus, in these cases, it may be inferred that an idea of representation implies some kind of double presence for/to the learner in which an object (an identity) and its equivalent (a near identity) are present together (*cf* Boynes, 1990, p93). But if this is the case then it is not clear how representations can play the role in learning the tacit theory of Cobb *et al* gives them. For to learn by representation would be to simply renew or re-present a concept already present - in other words learning would have already have taken place. If, for representation to function in a learning context it is required from the start that the object of instruction be present, then it is clear learning is not possible. Clearly this *paradox of representation* is similar to Bereiter's learning paradox (1985), and it is therefore ironic that in order to avoid the latter, the theory of social constructivism seems to embrace the former.

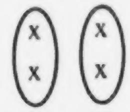
I believe an adequate treatment of the representation paradox shows one reason why we have to be most careful in how we approach thinking about how meaning is constructed for/to the learner in the process of learning. It would seem, for instance, that in order to avoid this paradox a second appearance of the social domain is required (*cf*

Lave and Wenger, 1991, p35). For Cobb *et al* the social domain appears as a domain of "acculturation", in contrast to the "cognitive aspect" of the aspiring individual (see p 28). In this, it's first appearance, it both constitutes and affirms the socially interactive component of learning as constructing taken-as-shared knowledge targets. Beyond this, however, I suggest that the social also appears intra-individually in order to allow for the personal construction of meaning. This second appearance, however, requires the promotion of the social domain as the third element of an adequate notion of individual representation. This would form what I shall call a *representational triad* consisting of object-representation-social. In the following sections, I will seek to further clarify and extend from perspectives in the philosophy of language, and general theory of signification (*semiosis*) the issues raised here. These lines of investigation will mark the boundaries of a second field of critical analysis of learning as social constructivism.

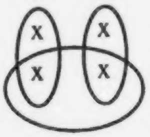
AN APPROACH THROUGH WITTGENSTEIN'S PHILOSOPHY OF LANGUAGE

Wittgenstein (1967, I-38) makes the following observation:

> "You only need to look at the figure to see that 2 + 2 = 4."



> - Then I only need to look at the figure to see that 2 + 2 + 2 are 4.



Wittgenstein's point succinctly reminds us that meanings are neither in the objects to which they refer, nor in the language which describes them, nor in the conventions which enable us to relate these two, but distributed amongst these domains, each one necessary but by itself insufficient. On one hand Cobb *et al* seem to endorse this observation, yet on the other, as my analysis of their tacit theory of representation has shown, they appear to reinstate assumptions consistent with mathematical essentialism by the demotion of the social within, or even its elimination from, the representational triad. As a consequence, individual interpretations in the theory of Cobb *et al* appear like movements in a private language. But is the idea of such a language intelligible? Wittgenstein himself struggled over this and related questions. In his early work (Wittgenstein, 1921/1974) for instance, he assumed that language was a perfect medium, it's operations did not distort as they painted to us a picture of the world. Later investigations (Wittgenstein, 1953/1991) however convinced him this view could not be sustained, language systematically distorts the world it reports; one instance for him of this view is the formulation of the notion of a private language, which he therefore rejects as unintelligible. Accepting Wittgenstein's argument would imply that individual interpretations are already social, and this throws into question the basis of the distinction between the social and the individual Cobb *et al* wish to preserve.

In studying the world, Wittgenstein came to insist it is necessary to also study the language in which it is represented. Such a study would need to focus on two related issues:

Issue 1. Language is organised by structure (*grammar*). How do the workings of grammar entangle, structure, distort the sense I make? (Hunnings, 1988)

Issue 2. Language is an act, a performance, a series of *language-games*. What is the purpose, and what are the particular circumstances of any given utterance? (Kanes, 1992)

These, together with the object world, form three elements in a triad of sense making involvements. Parallels with what above I have called the representational triad (object-representation-social domain) are to be noted: for instance, Issue 1 relates to a linguistic representation, Issue 2 the social domain. I suggest that future research into these (triads) and their interrelation within the multiple contexts of mathematical invention, application, and instruction could lead to the formulation of a learning theory alternative to the constructivist paradigm.

As Wittgenstein's later work testifies, finding a means to interrogate our language system, as it were, from the inside (Issue 1 above), is not a simple task either for the actor (the writer) or the observer *cum* actor (the reader). In general two lines have been followed: the first is shown by Wittgenstein himself, and follows the operations of natural language; the second, starting from quite different perspectives but following a roughly parallel path, is currently practised in the form of poststructuralist /deconstructive analysis as developed by authors such as Jacques Derrida and Julia Kristeva. Understanding the relationship between these two paths (Norris, 1983; Staten, 1986) will be an important task in fully articulating a theory of learning foreshadowed here. My present, very much limited aim, is to provide something of the flavour of the deconstructive line in order to signal key points of departure from the constructivist paradigm.

AN APPROACH THROUGH SEMIOTICS

At the turn of the century the American philosopher, C. S. Peirce, and a Swiss linguist, F. de Saussure independently proposed the need for a "science that studies the life of signs within society" (Saussure, 1983, p 16). Focus on such a science (*semiotics*, from the Greek, *seimon* meaning 'sign') will enable us to reframe the formulations of Cobb *et al* in order to see more clearly their implications and begin the development of a counter view.

For Peirce (Hawkes, 1977), the sign is a triadic relation between an *object*; a sign or *representamen* which stands for it; and an *interpretant* for whom the sign, *grounded* in a way particular to circumstances, makes sense. For example, in the case quoted by Wittgenstein above, the object is the diagram, the representamen is the arithmetic symbolisation, and the interpretant is the set of arithmetic codes and conventions which enable sense to be elicited. In contrast, Saussure (1983) takes a sign to be an object within the mind of an individual. It has two parts: the sense-impression of the *referent* called the *signifier*, and a concept, called the *signified*, which provides unity to the triad of elements (referent-signifier-signified). Kristeva calls these combined elements a *matrix* or *pyramid* of the sign. To illustrate using the previous example: the actual marks on the page are the referent; the impression these marks make on the mind, the signifier; the arithmetic concept which gives the signifier meaning, the signified.

Putting aside any interest in the differences between these approaches, three key points relevant to the study of learning mathematics emerge.

Dichotomy of language and the world:

In Umberto Eco's words (1984)

> The concept of the sign must be distinguished from its trivial identification with the idea of coded equivalence and identity; the semiosic process of interpretation is present at the very core of the concept of the sign. (p 1)

Kristeva (1989) notes the same thing when she says that signs

> all *replace* or *represent* something that is absent, evoked by an intermediary, and, consequently, included in a system of exchange: in a communication. (p 13, italics in the original)

It is therefore clear that representation includes a separation, or at any rate problematises a separation between what is *present* and what is *absent* in the process of representing. Viewed in this way the sign constitutes and affirms a radical break between language and on the other side, an actual, concrete world. In setting up a learning theory, we have to understand how *both* these operations (constituting, affirming) function; our goal is to grasp and scrutinise their full epistemological significance.

Epistemology of the *constructive notion of the sign*:

As just hinted, the sign, an "evoking intermediary", is not epistemologically neutral: signs both constitute and structure knowledge, they do not merely transmit it as if through a transparent medium. At the level of language this concern parallels Wittgenstein's focus on the operation of grammar (correlate of the signifier) to both distort and make meaning possible (the signified). Partially, at least, Cobb *et al* seem in agreement with this notion of the incapacity of language - they do, after all, rule out the possibility of transparency in instructional representations, and this confirms the language/world duality. As a corrective, mathematics is treated as both an individual and a socially organised process of "constructive activity".

As we have seen, the purpose of constructive activity for the constructivists is successful communication. But what can be said of its character? This question, which corresponds to the switch from accenting functionalism in behaviourist psychology to accenting structure in cognitivism, has not, I believed received due attention. My approach will be to reframe constructivist notions within Saussure's semiotic domain: it is hoped to then more fully grasp the epistemological implications of constructivist theory. The views of Cobb *et al* could thus be restated: Learners act on both a set of referents and signifiers in order to produce in collaboration with teachers and others, a viable (*ie* taken-as-shared) set of signifieds. I shall call this characterisation, the *constructive notion of the sign*. Of immediate note is the immobility and stiffness of this production/construction *versus* the apparently inexhaustible, multivalent nature of the particular ground or context upon which the sign is built. Duality between language and the world is radically asserted by this process. But for Julia Kristeva (1971, p 24), the significance of this duality is that *sense*, the signified, is constituted by it (see also Lave and Wenger, p104). She argues that sense is the construction within language of a mediation between the signifier and the referent. Sense appears as "hidden in the sign". But such an appearance is actually an illusion based on the way a signifier "elides the plurality of the outside (out-of-sign) that it begins by posing [in the form of the referent]" (p 26). Illusion of content in the signifier is derived from the way it operates. Wittgenstein comes to a similar conclusion about the illusions language establishes for us when we attempt to use it in any other way than as a (rather flawed and problematic) tool. It remains to add that the illusionary sense hidden in the sign, spoken of by Kristeva is, in this context, none other than the (re-)emergence of mathematical essentialism. The epistemological stance of the constructive notion of the sign, and therefore the social constructivist paradigm of Cobb *et al* privileges the operation of metaphysical realism within mathematics education, it does not deny it. Movement to the ideal, both endorsed and encouraged by the relatively immobile semiotic relation described here, is necessarily related to the elimination of differences and discrepancies - everything, in short, which is particular to the learner and the learning context. Kristeva aptly describes such a linguistic enterprise as "ideological": She does so because the elision of difference ultimately re-emerges on the political plane, as an expression of power (Lave and Wenger, 1991, p 38).

> Under the mask of socialisation or of mechanistic realism, ideological linguistics, absorbed by the science of signs, turns the sign-subject into a centre. The sign-subject [the learner, the teacher] becomes the beginning and the end of all translinguistic activity; it becomes closed up in itself, located in its own word, which is conceived of by positivism as a kind of 'psychism' residing in the brain. (Kristeva, 1969, quoted in Eco, 1984, pp25-26)

Thus we see again that the tenor of the Cobb *et al* argument re-endorses rather than overcomes the Cartesian mind/body dualism. At the heart of the approach sketched here is the suggestion that poststructuralist perspectives offer an effective critique of the constructive notion of sign. This observation will be further developed below.

An alternative to the *constructive notion of the sign*:

In thinking about linguistic activity Saussure distinguishes between *langue* (language) and *parole* (speech). *Langue* emphasises the structural aspect of language, that which is anonymous in the sense that it "exists perfectly only within a collectivity" (Saussure, 1983). It is

> the social part of language, external to the individual, who by himself is powerless to alter it or modify it. It exists only in virtue of a kind of contract agreed between the members of a community. (p 13)

Parole, on the other hand, being entirely a function of the speaker, belongs to the individual. Unlike *langue*, which "never requires premeditation", it is an "individual act of the will and the intelligence" (p 14), an active registration of the individual. Beyond these simple characterisations, including the suggestion of their duality, the question of the *langue/parole* relationship turns out to be of great significance to the questions of meaning, knowledge, and learning. Saussure argued that *langue*, viewed as "a fund accumulated by the members of the community through the practice of speech" (p 13), is an historical product constituted by *parole*; but the poststructuralists, in the tradition of Derrida and Kristeva, emphasise that an interplay operates on a deeper level, namely at every point the semiotic process is actualised, for example, when speaking, reading or writing. On the poststructuralist view, the distinctions implied by *langue* and *parole* (like those, also, between the signifier/signified itself) cannot be pinned down once and for all, they are constantly shifting ground. Making meaning is seen as an incorrigibly mobile process; *langue* and *parole* are locked into a never ending struggle in which one never has a necessary privilege over the other. This view provides fresh meaning to the dictum: The individual is inscribed within the social, the social is inscribed within the individual.

Returning to the problem of learning, Saussure's *langue/parole* distinction provides a good framework to *redirect* the thrust of the constructivist paradigm. If *langue* were simply identified with the "taken-as-shared interpretations of mathematically acculturated members of the wider community" and *parole* as the "students interpretations of instructional materials" (Cobb *et al*, p 17), then it becomes clear how germane to the debates about learning in mathematics, the issues at stake in the linguistic sphere are. Moves to postconstructivism are parallel to shifts towards poststructuralist thinking in other contexts. In each case, the kernel of the operation is the same: notions of identity are replaced with the notions of difference. From these perspectives, one problem with constructivism can be identified as the stiffness and immobility of 'construction' itself. Like *langue*, it is sensitive to the social collective, both in time and space; but as we have seen, *langue* is inextricably bound up with and entangled by *parole*, and therefore is unstable in every concrete instance. Likewise, the constructed artefacts of mathematics, what Cobb *et al* call mathematical knowledge, are locked into a permanent entanglement with instances of what I shall term *mathematical speech*, and this is the perpetually mobile and irreducible registration of the individual as one dealing in practice with mathematics.

In conclusion, my the aim has been to signal that an over-emphasis on mathematical *langue* currently exists both in theory and in practice; that the full consequences of this are anything but transparent; and that a new theory might favour the reorchestration of theory/practice in order to re-emphasise the importance, albeit problematic itself, of mathematical speech. At the level of *theory*, I have argued that emphasis on mathematical *langue*, leads to the endorsement of mathematical essentialism as an epistemological project, and that the work studied of Cobb *et al* is riddled with such a tendency. At the level of *teaching practice*, I argue that emphasis on *langue* leads to the algorithmisation of knowledge, in which "learning how to 'do' school becomes the object" (Scribner and Cole, 1981 quoted in Lave and Wenger, 1991, 107), not learning to participate meaningfully in a wide range of mathematics practices. As Lave and Wenger note

> The didactic use of language, not itself the discourse of practice, creates a new linguistic practice, which has an existence of its own. [Learning] in such linguistic practice is a form of learning, but does not imply that [learners] learn the actual practice the language is supposed to be about. (p 108)

Mathematical *langue*, and mathematical *parole*, one abstract and the other concrete, are engaged in an endless struggle; in the terms of Lave and Wenger (pp113-117), each provides *continuity* for the other and yet, paradoxically, also threatens to *displace* its opposite. For this reason, postconstructivist theories of learning do not make an end of abstraction; they aim to make it merely a means. As Lave and Wenger's invocation of Marx (p 38) reminds us, the goal of theoretical knowledge is not abstraction in itself, but rather to "ascend (from the particular and the abstract) to the concrete". In so doing it is sought to revoke the privileges of abstraction current in both theories of learning and classroom practice.

REFERENCES

Bereiter, C (1985). Towards a solution of the learning paradox. *Review of Educational Research*, 55, 201-226

Bourdieu, P (1977). *Outline of a Theory of Practice*. CUP

Boyne, R (1990). *Foucault and Derrida: The Other Side of Reason*. Unwin Hyman

Carr, W. and S. Kemmis (1986). *Becoming Critical: Education, Knowledge and Action Research*. Deakin University

Cobb, P, E. Yackel, and T. Wood (1992). A constructivist alternative to the representational view of mind in mathematics education. *Journal for Research in Mathematics Education*, 23(1), 2-33

Eco, U (1984). *Semiotics and the Philosophy of Language*. Macmillan

Ernest, P (1991). *The Philosophy of Mathematics Education*. Falmer

Hawkes, T (1977). *Structuralism and Semiotics*. Methuen

Hunnings, G (1988). *The World and Language in Wittgenstein's Philosophy*. SUNY

Kanes, C (1992). Knowing and Meaning in Mathematics Classrooms: Perspectives Drawn from Wittgenstein's Philosophy of Language. In B. Southwell (Ed.) *Proceedings of the Fifteenth Annual Conference ,of the Mathematical Research Group of Australasia.* Sydney.

Kristeva, J (1971). The semiotic activity. Translated by S. Heath and C. Prendergast in *Signs of the Times: Introductory Readings in Textual Semiotics,* Cambridge.

Kristeva, J (1989). *Language the Unknown: An Introduction to Linguistics.* Columbia University Press

Lave, J and E. Wenger (1991). *Situated learning: Legitimate Peripheral Participation.* CUP

Norris, C (1983). *The Deconstructive Turn.* Methuen

Seeger, F and H. Steinbring (1992). *The Dialogue between Theory and Practice in Mathematics Education: Overcoming the Broadcast Metaphor.* Materialien und Studien Band 38. IDM: University of Bielefeld.

de Saussure, F (1983). *Course in General Linguistics.* Translated by Roy Harris. Duckworth

Staten, H (1986). *Wittgenstein and Derrida.* University of Nebraska Press

Wittgenstein, L (1921/1974). *Tractatus Logico-Philosophicus.* Routledge and Kegan Paul

Wittgenstein, L (1967). *Remarks on the Foundations of Mathematics.* Basil Blackwell

Wittgenstein, L (1953/1991). *Philosophical Investigations.* Basil Blackwell

A COMPARATIVE ANALYSIS OF RESEARCH QUESTIONS, METHODS, AND METHODOLOGIES IN MATHEMATICS EDUCATION: 1979 AND 1993

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This paper reports on a comparative analysis of research presented to the 1979 and 1993 MERGA Conferences. Two main issues are considered: the nature of research methods and methodologies utilised by researchers; the content focus of principal questions made the subject of research. In considering the first of these, papers were stratified by research methodology (positivist, interpretivist, critical), data type (quantitative, qualitative), and research method (experiment, survey, clinical interview, discourse analysis, etc). Analysis of the data indicated that whilst the positivist paradigm predominated in both conferences, a significant switch to qualitative methods and a diversification of methodologies was observed in 1993. In addressing the second main issue, dominant research questions for each conference were identified in terms of subject domain clusters. Methodology used in this task involved the construction of a subject matter matrix, within which each submitted paper was allocated to a single cell. Papers were thus stratified by two variables accounting for educational agency (student, teacher, classroom, researcher) and content (knowledge, beliefs/attitudes, context). Major findings were an increased emphasis in 1993 on the student vis a vis teacher, and a switch of emphasis from beliefs/attitudes to the study of contextual domains of education. The paper concludes with a discussion of the limitations of the methodology used in this study, and an identification and discussion of principal findings.

The 1993 conference of the Mathematics Education Research Group of Australasia is the Sixteenth Annual Conference of the organisation and the second time the conference is held in Brisbane, the first being the Third Annual Conference held in 1979. During the fourteen years separating the two conferences, mathematics education research in Australasia has witnessed significant changes in the socio-political as well as theoretical contexts. Watson and Atweh (1992) for instance, in identifying a range of recent factors influencing mathematics education research, mentioned increases in research funding and numbers of researchers, and the growing diversity of international influences. The main aim of this paper is to discuss the major differences in mathematics education research reported at the two MERGA Brisbane conferences. This paper has two purposes: (i) to identify the foci of research interests during the 1979 and 1993 MERGA conferences and (ii) to identify major shifts in research interests between the two conferences. We will attempt to achieve these purposes by considering, firstly, the Plenary sessions and the Special Interest Groups at both conferences, secondly, the methods and methodologies used by authors of these papers, and thirdly, the subject matter analysis of the papers submitted at both conferences.

PLENARIES AND SPECIAL INTEREST GROUPS

The proceedings of the 1979 conference (Booker, 1979; Atweh, Booker, Irons & Jones, 1979) included 24 papers, while 93 papers were submitted to 1993 conference, representing an increase of 387%. There was one Plenary Address at the 1979 conference and five at the 1993 conference. The plenary at the 1979 conference was introspective dealing with the aims and future directions of MERGA. The plenaries at the 1993 conference dealt with research perspectives around the theme of the conference: Contexts in Mathematics Education. Two of the speakers at the 1993 conference are from overseas, representing the increasing international orientation of the organisation.

Special Interest Groups (SIGs) at the annual conference provide a reliable indication the major trends and interests of research. It is of interest to note that all but one of the SIGs at the 1979 has either been discontinued or have undergone changes in focus and/or name within the last fourteen years. The SIG Mathematics Learning in Early Childhood has changed into Number SIG which convened up to the 1992 conference. The SIG on Language, Reading and Mathematics has changed its title to Language and Mathematics and has remained active during the fourteen years. The SIG Attitudes to Mathematics and Diagnosis and Remediation have been discontinued at MERGA conferences. The SIG on Sex Differences in Learning has changed to Gender and Mathematics, and in the 1992 conference became the Social Justice SIG. The SIG on Problem Solving is still meeting under the same name. In addition to the above SIGs new groups have formed since the 1979 conference on Algebra, Space and Geometry, Ethnomathematics, Calculators and Computers, and on Adult Learner.

RESEARCH METHODS AND METHODOLOGIES

In this section we will consider the developments in methodologies and research methods used by Australasian mathematics educators in 1979 and 1993. A number of preliminary comments are needed. First, we accept the distinction made by Guba and Lincoln (1988) between methodology and methods of research. Methodologies refer to overarching conceptual paradigms used in framing a study, whereas methods refer merely to tools for research. Different methodologies may employ the same methods. Secondly, as Linn (1986) notes, although qualitative and quantitative alternatives in research are sometimes identified with paradigms, this view over-simplifies the issues concerned. Accordingly, in this paper, we regard the qualitative/quantitative choice as merely a matter concerning the type of data a study involves. Thirdly, whilst broad agreement may exist concerning the nature and methods appropriate to the traditional scientific paradigm for educational research, no such agreement exists for alternative paradigms. Each of the following labels, for instance, refers to various aspects of alternative models: naturalistic, interpretive, intuitive, subjective, ethnographic, field research. These terms may overlap with each other, but each refers to a distinct set of characteristics, hence each would lead to different classification of research. This section will deal with research paradigms (methodologies) and research methods separately.

Research Methodologies

Carr and Kemmis (1986) discuss three paradigms for educational research rather than two. The first paradigm, the positivist, involving a scientific approach, is based on the belief that human action is governed by "law-like relationships" (Candy, 1989, p2) which can be discovered by careful manipulation and measurement of variables. Research in the second paradigm, the interpretive, seeks to "explain how people attribute meaning to their circumstances, and how they develop and make use of rules which govern their behaviour" (p2). Lastly, Candy identified a "schism" within the interpretive paradigm which has given rise to a third alternative, namely the critical paradigm. This approach is based on the belief that "human beings do not live in worlds entirely their own devising and that people generally are subject to influences and pressures that share their attitudes and perceptions and yet of which they are often unaware" (p2). Research using this paradigm attempts to analyse the historical context and social interactions that give rise to meaning and behaviour.

Using Candy's definitions then, all of the papers from the 1979 conference that deal with empirical research fall under the positivist perspective. The papers in the 1993 conference, however, show a greater diversity. Although the majority are still in the positivist approach as defined above, 60 papers (65%), about 10 papers (11%) could be classified as interpretive, and 6 (6.5%) as critical. The remainder of the papers, 16 or 16%, could not be classified because they consist of reviews of literature, position papers or unfocussed research papers. In summary, alternative paradigms for research are in evidence within the mathematics education research community, although they are not predominant.

Research Methods

As noted above, the labels qualitative/quantitative are taken here to refer to a type of data rather than as paradigms for research. It is of interest to compare their relative use within the three paradigms outlined above. Table 1 provides such a comparison. It is of interest to note that while the majority of studies in the interpretive and the critical paradigms are qualitative, studies within the positivist approach tend to be divided between the two types of data. Further, there is a marked shift between 1979 and 1993 from studies using quantitative data towards those using qualitative data.

Table 1: Number(Percentages) of papers at the 1979 and ***1993*** *conferences by data and paradigm type.*

	Positivist	Interpretive	Critical
Quantitative.	13(81.3) **29(48.3)**	0(0) **2(20)**	0(0) **0(0)**
Qualitative.	3(18.7) **27(45)**	0(0) **7(70)**	0(0) **6(100)**
BOTH	0(0) **4(6.7)**	0(0) **1(10)**	0(0) **0(0)**

Table 2 below show numbers and percentages of papers classified according to their research method. In the vast majority of cases the papers themselves identify the method that they use. In those cases, the author classification is used, whether or not we agreed with their classification.

Table 2: Numbers(Percentages) of papers in the 1979 and ***1993*** *conferences by research method type.*

Experimental Design	Test Development	Survey	Action Research/ Teaching Experiment/ Program evaluation
5(21) **9(9.8)**	2(8) **0(0)**	4(16.6) **19(20.6)**	3*(12.5) **7(7.6)**

Classroom Observation	Ethnography / Case study	Clinical Interview	Critical / Discourse Analysis	Non-Analytic
1(4) **2(2.2)**	0(0) **7(7.6)**	1(4) **27(29.3)**	0(0) **6(6.5)**	8(33) **15(16.3)**

* Two papers are program description with little reflection

A number of patterns are of interest from this data. First, in the 1979 conference a large number of papers, about 32% did not provide empirical data. This category of paper was reduced by half in the 1993 conference. Second, there has been a noticeable decrease in the percentage of studies using experimental designs and test development research reported at the 1993 conference and a significant increase in the studies using clinical interviews. Although it might appear that the number of action research/teaching experiment type research has decreased, a closer examination of the papers, however, has revealed a shift from program exposition in 1979 to critical studies involving sophisticated theory and procedures in 1993. Thirdly, the new categories of ethnography, case study and critical analysis have emerged since the 1979 conference.

ANALYSIS OF RESEARCH SUBJECT MATTER

The purpose of this analysis was to obtain an overall view of the subject matter concerns represented by submitted papers to each of the conferences under review. Identification and comparison of lines of research activity within and between the target years may thus be facilitated. Data was stratified by 4 variables representing educational agency (Student, Teacher, Classroom, Researcher); and by 3 content variables (Knowledge, Beliefs/Attitudes, Context) representing critical aspects in the 'life' of the agent. Allocations between content variables were made with reference to the following definitions: Knowledge, consisting of conceptual knowledge (content knowledge), procedural knowledge(specific skills, higher order processes) and accessed by the user either tacitly or explicitly; Beliefs/Attitudes, involving statements which are taken-as-true by the agent either tacitly or explicitly (Clark and Peterson, 1986; Peterson, Fennema, Carpenter, Loef, 1989); Context, thought of as a domain of influence beyond mathematical or pedagogic knowledge narrowly defined.

Utilising these stratifications, a matrix (4x3) was constructed. Each paper was then allocated to a single cell position, and this was taken to represent the paper's chief focus of research involvement within the matrix.

Data was further stratified within matrix cells. In the case of the Teacher/Knowledge cell, initial use was made of Shulman's taxonomy (1986, 1987) of the teacher knowledge base. Salient elements in this were adapted in order to construct the following 5 sub-stratifications:

- Mathematical content knowledge
- Pedagogic knowledge: Knowledge of teaching strategies, management, organisation *independent* of content/topic.
- Pedagogic content knowledge: Knowledge of teaching strategies, management, organisation *dependent* on content/topic.
- Curriculum knowledge: Knowledge of materials and programmes, including technologies, assessment instruments *etc*.
- Professional development: Development of teacher knowledge.

Of the papers reviewed, 1 paper in 1979 and 2 papers in 1993 were excluded from this analysis as they were peripherally related to research questions in mathematics education.

Subject matter cluster analysis results are set out in Table 3 below.

Table 3: Number (percentage) of papers in 1979 and 1993 with matrix showing subject matter clustering.

	KNOWLEDGE	BELIEFS/ ATTITUDES/	CONTEXT	*TOTAL*
TEACHER	9(37.5) **34(37.4)**	4(16.7) **6(6.6)**	0(0) **1(1.1)**	*13(54.2)* ***41(45.1)***
STUDENT	6(25.0) **31(34.1)**	2(8.3) **4(4.4)**	1(4.2) **6(6.6)**	*9(37.5)* ***41(45.1)***
CLASSROOM	-	-	0(0) **3(3.3)**	*0(0)* ***3(3.3)***
RESEARCHER	1(4.2) **4(4.4)**	-	-	*1(4.2)* ***4(4.4)***
TOTAL	*16(66.7)* ***69(75.8)***	*6(25.0)* ***10(11.0)***	*1(4.2)* ***10(11.0)***	

These results indicate that research focus on the educational agent variables of Teacher and Student claimed the principal attention of about 90 percent of studies in both 1979 and 1993. However, in 1979, focus on the teacher (54.2% of total) was greater than for the student (37.5%), whereas in 1993 the share between teacher and student was exactly even. Classroom and Research variables also claimed more attention in 1993. Analysis of content variables shows that focus on knowledge predominated in both conferences, although in 1993 it received increased attention compared to 1979 (66.7% *versus* 75.8%). On the other hand, there has been a marked switch in focus between the Belief/Attitude and Context variables. Interest in the former cell declined from 25% to 11%; interest in the latter rose from 4.2% to 11%. Studies focussing on the Context variable have involved investigations of gender, ethnicity, institutional frameworks and other factors within the social domain.

Tables 4a and 4b below sets out data relating to the sub-stratification of the Teacher/Knowledge and Student/Knowledge cells.

Table 4a: Number (percentage) of papers in 1979 and 1993 showing Teacher/Knowledge cell sub-stratifications.

Mathematical knowledge	Pedagogic knowledge	Pedagogic content knowledge	Curriculum knowledge	Professional development
0(0) **1(1.1)**	2(8.3) **9(9.9)**	3(12.5) **9(9.9)**	4(16.7) **8(8.8)**	0(0) **7(7.7)**

Table 4b Number (percentage) of papers in 1979 and ***1993*** *showing Student/Knowledge cell sub-stratifications.*

Mathematical concepts	Mathematical processes	Language
3(12.5) **18(19.8)**	3(12.5) **11(12.1)**	1(4.2) **2(2.2)**

The most notable feature in the Teacher/Knowledge cell (Table 4a) is the rise of interest in a focus on professional development. In the Student/Knowledge cell (Table 4b) an increased focus on student conceptual domains is observed (25% *versus* 34.1%). In contrast to 1979, attention is more diversely spread across the content domains of arithmetic, algebra, calculus, geometry, and statistics. Studies in problem solving in 1993 show a highly focussed utilisation of the relevant theories of cognition compared with those in 1979.

DISCUSSION AND CONCLUSIONS

In considering the results of this paper a number of important qualifications need to be kept in mind. Firstly, this has not been a longitudinal study of mathematics education research. Data, covering two years only, are considered insufficient to indicate research trends. Our more modest aim has been to closely study two temporally separated 'snapshots' of research in order to draw out perceived differences and similarities. A second qualification involves the concern that our methodology may have distorted or in some cases obscured notable themes within the bodies of research studied. For instance, the subject matter matrix used to obtain a view of principal research foci does not adequately deal with instances where papers have dual concerns, or where no single concern is easily determined. We are conscious that emergent themes may have suffered the most: for example, tertiary studies in mathematics education has emerged as a theme in 1993, although this has not been noted in our analysis above. Similarly, there is evidence for the emergence of concern for themes of difference such as gender, ethnicity, social class *etc*. In our third area of concern, we note that our categories of analysis have largely been drawn from the research literature, rather than the data themselves. Arising from this, the stability of the taxonomies in relation to the bodies of research studied in this paper, may need to be more thoroughly considered. Notwithstanding these qualifications, however, we believe significant conclusions concerning outlines of research and their comparison in 1979 and 1993 are possible, and we briefly summarise these below.

First, concerning research methodologies, we note that although the positivist paradigm remains predominant in both conferences, the alternative interpretivist and critical paradigms have made a substantial contribution to the 1993 programme. A second important observation indicates an increased reliance on qualitative over quantitative research methods between the years studied; we caution, however, that this observation, by itself, does not imply a switch away from the positivist paradigm. We note that although there has been a marked reduction in the use of experimental design, surveys and clinical interviews remain the most widely used methods in research. The ratio of descriptive or non-analytical studies is also markedly less in 1993.

Turning to consider subject matter of research questions pursued by researchers, our first conclusion is that for both conferences the attention of most studies is mainly focussed on teachers and students. In 1979, however, more studies focussed on the teacher than the student, whereas conversely, in 1993, emphasis fell more heavily on the student. We note that this finding is consistent with a shift in ethos within the mathematics education community towards obtaining a more even balance between these particular agents of the educational process. Secondly, there are signs of a movement to reconceptualise research beyond the usual teacher/student dichotomy; in this regard classroom interaction has emerged as a unit for investigation in its own right. Thirdly, we found that there has been a significant switch of focus from the study of beliefs/attitudes, toward a study of contexts and with this, as noted above, a study of relations of difference. We consider this a significant finding, and note it is consistent both with the known strengths of the interpretivist and critical paradigms and an apparent shift toward the increased utilisation of these paradigms in mathematics education research, as has been noted above. Another important conclusion is the strong concern for professional development in the 1993 conference, and this is also consistent with shifting emphases in policy and needs recently identified by the teaching profession at large. We note in closing that there has also been a marked increase in the range and level of theoretical involvements in the studies reported in the 1993 conference. In this connection, it would be informative to any future investigation into research trends to analyse the depth and variety of theoretical concerns accessed by researchers in their studies.

REFERENCES

Atweh, B, Booker, G. Irons, C, and G. Jones (1979). *Research in Mathematics Education in Australia, 1979, Volume 1.* Brisbane: Brisbane College of Advanced Education

Booker, G (1979). *Research in Mathematics Education in Australia, 1979, Volume 2.* Brisbane: Brisbane College of Advanced Education

Candy, P. (1989). Alternative paradigms in educational research. *Australian Educational Researcher, 16*(3), 1-11

Carr, W. and S. Kemmis (1986). Becoming Critical: Education , Knowledge and Action Research. Geelong: Deakin University Press.

Clark, C. and P. Peterson (1986). Teachers' thought processes. In M. C. Wittrock (Ed.), *Handbook of Research on Teaching* (3rd ed., pp255-296). NY: Macmillan

Guba, E. and Y. Lincoln (1988). Do inquiry paradigms imply inquiry methodologies? In D. Fetterman (Ed.), *Qualitative Approaches to Evaluation in Education: the Silent Scientific Revolution* (pp89-115). NY:Praeger

Linn, R. (1986). Quantitative methods in research on teaching. In M. C. Wittrock (Ed.), *Handbook of Research on Teaching* (3rd ed., pp92-118). NY: Macmillan.

Peterson, P., Fennema, E., Carpenter, T., and M. Loef (1989). Teachers' pedagogical content beliefs in mathematics. *Cognition and instruction, 6*(1), 1-40

Shulman, L (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher, 15*(2), 4-14

Shulman, L. (1987). Knowledge and teaching: Foundations of the new reform. *Harvard Educational Review, 51*, 1-22

Watson, J. and B. Atweh (1992). An Overview of Australasian Mathematics Education Research. In Atweh, B. and J. Watson (Eds.) *Research in Mathematics Education in Australasia: 1988-1991.ß*

"CONSTRUCTIVISM" IN PRE-SERVICE TEACHER EDUCATION: PANACEA OR PANDORA'S BOX?

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This paper reports an action research project that spans more than two years and chronicles the transition from an unquestioning acceptance of constructivism as the "one best solution" for the problems of student-teacher empowerment in mathematics education to a questioning of its intuitively convincing canons. One of the many outcomes of the study centres around my changing notions of "empowerment" of students to construct their own knowledge, to an interrogation of constructivism itself for its evasion of consideration of the historical and social situatedness of the nature of knowing and learner subjectivity. Implicated as I am as the teacher in the project, I offer one interpretation of the data which to me renders constructivism vulnerable to censure for a perceived lack of recognition of the effects of culture and power inherent in all discursive practices. At the completion of two cycles and upon entering the third it appears that students, after participation in "constructivist" practices, are getting the same epistemological messages concerning mathematics and mathematics teaching as previously established during the reconnaissance phase.

Constructivists generally would agree that learning means to create progressively more powerful conceptual structures. Representative of such educators and researchers is Paul Ernest (1992, p.99) who asserts that "the aims of teaching mathematics need to include the empowerment of learners to create their own mathematical knowledge". A second aim of teaching mathematics Ernest (1992, p.99) believes "is to facilitate confident problem posing and solving; the active construction of understanding built on learners' own knowledge; and the exploration and autonomous pursuit of the learners' own interests". In this paper I explore further, through reference to an action research project in initial teacher education, the problematic notion of empowerment itself. As well, the epistemological foundations of constructivism are ultimately interrogated for non-theorisation of the subject and the unquestioned acceptance of a technico-rational conception of learning for behaviourial outcomes such as problem-solving, and "autonomous pursuit of ...own interests" (Ernest, 1992). Finally, the vulnerabilities and possibilities of the adoption of constructivist policy in pre-service teacher education are summarised.

The initial work on constructivist teaching had its genesis in Piagetian epistemology. Piaget reportedly believed "...the structure of the mind is the source of our understanding of the world" (Venn and Walkerdine, 1977, p.73). von Glasersfeld (1991) enlarged upon the two major assertions of Piaget's developmental psychology: (a) that knowledge acquisition is a process of coming to know that can be likened to biological adaptation of an organism or species to its environment; and (b) that cognition produces conceptual structures through a process of reflective abstraction. Thought is fundamental and language is merely a tool which facilitates the development of conceptual structures in the mind. It was with this understanding of knowledge as personally constructed that I undertook the first cycle of the project.

CONTROL CYCLE

Evidence from the first of the three cycles of the action research project demonstrated an unqualified, unquestioning acceptance on my part of constructivism premised on the notions of developmental psychology as portrayed by Piaget and elaborated by von Glasersfeld. Although these theorists had not posited pedagogical practice to support their views on cognitive development, my interpretation was that if students were to be "empowered" to construct mathematical knowledge my role was to engineer an environment for learning in which this might happen. I saw constructivism as having pedagogical applications for theoretically informed practice. It was depicted in the data from this first action cycle as the opposite of "traditional" teaching, this evident from my placing each at opposite ends of a continuum in several lectures. My journal entries recorded distaste for, and the uselessness of, the lectures I was forced to conduct because of large numbers of students. This data demonstrated the dialectics I perceived between good and bad practice which constituted my understanding at the time. Lack of teaching and learning time shows up consistently in the data as an issue: to be "empowered" students needed to learn\construct many mathematical and pedagogical concepts as the syllabus covered an exhaustive list of topics. Constructivism was utilised to develop "the" concepts of maths and maths method and learning was purely a cognitive act.

Looking at the data now I see myself well in control of the students and myself kept in control by the "common-sense" nature of constructivism and institutional and other constraints. Though a collaborative learning environment was established, tapes of discussions on set readings show my steering of students towards "correct" interpretations. Although purporting to enact emancipatory action research within a critical social science interpretive framework, at this first stage I neither sought nor found any ambiguities or uncertainties evident in the social (power) relationships. The social was merely a given and there was little time for deviation from the task of building and assessing cognitive structures.

Reading my written reflections on this first cycle, I discern the apprehension I felt at personal control of most aspects of the teaching/learning situation. Students still thought of mathematics as knowledge to be transmitted by the teacher and I regretted my lack of "efficiency" in having them construct alternative conceptions of the nature of mathematical knowledge. That I was using action research in a technical-rationalist way became evident and I felt a vague dis-ease that engagement in the process wasn't proving emancipatory. It was at this point that my readings on socio-constructivism and the importance of peer collaboration (Cobb, 1988, 1991; Lerman, 1992) came together with Habermas' (1971) concept of communicative action within critical social science. This stress on communication in learning translated into a pedagogical agenda which once again left constructivism itself unproblematised and unchallenged.

POWER CYCLE

The second action cycle was distinguished in the data by a concern for the ideological. In this cycle I hoped to foster learning through group involvement and collaboration (socio-constructivism) and to enforce a sort of critical dialogue (Habermas' critical theory) which I presumed would provide "perturbation" for the development of new and "better" conceptual structures concerning the teaching and learning of mathematics. The context for learning was signified by an emphasis on critique of the dominant "traditional" discourse of mathematics education presumed to be ideological where constructivism was represented as inherently liberating and empowering. It is worth noting that my personal understandings evidenced in the data in many lessons throughout the semester were not challenged by students. The students were to be "rational" participants in critical communication (Habermas, 1971) who through being made aware of injustices would act to change them when later themselves teaching. I thought that by changing students' consciousness (concepts about) mathematics teaching I would enable them to change the system. Concern was still centred around individual change within the system.

I looked at the video and audio tapes of cycle 2. I read and re-read student writings about the course. Nothing had changed from the first cycle except the students. These students like those the year before enjoyed the course - they evaluated it positively but the data revealed little evidence of students constructing anything other than pre-determined and pre-specified knowledge. Even a particular view of "rationality" was enforced as students didn't willingly participate in critical dialogue and I inevitably ended up telling them of perceived problems in the system and pointing out injustices. Critical theory translated into a pedagogical agenda didn't seem to further the cause of students' genuine construction and co-construction of meaning. Looking at the data, I was forced to question the alleged superiority of the constructivist classroom over the traditional didactic model given that the authoritative voice of teacher and text remained unchallenged and students continued to receive unchanged epistemological messages concerning mathematics and its relation to their world.

I was now in a state of disillusionment given the apparent impossibility of breaking through the constraints imposed upon teachers and students by virtue of inscription in institutional contexts. I turned to the social theory of various feminist and post-structural writers for enlightenment and renewed hope. These, considered alongside the data from the second action cycle, convinced me that something more than negativity would be needed to empower students and myself to change and ultimately bring about change (See Luke, 1992; Ellsworth, 1992; Delpit, 1988; Walkerdine, 1984, 1992). It was at this time too that disenchantment with the framework of Habermas' critical social science was replaced by the analytics of Foucault (1980) and his notion of the "capillary" form of existence of power.

Several points were given new clarity. First, that history and the discursive practices including power inscriptions were implicated in the emancipatory space that might eventually be afforded each of us. Second, that we needed to make problematical "the" mathematics and "the" teaching methods we were seeking to understand - it was not a simply a question of which methodology should be favoured after all. Methodology implies set unquestioned means towards a fixed and unproblematic end. Constructivism is itself a discursive practice which privileges some voices and some knowledges over others. Third, it is not a question so much of "what" is said within the discourse as to who is doing the speaking and whose voice is authoritative and privileged. Nor is it a question of who holds the power but rather of how power operates within the discursive practice to legitimate some voices and silence others; to valorise some statements and marginalise others; to license some as holders of valid information and knowledge and to marginalise others. My practice included aspects of all of these in the name of constructivism.

Various aspects of Foucault's conception of power are especially helpful in enabling a reconceptualisation of mathematics which might prove emancipatory for students and teacher. Firstly, power is not essentially repressive and can be productive. Teachers in institutions will always hold some advantage over students, they can use this in productive ways towards student emancipation. It is not a matter of the authoritative use of power - nor the liberal denial of our power - but an explicit acknowledgment of our power and making it problematic. So teachers can make power relations obvious to students, (Delpit, 1988) and act to open up spaces of freedom where possible. Secondly, power in capillary form forces us to take account of local contexts for possibilities of change and to forgo the pre-occupation with power of the state and economic relations. Whereas Foucault's genealogy of modern power asserts that power touches individuals more through practices than through beliefs, we need to change our practices in teaching mathematics - the use of space, text and examinations to name but a few. This does not mean to raise the question of superiority of one method over another, for no method within a discursive practice can be always and inherently liberatory. It means to suspend the question of epistemic justification, the right or wrong, the true or false, and to take note of how in the local context practices in mathematics serve to facilitate learning in some students but not others.

PROBLEMATISING CYCLE

In the third action cycle now in progress I encourage students to speak as the socially constructed subjects they are through involvement in a lifetime of cultural and social discursive practices and relationships. They speak about the mathematics we are doing and the projects on which they are working. Discussion is concerned more with the immediate context and student autobiographics of mathematics learning inform the possibility of a different (not necessarily "better") mathematics. I give lectures on topics covering the teaching of mathematics, on the various syllabus and policy documents, history, theoretical foundations, possible marginalisation of various cultural groups, women and girls and the poor. I stop every ten minutes or so for the students to discuss points just made and to list points about which they need to know more.

In tutorials students work together on projects covering various mathematical topics. I give them the concept area to be covered and some basic ideas, they extend these and include applications. As this is a "maths method" subject they develop ideas on the teaching of various topics, discuss and list possible problems. For assessment, students present to me the projects they have completed and discuss whether they think they have developed a useful non-discriminatory mathematics.

In a way I regret that for me the Pandora's box on constructivism was ever opened. There is something reassuring in being able to tell students about the "best" way of teaching mathematics, denied educators in the post-modern era. Whereas in the reconnaissance phase of the study student questions could be answered in a word or short sentence, replies now are more often "Let's talk about it". The word constructivism will not be so often in evidence in the data collected on cycle three as students and teacher together develop a mathematics where the discourse of constructivism, as all discourses of mathematics education, are problematised.

CONCLUSION

This research was carried out to adjudge the problems and possibilities of the acceptance of constructivist teaching practice as previously summarised by Ernest (1992) into the tertiary context. Recent policy documents such as *The Discipline Review of Teacher Education in Mathematics and Science* (1989) and the *National Statement on Mathematics for Australian Schools* (1990) support pedagogical practice emphasising personal knowledge construction through problem solving, working collaboratively and communicating mathematically.

My research challenges the technical rationality typical of policy makers who tend towards the "one best solution" to address problems perceived in contemporary practice. I have intuited through this action research project that knowledge ostensibly "constructed" is about as damaging as knowledge "transmitted" when such learning attends the disenfranchisement of many students. Learners will construct knowledge no matter what the particularities of the environment, but while teacher educators continue to set the text and the exam knowledge constructed includes the fact that personally constructed knowledge is worthless if it does not conform with that in the exam, the text and the teacher's head. Spivak (1990, pp.18,19) cautions against the ready acceptance of policy statements and specifies their vulnerability in that "...when a narrative is constructed, something is left out. When an end is defined, other ends are rejected, and one might not know what those ends are".

REFERENCES

Australian Education Council (1990). *A national statement on mathematics forAustralian schools.* Carleton, Victoria: Curriculum Corporation.

Cobb, P. (1988). The tension between theories of learning and instruction in mathematics education. *Educational Psychologist, 23*(2), 87-103.

Cobb, P. (1991). Reconstructing elementary school mathematics. *Learning Problems in Mathematics, 13*(2), 3-32.

Delpit, L. (1988). The silenced dialogue: Power and pedagogy in educating other people's children. *Harvard Educational Review, 58*, 280-298.

Ellsworth, E. (1992). Why doesn't this feel empowering? Working through the repressive myths of critical pedagogy. In C. Luke & J. Gore (Eds.) *Feminisms and Critical Pedagogy.* (pp.90-119). New York and London: Routledge.

Ernest, P. (1992). The nature of mathematics: Towards a social constructivist account. *Science and Education, 1*, 89-100.

Foucault, M. (1980). Prison talk. In C. Gordon, (Ed.), *Michael Foucault: Power\Knowledge.* (pp.37-54). New York: Harvester Wheatsheaf.

Gee, J.P. (1992). *The social mind: Language, ideology and social practice.* New York: Bergin & Garvey.

Habermas, J. (1971). *Knowledge and human interests.* Boston: Beacon Press.

Kemmis, S. (1988). *The action research planner* (3rd ed.). Deakin: Deakin University Press.

Lerman, S. (1992). The position of the individual in radical constructivism: In search of the subject. In J.A. Malone & P. Taylor, (Eds.) *Constructivist Interpretations of Teaching and Learning Mathematics.* (pp.105-112). Perth: National Key Centre for School Science and Mathematics.

Luke, C. (1992). Feminist politics in radical pedagogy. In C. Luke & J. Gore, (Eds.), *Feminisms and Critical Pedagogy.* (pp.25-53). New York: Routledge.

O'Loughlin, M. (1992). Rethinking science education: Beyond Piagetian constructivism towards a sociocultural model of teaching and learning. *Journal of Research in Science Teaching. 29*(8), 791-820.

Speedy, G.W. (1989). (Chairperson) *The discipline review of teacher education in mathematics and science. Vol. 1*. Canberra: AGPS.

Spivak, G. (1990). *The post-colonial critic: interviews, strategies, dialogues.* New York: Routledge.

Venn, C. & Walkerdine, V. (1977). The acquisition and production of knowledge. *Ideology and Consciousness. 3*, 67-94.

von Glasersfeld, E. (1991). Knowing without metaphysics: Aspects of the radical constructivist position. In F. Steier, (Ed.), *Research and Reflexivity,* (pp.12-29). London: Sage.

Walkerdine, V. (1992) Progressive pedagogy and political struggle. In C. Luke & J. Gore, (Eds.), *Feminisms and Critical Pedagogy*. (pp.15-24). New York: Routledge.

SOME PROBLEMS IDENTIFIED WITH MAYBERRY TEST ITEMS IN ASSESSING STUDENTS' VAN HIELE LEVELS.

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In the early 80s Mayberry (1981) developed a diagnostic instrument to be used to assess the van Hiele levels of pre-service primary teachers. The test which was carried out in an interview situation, was designed to examine seven geometric concepts. There has been no reported attempt to (a) replicate this work in Australia; (b) consider the items in some alternative format; or (c) analyse the validity of the test questions.

To address these issues, a detailed testing and interview program of 60 first year primary-teacher trainees was undertaken at the University of New England. This paper considers one aspect of the findings of this study. It concerns the potential for certain aspects of Mayberry's work to lead to an incorrect assessment of a student's level of understanding in geometry. In particular, four main features were found to account for major problems to the test validity.

The ability to be able to instruct students at their level of understanding is dependent, in part, on the teacher being able to assess students' levels of understanding. In order to make this assessment, there needs to be available a reliable diagnostic instrument. In the early 80s Mayberry (1981) in her work with pre-service primary teachers, developed such an instrument that could be used in an interview situation. While her work has been used as a basis for other research projects (e.g., Denis, 1987), there appears to have been no critical evaluation of the questions used. Before addressing this issue it is appropriate to provide a brief background to the important ideas underpinning her work.

BACKGROUND

The van Hiele Theory

In the 1950s, Pierre van Hiele and Dina van Hiele-Geldof completed companion PhDs which had evolved from the difficulties they had experienced as teachers of Geometry in secondary schools. Whereas Dina van Hiele-Geldof explored the teaching phases necessary in order to assist students to move from one level of understanding to the next, Pierre van Hiele's work developed the theory involving levels of insight. He identified five levels (Levels 1 to 5) and brief descriptions of these are:

Level 1 Perception is visual only. A figure is seen as a total entity and as a specific shape. Properties play no explicit part in the recognition of the shape.
Level 2 The figure is now identified by its geometric properties rather than by its overall shape. However, the properties are seen in isolation.

Level 3 The significance of the properties is seen. Properties are ordered logically and relationships between the properties are recognised.

Level 4 Logical reasoning is developed. Geometric proofs are constructed with meaning. Necessary and sufficient conditions are used with understanding.

Level 5 The logical necessity of deductive argument is accepted. Insight into the nature of logical laws is acquired.

The van Hieles saw their levels as forming a hierarchy of growth. A student can only achieve understanding at a level if he/she has mastered the previous level(s). Also they saw the levels as discontinuous, that is, students did not move through the levels smoothly. Instead they saw the necessity for a student to reach a "crisis of thinking" before he/she could proceed to a new level. In addition to the descriptions provided above that highlight clear differences between levels, the van Hieles believe that students at different levels speak a "different language" and have a different mental organisation associated with each level.

Mayberry's Research

JoAnne Mayberry's study (1981) investigated, in part, whether the van Hiele level at which a student is functioning in geometry, can be discerned. To carry out this investigation, Mayberry created a diagnostic instrument consisting of 62 items designed to the operational definition of each of the five levels. Many of the items contained separate question parts. The items covered seven geometric concepts, namely, squares, right triangles, isosceles triangles, circles, parallel lines, congruency, and similarity. These concepts all occur in the elementary curriculum in the USA. A matrix/grid was used to develop questions by level and concept so that the questions would have parallel forms. One or more questions were developed for each cell in the grid.

Experts in the fields of mathematics and mathematics education, among them Pierre van Hiele, were asked to validate the items by judging whether the items satisfied the following criteria (Mayberry, 1981, p. 52):

1. Is the question suitable?
2. Does the question appear to test the given van Hiele level?
3. Does any aspect of the question seem to test a higher level?
4. What comments or suggestions can you give to help with evaluation, clarity, reformulation?

The final form of the diagnostic instrument was then used in an interview situation to investigate the undertakings of geometric 19 preserve elementary education students at Georgia College, Milledgeville, Georgia. At the commencement of the interview, the students were told that: the reasons behind an answer were of more interest than the particular answer. During the interview, the researcher could ask further questions depending on the responses. The interviews lasted approximately two hours.

DESIGN

In order to consider Mayberry's work in an Australian context, a detailed study of the geometric understanding of 60 first year primary-teacher trainees was carried out at the University of New England. The study aimed, in part, to provide a written test based on the Mayberry interview schedule Follow-up interviews were conducted with students to validate the levels of thinking as determined in the written test. Conversion of the Mayberry items to a written test involved the following modifications:

1. Where necessary, the wording of the Mayberry items was amended to ensure that the intention of each question was clear.

2. Level 5 items were omitted, hence the written test assessed van Hiele Levels 1 to 4 (Mayberry items 1 to 57).

3. So as to reduce the time taken by the test to an acceptable length, the concepts were divided into two groups, each group consisting of four concepts (the concept squares was included in both tests). Each of the two written tests took approximately 50 minutes.

IDENTIFIED PROBLEMS

When collating results in the assessment of van Hiele levels in the Australian study, inconsistencies in some of the students' reasoning emerged. Interviews did not appear to clarify these inconsistencies. On analysis of the results by concept and by level, it was considered that certain aspects of the Mayberry items had the potential to lead to incorrect assessment of a student's level of understanding in geometry. In particular, four main features were found to account for major problems to the test validity. They were:

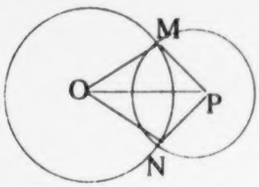
1. incorrect assignation of a level to certain items;
2. unequal treatment of concepts across levels;
3. uneven distribution of questions across levels; and,
4. unbalanced distribution of question focus within levels.

RESULTS AND DISCUSSION

FEATURE 1 (Incorrect assignation of a level to certain items).

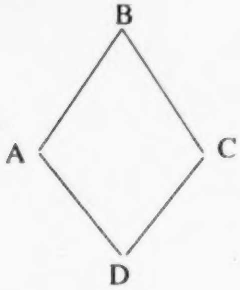
Some items did not appear to be measuring the level for which they had been designed. This was identified when large differences were exhibited by students on questions supposedly at the same level. It is possible that some teaching effect or rote learning may have influenced these results but this was not confirmed by interview. An example of this phenomenon can be demonstrated by examining and comparing two of the eleven items which Mayberry designed to test Level 4.

Item 56.



These circles with centres O and P intersect at M and N.
Prove: triangle OMP is congruent to triangle ONP.

Item 55



In this figure AB and CB are the same length.
AD and CD are the same length.
Will angle A and angle C be the same size?
Why or why not?

In Item 56, triangles OMP and ONP are clearly delineated. The solution solely requires identification of three equal pairs of corresponding sides to prove congruency of the triangles. By contrast, Item 55 can be solved by a number of different techniques. One solution to the problem involves the use of congruent triangles. To do this, a decision is needed concerning a suitable construction, i.e., join BD, which will produce the required pair of triangles, (triangles ABD and CBD). The proof of congruency of these triangles then becomes an instrument used within the solution of the problem.

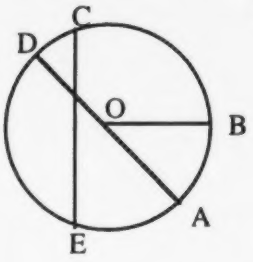
In the Australian study, of the nine students correctly answering Item 56, only three were also correct for Item 55. No student was incorrect for Item 56 yet correct for Item 55. Of these nine students, only three showed competence at Level 4.

The spontaneous recognition of the need to construct triangles before undertaking congruency requires a deeper overview of the power of congruency. This problem begs the question: Is the ability to give a proof of congruency working at Level 4, or only at Level 3? Van Hiele summarises from his dissertation that a student will have reached Level 3 thinking "if, on the strength of general congruence theorems, he (she) is able to deduce the equality of angles or linear segments of specific figures" (1957, in Fuys, Geddes, and Tischler, eds., 1982, p. 239). The very real difference between using the idea when it is apparent and recognising the need to use the idea is highlighted by the comparison of performances for these two Mayberry items.

FEATURE 2 (Unequal treatment of concepts across levels).

The seven geometric concepts used in Mayberry's work do not appear to be treated in an equal manner. Investigation of the results across all the concepts reveals that either the students in both USA and Australia had achieved a much greater understanding of the concept circles, or else the items designed for that concept were not true to level descriptions. In Mayberry's research, of the 33 occasions when a question of a Level 4 standard was asked, 9 (27 per cent) were for the concept circles. The results for Level 3 were of a similar standard. This can be illustrated by examining two of Mayberry's items, Item 35 and Item 52. According to Mayberry, a student answering Item 35 needs to be working at Level 3 in order to answer each of the four parts of the question correctly:

Item 35.



This figure is a circle with centre O.

Would the following be:

(a) certain
(b) possible
(c) impossible

Give reasons for your answers.

1) OB =OA ________________________________
2) OD =OA ________________________________
3) 2OB=AD ________________________________
4) AD =EC ________________________________

It could be argued that the correct answering of the first three parts of the question solely requires Level 2 knowledge of the properties of radii, that all radii are of equal length, and that the diameter is equal in length to two radii. In comparison, the fourth part of this item requires the understanding that a chord passing through the centre of a circle is the longest possible chord of a circle, i.e., Level 3 understanding of the relational properties of the diameter.

Mayberry lists Item 52 as requiring Level 4 reasoning for a student to provide the correct solution:

Item 52.

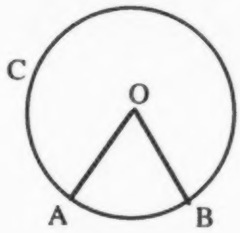


Figure C is a circle
O is the centre

Prove that Δ AOB is isosceles.

Proof of this item requires the identification of equal radii, OA and OB, as equal sides of triangle AOB. It is considered this solution only incorporates the use of the relating of properties, i.e., Level 3, and that a student is unable to display formal skills with this item.

This focus on one concept at the expense of others raises clear questions about the allocation of levels within this concept. Compounding this problem is the difficulty of considering the growth of student understanding about aspects of a circle. Properties and the relationships of properties of circles at Levels 2 and 3 are not as clear as in the case of quadrilaterals. Further, there is no evidence in van Hiele's writings to guide the development.

The difficulties became obvious when students in the Australian sample were able to score much higher on circle questions than on other concepts. This could not be rationalised in terms of greater experience or familiarity with circles.

FEATURE 3 (Uneven distribution of questions across levels).
The test items are not evenly distributed throughout the cells of the matrix/grid. This results in an imbalance between levels within a concept, and has the potential to lead to response-pattern errors. This can best be illustrated through the comparison of criteria requirements for Levels 2 and 3. In her design, Mayberry has allocated between three and seven items per concept to test for Level 3, however, she has allocated only one or two items to test for Level 2. Five concepts, right triangles, isosceles triangles, parallel lines, similarity and congruency, are tested by a single item at Level 2. For example, the most obvious case concerns the concept isosceles triangles. Whereas seven separate items (Items 28 - 32, 42 and 49) test at Level 3, only a single item (Item 18) determines whether or not a student displays mastery at Level 2. Thus the criteria for Level 2 in isosceles triangles is a perfect score, one out of one:

Item 18.
What can you tell me about the sides of an isosceles triangle? ______________________________
What can you tell me about the angles of an isosceles triangle? ______________________________

Should a student have misunderstood the thrust of this single item, answering, for example, "there are three", or "the angles sum to 180 degrees", or have incorrectly answered " they are all less than 90 degrees" (an answer commonly resulting from frequent exposure to acute-angled diagrams), he/she is deemed not to have shown mastery at that level. Often such students can still correctly display mastery of Level 3. Twelve out of a total of twenty-four response pattern errors in Mayberry's results (50 per cent) occur at Level 2.

FEATURE 4 (Unbalanced distribution of question focus within levels).
In the Mayberry scoring, it would appear that a subject can be adversely affected through the lack of exposure to a particular aspect of a form of reasoning. In the testing of the concept 'squares' at Level 3, the notion that a square is also a rectangle accounts for three of the nine possible scores, (see items 9a, 25b, and 42d). Criteria for this level is a score of six out of nine, hence, lack of exposure to the above notion means that a student must score correctly for all other questions in order to register success at Level 3. Should a student not have been exposed to, for example, class inclusion, a Level 3 concept, the Mayberry scoring could assess that student as having mastery only of Level 2. Pegg (1992, p.24) in his investigation of recent research into properties of levels, summarises:

> It is not sufficient to say that a student is not at Level 3 is he/she does not believe a square is a rectangle. Class inclusion is not simply a part of a natural mathematical development. It is linked very closely to a teaching/learning process. It depends upon what has been established as properties. ....The main feature of Level 3 should not, in my view, be the acceptance of class inclusion but the willingness, ability and the perceived need to discuss the issue.

CONCLUSION

This analysis not only gives us a clearer perspective about the Mayberry test and the results, but also provides further insight into the van Hiele Theory itself. In particular it provides further empirical evidence about what it means to work at a particular level.

REFERENCES

Denis, L. P. (1987). *Relationships between stage of cognitive development and van Hiele level of geometric thought among Puerto Rican adolescents* Doctoral dissertation, Fordham University, (University microfilms, no. DA8715795).

Mayberry, J. W. (1981). *An investigation of the van Hiele levels of geometric thought in undergraduate preservice teachers.* Doctoral Dissertation, University of Georgia, (University microfilms, no. 8123078).

Pegg, J. E. (1992). *Students' Understanding of Geometry:* Theoretical Perspectives. *Proceedings of the 15th Annual MERGA Conference*, University of Western Sydney. (p. 18-35)

van Hiele, P. M. (1984). The problem of insight in connection with school children's insight into the subject-matter of geometry (Summary of doctoral dissertation, University of Utrecht, 1957). In D. Fuys, D. Geddes, and R. Tischler (Eds. and Trans.), *English translation of selected writings of Dina van Hiele-Geldof and Pierre van Hiele* (p. 3237-241). (Professor Dorothy Geddes, School of Education, Brooklyn College, C.U.N.Y. Brooklyn, NY 11210).

CHARTING A FRACTION ENVIRONMENT FOR CLASSROOM LEARNING

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The fraction program described in this paper is characterised as an environment with resources of knowledge and skill that are available and appropriate for classroom use. The main goal of the program is that learners might be enabled to use a fraction environment confidently and in appropriate ways. The program presents activities for classroom groups to develop, select and manage fraction knowledge in purposeful tasks. It aims to encourage a learning partnership among 10- to 13-year-olds and their teachers. The intention is that knowledge and skill in fractions are developed interactively and in authentic contexts.

The project is informed by research into cognition, especially as it is manifested in social and workplace activity. Three stages completed so far have involved (1) an investigation into perceived classroom learning needs for the selected age group and their teachers, (2) the charting of a fraction environment in terms of relationships between concepts and procedures, and (3) the design and publication of a book presenting tasks involving fractions and suggestions as to how the activities might be managed in classroom groups. This paper indicates how theoretical and empirical information gathered have been integrated to provide a framework and action plans for classroom implementation.

The project reported in this paper addresses concerns about students' understanding of mathematical concepts and ways of thinking, and their ability to use mathematics as a tool in solving problems. Traditional classroom mathematics has typically been presented (and practised) as a collection of facts and rules to be memorised, and exercises to be done. Alternatively, some more recent approaches have put students into the position where they are expected to solve problems without necessarily being given informed and supportive access to the necessary mathematical tools required.

Both of these approaches present mathematics and its uses in narrowly-defined terms. Both are limiting to students' mathematical opportunities and potential. Under-developed abilities in formulating and solving problems can deprive students of much of the richness of mathematical experience. Similarly, lack of knowledge in mathematics limits students' access to the range of resources available for use in problem situations and reduces their chances to construct sophisticated mathematical meanings for ideas that might be employed.

THE PROJECT: CHARTING OF A FRACTION ENVIRONMENT FOR CLASSROOM LEARNING

In this study, a physical environment provides a metaphor for a classroom program in a mathematical domain (Greeno, 1991). An environment is a centre for human activity, involving purposeful and productive use of resources that are balanced by responsibility for their maintenance and development. In a mathematical environment learners need access to meanings and uses that enable them to participate fully in these activities: students need to develop a comprehensive range of appropriate knowledge and skill.

The task of charting an environment involves defining parameters for the area under scrutiny. Key resource bases and constraints and their relationship to each other need to be identified, along with ways to encourage flexible and productive use of the environment. In mathematics, due attention must be paid to relevant concepts and the procedures commonly employed in solving related problems. Human use of the environment involves experiences that give texture to concepts through deployment in a variety of situations. Thus effective skills in using mathematics provide a key means to enabling further growth in the understanding and use of mathematics. In the program designed, fractions have been selected as an area of mathematics for classroom implementation of these practical considerations.

A FRAMEWORK FOR THE PROJECT'S ACTIVITIES

Several interrelated strands of thinking and research activity in cognition have informed the classroom approaches being tested. A study of the relevant literature suggests that a comprehensive program for classroom learning needs to address instructional aspects of knowledge development, cognition and learning, and social interaction. These aspects have provided a base plan for the charting activity.

Knowledge Development: Cognitive research shows that understanding and competence in complex, thinking skills involves a critical and close relationship between reasoning processes and knowledge that is specific to the area (Glaser, 1984; Glaser & Chi, 1989). For example, skill in working through a problem or complex task involves activity that is underpinned by a well-organised base of relevant knowledge. Students' knowledge in fractions is therefore seen as central to fraction use in formulating questions, communicating ideas, reasoning about problems, justifying solutions, and so on. In regard to mathematics instruction, the suggestion is that the development of such fraction skills should go hand-in-hand with the systematic development of knowledge in fractions (Greeno, 1991).

Cognition & Learning: The study draws on the view that learning is situated in contexts (Rogoff & Lave, 1984; Brown, Collins & Duguid, 1989; Collins, Brown & Newman, 1989; Greeno, 1991). The suggestion is that social and physical contexts provide a 'scaffolding' for situating learners' representations of knowledge. This represents a move away from the information processing view of reasoning as mental manipulation of *symbols* of objects and events. Instead, learners' perceptions and reasoning processes are considered in relation to the learning situation, the objects in the situation and to others who are involved in the activity: mathematical meaning is developed in the context of its use.

Mathematics learning in these terms involves the development of abilities in finding, selecting, interpreting and using mathematical ideas in activities and situations: classroom mathematics is seen as belonging to the world, not removed from it. Fraction learning therefore needs to be embedded in a wide range of problems, tasks and situations that are actually found in familiar activities. In this way, concepts developed are shaped and textured by a network of meanings that are grounded in their use. Furthermore, learners need to have experience in interpreting everyday problems, tasks and situations from a *mathematical* point of view. This involves deciding on the concepts that are appropriate for application, and using mathematical concepts in authentic (and increasingly complex) problem solving tasks (Collins, Brown & Newman, 1989). The intention is that mathematics is presented as a dynamic, flexible tool for human use.

Social Interaction: Studies that have focused on the development of thinking skills indicate that at even apparently elementary knowledge levels mental activities involve complex thinking processes (Scardamalia & Bereiter, 1985; Resnick, 1987). From this position Resnick presents a case for embedding instruction in thinking skills in the academic disciplines of the school curriculum, from the very earliest stages onwards. Resnick furthermore advocates that classroom instruction should seek the development of thinking skills through socially-interactive means. She carried out a study that examined a wide range of teaching programs designed to encourage and transfer 'higher order' thinking, summarised as the skills needed for complex activities requiring responsibility and judgement (Resnick, 1987). Of the programs that demonstrated some transfer of higher order skills, Resnick noted a major, common theme: the programs saw the social setting and students' social interaction as central to much of the teaching and practice. Resnick's pragmatic view is that it is therefore useful to situate skill development teaching in a deliberately planned social context. The view sits well with the Vygotskian notion that the internalisation of social activity is the basis and stimulus for growth in cognitive competence.

CHARTING OF A FRACTION ENVIRONMENT: IDENTIFYING CLASSROOM NEEDS

The first stage of the project involved an investigation into perceived classroom learning needs for the selected age group and their teachers. This comprised observation of classroom experiences involving fractions, interviews with teachers, and monitoring of students' approaches to problem solving in fractions. Relevant mathematics curricula were also reviewed. Analysis of the information collected suggested that a classroom program in fractions might appropriately focus on: (1) learners' thinking about fractions in relation to fraction knowledge as used by mathematicians (to broaden limited conceptions of fractions and to confront misconceptions); (2) fundamental concepts involved in fraction thinking (for example, the central notion of 'comparison', the distinction between a whole in terms of one object and a whole as a set of objects, and an understanding of fractions as rational number); (3) authentic and commonplace uses of fractions (to give a clear message that fractions and fraction thinking are an important part of the everyday world); (4) appropriate uses of fractions (for example, to encourage a recognition of when decimal fractions and/or calculators might rather be used).

CHARTING OF A FRACTION ENVIRONMENT: IDENTIFYING FRACTION RESOURCES AVAILABLE

In the second stage of the project, a fraction network was constructed in an effort to capture the nature, functions, and means of representation of the resources available to learners for the development of fraction thinking (see figure 1). Freudenthal's (1983) identification of fraction concepts represented in phenomena provided a starting point for the development of the network. In line with the learning aspects identified in the theoretical framework, the network was constructed to reflect the interrelated use of fraction concepts, skills and applications that is seen as critical to working effectively in the environment. The network served as a groundwork map to guide the charting activity.

CHARTING OF A FRACTION ENVIRONMENT: DESIGNING CLASSROOM ACTIVITIES

Human activity is integral to a mathematical environment. Thus the charting activity needed to reflect the close yet variable connections between fraction knowledge and learners' use of it. The outcome, a classroom book entitled *Knowing your way around the Fraction Environment* is presented as an ongoing program for fraction learning. Its contents feature central concepts and functions of fractions: these range from the *fracturing* (or partitioning) notion—the best understood and most frequently taught aspect of fractions in the classrooms observed—to *fractions as rationals*, a characteristic that was not found to be evident in learning activities or student knowledge in the classroom study. Other concepts and functions addressed include estimation, equal sharing, describing with fractions, operations on fractions, equivalent fractions, ordering of fractions and fractions as decimals.

Within the content framework presented, the book provides a folio of exemplars for the planning and implementation of classroom developmental and assessment activities. The activities are designed for a social setting where fraction knowledge is talked about, interpreted in different ways, applied flexibly and in different contexts, and used to formulate and solve problems. Specifically, knowledge and skill development are addressed through task-based approaches that seek to engage learners' thinking and interest (eg, figures 2 & 3).

The program throughout includes notes to teachers regarding classroom organisation and learning approaches. The suggestions made are derived from the theoretical framework used. It is believed that teachers as well as students will benefit from a learning program involving fractions. In an initial testing of the program, teachers' judgements of student performance in fractions are being monitored and documented.

REFERENCES

Brown, J.S., Collins, A. & Duguid, P. (1989). Situated cognition and the culture of learning. *Educational Researcher*, 18(1), 32-41.

Collins, A., Brown, J.S. & Newman (1989). Teaching the craft of reading, writing and mathematics. In L.B. Resnick (Ed). *Knowing and learning: Issues for a cognitive science of instruction. Hillsdale, New Jersey: Erlbaum.*

Freudenthal, H. (1983) *Didactical Phenomenology of mathematical structures.* Dordrecht: D. Reidel.

Glaser, R. (1984). Education and Thinking: The role of knowledge. *American Psychologist*, 39, 93-104.

Glaser, R. & Chi, M.T.H. (1988). Overview. In Chi, M.T.H., Glaser, R.& Farr, M.J., *The Nature of Expertise.* Hillsdale, New Jersey: Erlbaum.

Greeno, J.G. (1991) Number sense as situated knowing in a conceptual domain. *Journal for Research in Mathematics Education,* 22:3, 170-218.
Resnick, L.B. (1987). *Education and learning to think.* Washington, DC: National Acadamy Press.
Rogoff, B. & Lave, J. (Eds.) (1984). *Everyday cognition: Its development in the social context.* Cambridge, MA & London: Harvard University Press.
Scardamalia M. & Bereiter C. (1985). Fostering the development of self-regulation in children's knowledge processing. In S.F. Chipman, J.W. Segal & R. Glaser (Eds) *Thinking and learning skills: Vol. 2. Research and open questions* (pp563-578). Hillsdale, New Jersey: Erlbaum.
Vygotsky, L.S. (197). *Mind in Society.* Cambridge, MA: Harvard University Press.

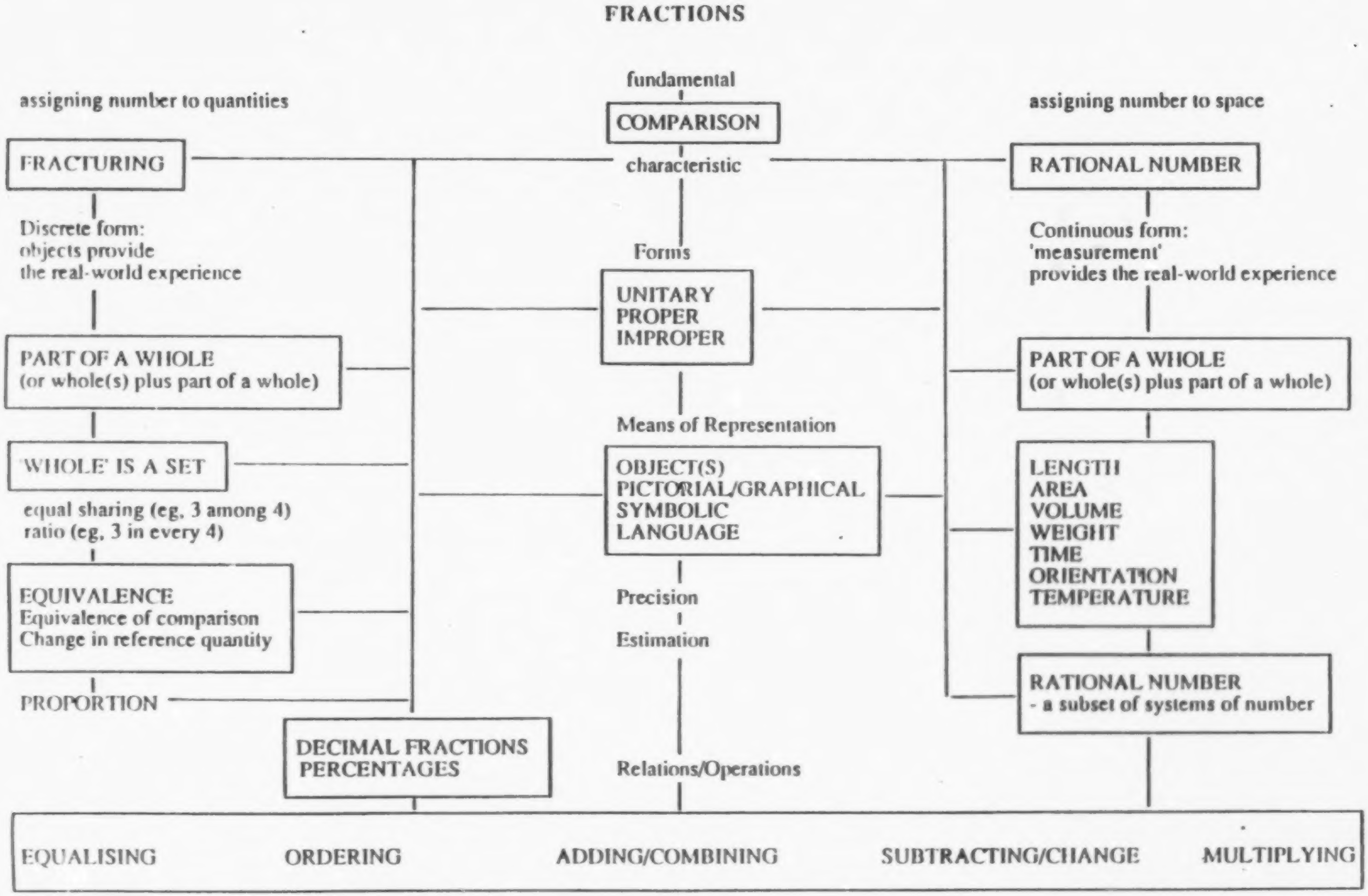
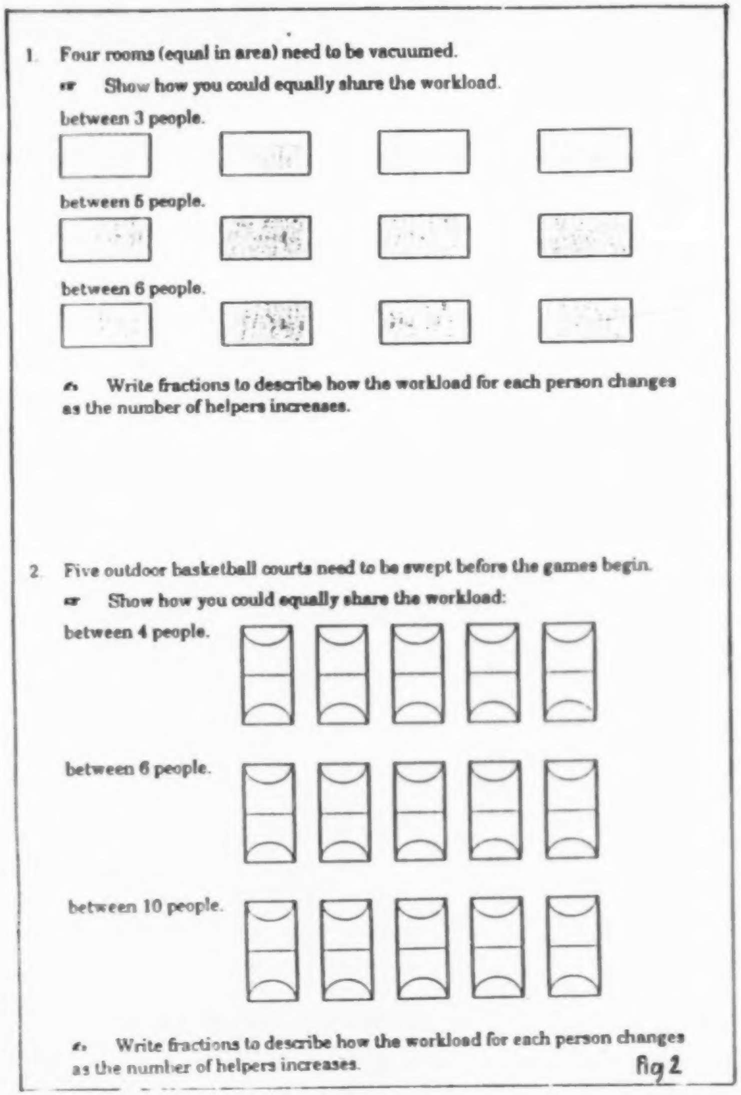

FRACTIONS
fundamental
COMPARISON
characteristic
assigning number to quantities
assigning number to space
FRACTURING
Discrete form:
objects provide
the real-world experience
PART OF A WHOLE
(or whole(s) plus part of a whole)
'WHOLE' IS A SET
equal sharing (eg. 3 among 4)
ratio (eg. 3 in every 4)
EQUIVALENCE
Equivalence of comparison
Change in reference quantity
PROPORTION
DECIMAL FRACTIONS
PERCENTAGES
Forms
UNITARY
PROPER
IMPROPER
Means of Representation
OBJECT(S)
PICTORIAL/GRAPHICAL
SYMBOLIC
LANGUAGE
Precision
Estimation
Relations/Operations
RATIONAL NUMBER
Continuous form:
'measurement'
provides the real-world experience
PART OF A WHOLE
(or whole(s) plus part of a whole)
LENGTH
AREA
VOLUME
WEIGHT
TIME
ORIENTATION
TEMPERATURE
RATIONAL NUMBER
- a subset of systems of number
EQUALISING
ORDERING
ADDING/COMBINING
SUBTRACTING/CHANGE
MULTIPLYING

Fig 1



1. Four rooms (equal in area) need to be vacuumed.

☞ Show how you could equally share the workload.

between 3 people.

between 5 people.

between 6 people.

✍ Write fractions to describe how the workload for each person changes as the number of helpers increases.

2. Five outdoor basketball courts need to be swept before the games begin.

☞ Show how you could equally share the workload:

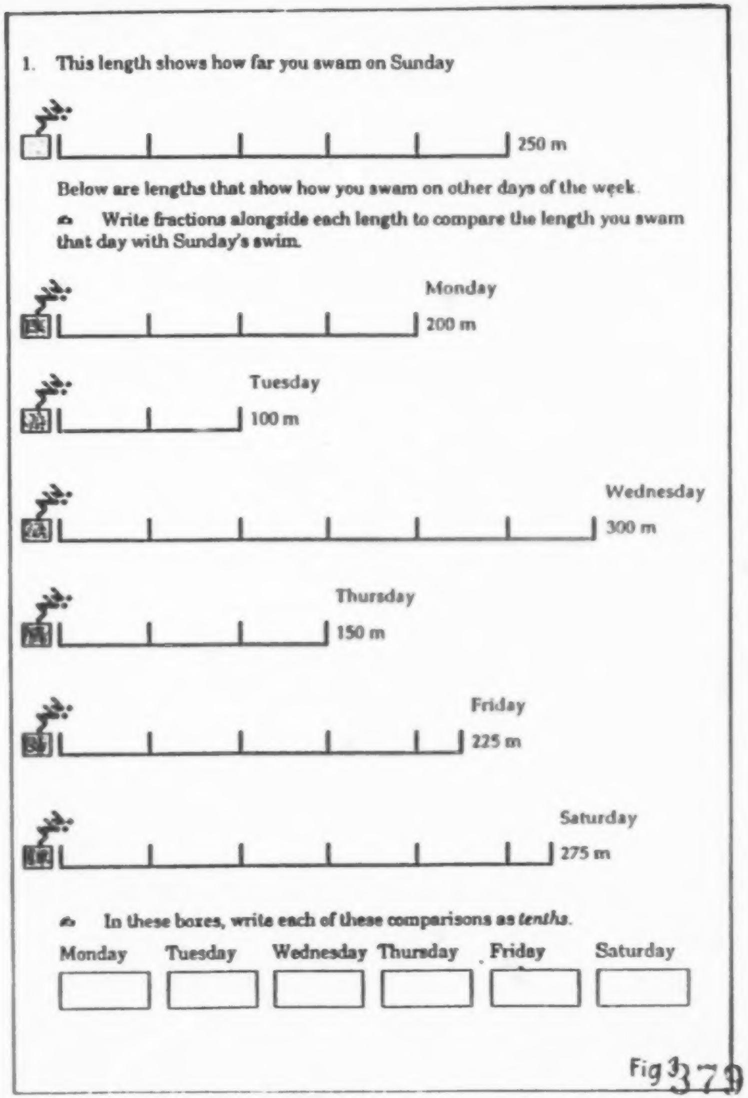
between 4 people.

between 6 people.

between 10 people.

✍ Write fractions to describe how the workload for each person changes as the number of helpers increases.

Fig 2



1. This length shows how far you swam on Sunday

250 m

Below are lengths that show how you swam on other days of the week.

✍ Write fractions alongside each length to compare the length you swam that day with Sunday's swim.

Monday
200 m
Tuesday
100 m
Wednesday
300 m
Thursday
150 m
Friday
225 m
Saturday
275 m

✍ In these boxes, write each of these comparisons as *tenths*.

Monday	Tuesday	Wednesday	Thursday	Friday	Saturday

Fig 3

AN EVALUATION OF THE EFFECTS OF THE VICTORIAN CERTIFICATE OF EDUCATION ON MATHEMATICS TEACHERS[1]

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Ballarat University College

Secondary mathematics teachers have historically taught centrally based, skills oriented courses at Year 11 and 12 levels. Introduction of the Victorian Certificate of Education (VCE) mathematics study design represented a significant shift in personal philosophies and practices for many mathematics teachers. This report focuses upon the effects that VCE implementation is having on mathematics teachers in the Ballarat area.

Individual teachers, and schools, were coping with implementation in different ways, and at differing rates, supporting the Northfield findings. (1992) The major concerns of mathematics teachers centred around the needs of more time, more consistent and specific advice from the Victorian Curriculum and Assessment Board (VCAB), and better in-service for country teachers.

Many have readily accepted the underlying philosophies of the VCE but are unhappy with the way in which it is being implemented. There is evidence of a growing contempt and cynicism of VCAB brought about by a perceived failure to acknowledge the importance of the feelings and attitudes of mathematics teachers. The resulting loss of faith, if unchecked, may well see little effective change in mathematics education at these levels in the long term.

THE EMPIRICAL STUDY

The impact upon teachers of mathematics of the introduction of the VCE into Victorian schools is expected to be great. This is because both the content and the teachers themselves are the product of a precisely structured, skills-oriented form of thinking. The VCE mathematics program features a commonality of elements in mathematics courses with attention focused upon problem-solving, modelling and project work, with assessment procedures involving both internal and external components. Many VCE mathematics teachers have been required to change direction drastically with regard to personal philosophies and practices.

This study, involving teachers from schools in the Ballarat region focused upon VCE mathematics teachers as they came to grips with the challenge of implementing major curriculum change. Issues of qualifications, previous experience, confidence, competence and support, were considered in an attempt to build up a characterisation befitting a typical VCE mathematics teacher.

The information was collected from two sources: (a) replies to a survey form; (b) personal interviews. The survey was completed at the end of the first term, and at the beginning of the second term 1992. The personal interviews were conducted at the end of the third term, 1992.

Seventy survey forms were sent to 13 schools, ten in Ballarat, and three schools within a 90 kilometre radius of Ballarat. Responses were received from 33 teachers (i.e. 47%) representing nine of the 13 schools. Six of the schools that were represented were State Government schools (21 respondents), while three were Private schools (12 respondents).

Each interview was approximately 40 minutes in length and was recorded on audio cassette. The five teachers interviewed were each involved in teaching VCE mathematics at Years 11 and/or 12. Interview questions revolved around three main issues: (a) personal feelings about the VCE mathematics program; (b) personal effects of VCE implementation; and (c) support services. Interviewees were chosen because their positive activity throughout implementation was considered likely to produce informed comment.

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[1] This paper is a summary of a study entitled *The VCE mathematics experiment: An evaluation* (Martin, P. J., 1993).

SUMMARY OF SURVEY RESULTS

Staff involved in teaching VCE mathematics programs typically had ten years of experience teaching at senior secondary levels. There were significant differences between teachers from private schools and state schools, in that private school teachers were more experienced at both Year 11 and Year 12 levels. At the Year 12 level private school teachers often had two to three times the experience of their state school colleagues.

The issue of teachers being properly qualified to teach senior level mathematics remains a cause for concern, as it was in the sixties. Thirty percent of the sample were considered to be formally unqualified to teach mathematics at this level, assuming that the major emphasis of their degree was not mathematics. More than half indicated that they were interested in studying more mathematics. There was a tendency towards a higher proportion of state school teachers being interested in further study than those in private schools, and the majority of those who expressed an interest in further study were those with less teaching experience.

The typical VCE mathematics teacher is most likely to teach Space and Number and/or Reasoning and Data. Private school teachers were far more likely to be teaching across two or three VCE mathematics units than state school teachers.

Slightly less than 25% of those surveyed were not able to express confidence in what they were doing. All taught Space and Number, and were mostly the senior experienced teachers. The difficulties of senior experienced staff adjusting to change was again evident when dealing with teachers' feelings about being prepared to teach the required units.

Generally, teachers felt that the VCE programs in their schools were successful. Practically all of the 30% who felt otherwise came from two schools only. Typically they were involved in teaching Space and Number and had had more than ten years of teaching experience at both Years 11 and 12 levels.

The major concerns of the teachers centred around the need for more time for personal adjustment, the need for consistent and specific direction in course content and assessment, and the need for better access to, and provision of, suitable in-service activities for country teachers.

While local network meetings and workshops received the highest support from the point of view of attendance, teachers rated their colleagues as the most helpful source for teaching ideas by a factor of more than two-to-one.

IMPRESSIONS FROM TEACHER INTERVIEWS

It was difficult to come away from the teacher interviews with any feelings of hope. The tiredness that Northfield (1992) found was clearly evident in most of the teachers interviewed. But there was more than just tiredness. There seemed to be a resentment towards VCAB because of its failure to provide the necessary resources and teacher-support that many mathematics teachers felt was required. The major concerns seemed to be associated with the implementation process.

Assessment: Each teacher interviewed commented upon assessment in some form or another and, if their comments are any guide, VCAB clearly needs to do a lot more work in this area. The original S/N grading was, and still is, condemned by many mathematics teachers.

T.1 The S/N system is laughable! To get an S you have to stay alive. We used to joke about it in the first couple of years, but it isn't a joke, S really stands for stayed alive.

It was often remarked that the S/N grading system provided very little incentive, particularly for the brighter students.

Critical Criteria: The guidelines introduced by VCAB to assist teachers in the grading of their students' work were not problem-free. Mathematically strong students were, in some cases, found to be receiving the same grades as students considered to be significantly weaker. Clarification from VCAB was found by some teachers to be unreliable and inconsistent.

T.2 I know of some very able mathematics students who have got the same grade as some of my very weak students and they don't deserve that. One of the problems is the way the grades are allocated.

T.3 They change the rules a little bit but we don't get informed about that though. You've got to read about it somewhere, you know, just feel, just absorb, otherwise you get caught.

Subjective Assessment: Concern was expressed by most of those interviewed regarding the subjectivity of the moderation process. This was most alarming given that uniformity across the state was one of the objectives of the moderation process.

T.1 The only thing that's surprised me more than the kids' work sometimes is the mark they have ended up getting. We've had some exceptionally good pieces of work and it's rare that any one of them has ever survived to an A in the final analysis.

It is difficult to avoid the issues of subjectivity and personal bias in instances like this. The matter cannot simply be dismissed by saying that some teachers clearly do not understand the critical criteria. If in fact this is the case, and I suspect to a large degree that it is, then why haven't these teachers been provided with more effective professional development?

Time and Money: The Essence of Implementation: There is little doubt that time and money had a major impact on how the VCE was implemented. Unfortunately, there has been an insufficient supply of both, according to many teachers. The resulting pressure was particularly felt by mathematics teachers because of their past training, the traditional content-driven courses and historical teaching practices. Several mentioned that the VCE was being "done on the cheap" and that they felt they had been used. The sad thing is that teachers feel they are the ones who are ultimately responsible for their students' performances, which naturally brings considerable pressures to bear on them.

The Underlying Philosophy: All of those interviewed believed that the underlying philosophies of the VCE were sound and that the mathematics study design had attempted to address issues that needed to be addressed. All were able to express some positive thoughts on what the VCE mathematics study design was trying to do.

T.5 As far as producing thinking people then I think VCE absolutely outdoes what HSC did.

T.2 ...for the first time how and why things are taught has been really looked at. ...I find that I've thought a lot more about how I approach topics and how I teach.

T.1 ...It has enabled some very very weak students to meet with some measure of success.

While many agreed with what the VCE was trying to do, several teachers believed that some of the original objectives were being lost. There was concern that the VCE had come to be driven by the demands of the CATS. Problems of students absenting themselves from classes as they stumbled from one CAT to the next were mentioned by several teachers. Under such conditions it is difficult to promote the idea of continuity within the courses. Others questioned the sorts of skills that are really being assessed. The following comments reflect these concerns:

T.4 ...last week was Literature and Biology and Chemistry CAT week, so no one was tuned in. If they came to class at all, well and good, but they were just taking notes. They were barely paying attention...

T.1 ... They're at school, they're working their butts off, but not in your class. The other occasions when they are there they're staring at you vacantly because they've been up all night. So the efficiency of the students has gone downhill.

T.3 The trouble here is that some of the skills it's teaching them aren't the ones that were intended. The skills now are to dig out the student who's done the work, dig out the right answer at all costs, find somebody who has the right answer, get it off them and then write it up so it looks like your own work.

Some of these problems were forecast by Northfield (1992), particularly in regard to the possibility of students becoming over-whelmed by the constant demand for CATS throughout the year. Problems associated with authentication have long been an issue, particularly with the specific nature the solutions to some of the problem-solving tasks set.

A Terribly Long Hard Grind Physically and mentally those interviewed showed signs of an unhealthy tiredness. One teacher ruefully described the situation at his school as follows:

T.1 The lunch room is empty after everybody has eaten. All they have time to do is really just go there and eat, and then they're back at their desks. Most of the staff would be at their desks for all of the lunch break. Social activities have almost come to a grinding halt. Staff don't seem to have the time to organise them to get together.

The despair implicit in the above comment is most alarming. The tiredness that Northfield (1992) found was clearly evident in every teacher interviewed except, the one teacher who had recently returned to the classroom from a consultancy position.

T.4 When you get to the stage of being crying tired by the end of the week, or Sunday night comes and you haven't done everything you had to do...

T.3 It's still a terribly long demanding hard grind. I honestly don't know how many years I can survive working at this rate.

Listening to these comments I found myself wondering just whether or not all of this was worth it in real terms. There was an air of desperation surrounding these people, but when asked if they would teach it again next year given the choice they all said "yes."

Serious concerns and doubts have been raised with respect to the difficulties being faced by VCE mathematics teachers. If nothing constructive is done to provide assistance then many will lose faith completely in the VCE and resort to the "tried and trusted" ways of the past. You cannot continue to grind teachers into the ground. There are some similarities with what happened regarding the "new maths" of the sixties, not so much with the content, but rather, in the manner of implementation.

The VCE and Mathematics: The VCE Mathematics Study Design has stripped teachers of their traditional roles of "setting learning objectives, transmitting the necessary knowledge and skills, then assessing student performance" (Mousely, 1990, p. 53). This unfamiliar role has, as expected, created problems associated with the required adjustment. In order that teachers develop the required skills they need to be supported with ample opportunity for professional development. The two areas where support and guidance for mathematics teachers have been lacking include professional development and direction. Not enough time for professional development has been provided for mathematics teachers, and the lack of specific and consistent direction from VCAB has created confusion, undue stress, increasing contempt, and cynicism.

T.2 You know, we've been supportive, as supportive as we could have been. We've been to professional development, we're doing post graduate studies so that we can keep up with these things. We've come in over holidays and after school, on Friday nights and things like that to help students to complete these tasks. We've done everything to try and go with it, and yet VCAB still haven't come clean on some of the criteria on what people need to do to meet things like that, and in the end that's unacceptable. They have got to be honest about these things. Now we're crying out for some sort of advice in these sorts of things. They know that we don't know, and yet they have done absolutely nothing in response to that sort of thing. Absolutely nothing!

Surely, if we want the teachers to prepare students as best they can, then there has got to be time provided to enable them to do so without mentally and physically grinding them into the ground in the process. The policy-makers have to realise that they cannot radically alter schooling by simply getting teachers to change their habits. Such reforms are expensive, both in financial and human terms.

Breaking Down the Resistance: Initial resistance to innovative change is to be expected and is understandable. While they may not dispute the potential of the underlying philosophies of the VCE mathematics, many have not been able to make a full commitment to the VCE. For some there is anxiety about coping with the considerable time demands imposed by the VCE. For others, the problems associated with assessment are serving to create a barrier between themselves and successful implementation. For still others, there is scepticism and cynicism towards VCAB, and a feeling of having been used in an attempt to implement the changes as quickly and as cheaply as possible.

MAINTAINING THE MOMENTUM

The motivation and enthusiasm of mathematics teachers alone will determine the quality and the degree of change resulting from the introduction of the VCE. This study suggests that more guidance and support is needed to enable mathematics teachers to move through the initial tentative stages and become more confident and independent in their approaches to teaching mathematics effectively at this level. The Ministry realises that to achieve improved student understanding in mathematics there needs to be changes in the following three areas: (a) teaching practices; (b) assessment procedures; (c) curricular emphasis (Leder & Forgasz, 1991). Change in teaching practice will be short-lived if it is not supported by a corresponding change in philosophy.

Smith (1985) refers to an increasing body of writing suggesting that "enduring reforms are those which change structures, create large supportive teacher constituencies, and are easily monitored" (p. 148). The VCE has achieved the first of these points only so far as the mathematics study design is concerned. But whatever stage has been reached, there has certainly been a considerable amount of unlearning and relearning for many mathematics teachers. This adjustment has impacted upon the lives of the teachers, both in and out of school.

> T.2 We are now doing twice what we used to do. You didn't have course development to do. You didn't have the interpretation of the study design to worry about. You didn't have these new CAT 1 and CAT 2's to worry about. You didn't have interpretations of the criteria to worry about. All of those things, and changing your seven to ten courses, and yet we're just expected to do all of that alongside what people in the past have found quite demanding to do anyway.

Adequate time and opportunities for training should have been provided to enable this re-orientation to occur. The overall long term success of the program will depend upon continual provision of support and guidance for teachers until the changes have become part of the normal daily routine of teachers and schools. Many writers refer to this need for ongoing support and guidance through major curricular change (see for example Clements, 1990; Leigh-Lancaster, 1991; Northfield, 1989, 1991, and 1992).

The fact that the problems and the concerns identified in this study still exist surely indicates that not enough time, effort, or money was given to anticipating the problems which have arisen. Further, there has been insufficient attention given to the problems which VCE teachers are experiencing as a result of the changes. If teachers do not know precisely what they should be doing, if they are having difficulty interpreting the study design or assessment criteria, *then it is not their fault,* and they ought not be held ultimately responsible.

This study has found that many mathematics teachers are concerned with the manner in which implementation of the VCE has occurred. They are not happy with the assessment procedures, they object to the pressures which the VCE has had on their personal lives, and they deplore the role that VCAB has played throughout the implementation process. In this context, it needs to be remembered that in Victoria we have an ageing teaching force, where the average age is nearly 39, not an age noted for flexibility.

VCAB must now move quickly and decisively to calm the troubled waters. This study shows that mathematics teachers are losing faith, and evidence from the 1960's and 1970's reminds us that when this occurs worthwhile change can easily be reversed. It would be a pity if all the positive aspects from the VCE mathematics study design were to be abandoned because of a failure on the part of the policy-makers to acknowledge the importance of the feelings and attitudes of those in the front line of implementation.

REFERENCES

Clements, M. A. (1990). 1990 - A watershed in the history of mathematics and mathematics education. In K. Clements (Ed.), *Whither Mathematics* (Preface). Melbourne: Mathematical Association of Victoria.

Leder, G., & Forgasz, H. (1991). Learning mathematics: some student perspectives. In J. O'Reilly & S. Wettenhall (Eds.), *Mathematics: Ideas* (pp. 3-9). Melbourne: Mathematical Association of Victoria.

Leigh-Lancaster, A. (1991). A school based reflection on recent changes in the maths curriculum. In J. O'Reilly & S. Wettenhall (Eds.), *Mathematics: Ideas* (pp. 87-90). Melbourne: Mathematical Association of Victoria.

Martin, P. J. (1993). *The VCE mathematics experiment: An evaluation.* Geelong: Deakin University National Centre for Research and Development in Mathematics Education

Mousley, J. A. (1990). Chaos in the VCE. In K. Clements (Ed.), *Whither mathematics* (pp. 47-60). Melbourne: Mathematical Association of Victoria.

Northfield, J. (1989). *Responding to the challenge: An evaluation of the VCE pilot program.* Melbourne: Monash University.

Northfield, J. (1991). *Meeting the challenge: An evaluation of the VCE pilot program after two years.* Melbourne: Monash University.

Northfield, J. (1992). *Understanding the challenge: An evaluation of the VCE implementation after three years.* Melbourne: Monash University.

Smith, R. (1985). The policy of education change by changing teachers: comments on the democratic curriculum. *Australian Journal of Education*, 29(2), 141-149.

Victorian Curriculum and Assessment Board, (1989). *Mathematics study design.* Melbourne: VCAB.

WRITING IN MATHEMATICS: SOME INFLUENTIAL VARIABLES

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Abstract*

The primary purpose of this study was to investigate students' ability to communicate their understanding of mathematics terms in writing. A second goal was to examine the performance of students in rural environments as compared to urban students. Consequently locality of upbringing, gender and, to a lesser extent, culture were variables which were considered as influential on the set task. Data was collected from 2371 year 8,9,10 and 11 students from twelve urban and rural schools in Western Australia and the Northern Territory. Results indicate that students in urban environments were better able to define mathematical terms that their rural peers. Both urban and rural male and female student results were very similar with the exception of the Year 10 group where female students outperformed their male counterparts.

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* A version of this paper is to be published in Clarke, D.J.; Stephens,M. and Waywood, A. (Eds). (In press). Communicating Mathematics - Perspectives from current research and classroom practice in Australia. Australian Council for Educational Research. Melbourne.

ABSTRACTING THE ANGLE CONCEPT

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Twenty four Year 2 children were presented with realistic models of either cricket or tiling and asked firstly, to indicate which of a set of 10 abstract angle models could represent the path of the ball or the corners of the tiles and secondly, to draw the paths or corners. Responses were analysed to indicate obstacles to abstracting the angle concept. In the cricket context, children needed to conceptualise the path of an object as a straight line segment and then to link the segments to the sequence of actions. In the corners context, children needed to abstract the two sides and ignore the shape at the point; ideas of size were apparently abstracted concurrently. Similar investigations in further contexts promise to uncover other difficulties which children face in abstracting the angle concept.

The research to be described below arises from a model of conceptual development in mathematics developed by Paul White and myself (White & Mitchelmore 1992). This model is based on the assumption that children develop mathematical concepts by abstracting the common features of various situations and learning to ignore the specifics (Skemp, 1971). But concept formation is not a once and for all process; as more and more dissimilar situations are seen to contain the same common elements, the concept becomes more and more general (Mitchelmore 1992). A common path is for the same concept to develop separately in different contexts which the learner does not link together because of superficial differences; generalisation occurs when the superficial differences are seen to be less important that the deep structure.

The angle concept is a case in point. Elsewhere, I have described 14 categories of angle contexts which would seem *a priori* to be superficially different (Mitchelmore 1993). In an initial investigation into how children abstract and generalise the angle concept from these contexts, six were selected and presented to a sample of Year 2 students. This paper reports on their understanding of angle in two of these contexts: cricket and tiling.

METHOD

The cricket context was presented in a model in which a ball was rolled along a groove to a "batter" - a block of wood, faced in plastic foam, which could be rotated to reflect the ball to roll in various directions. A second block of wood representing a fielder was used to provide a target. Children were asked to adjust the batter to hit the ball to the fielder, once on each side of the field and once back to the bowler.

The tiling context employed plastic rhombuses of side 5 cm and angles 60°, 75° and 90°. After making a flooring pattern using nine 75° tiles, children were asked why neither of the other two tiles would fit the pattern, the interviewer steering the discussion towards the corners of the tiles if necessary. A deformable model of a 5 cm rhombus made from meccano was then presented and children were asked to use it to show each tile; they were also asked what the corners of "the funniest tile they could imagine" would look like.

After these different initial introductions, intended to test children's concrete understanding of each context, the interview proceeded in the same way in both contexts. Firstly, in an attempt to assess the extent of each context presented, the interviewer asked children to name examples of "anything else where something is hit like this" (demonstrating the path of the cricket ball) or "corners like these" (holding up the acute-angled corners of the three tiles). Secondly, children were shown the ten abstract models illustrated in Figure 1. The interviewer asked children which of the models could be used to show paths or corners, and to select the one they thought was the best model for doing this. Children were asked to demonstrate how each selected model showed a cover drive or a 75° corner, and how their best choice showed the other hits or tiles. Thirdly, children were asked to draw the various hits or corners.

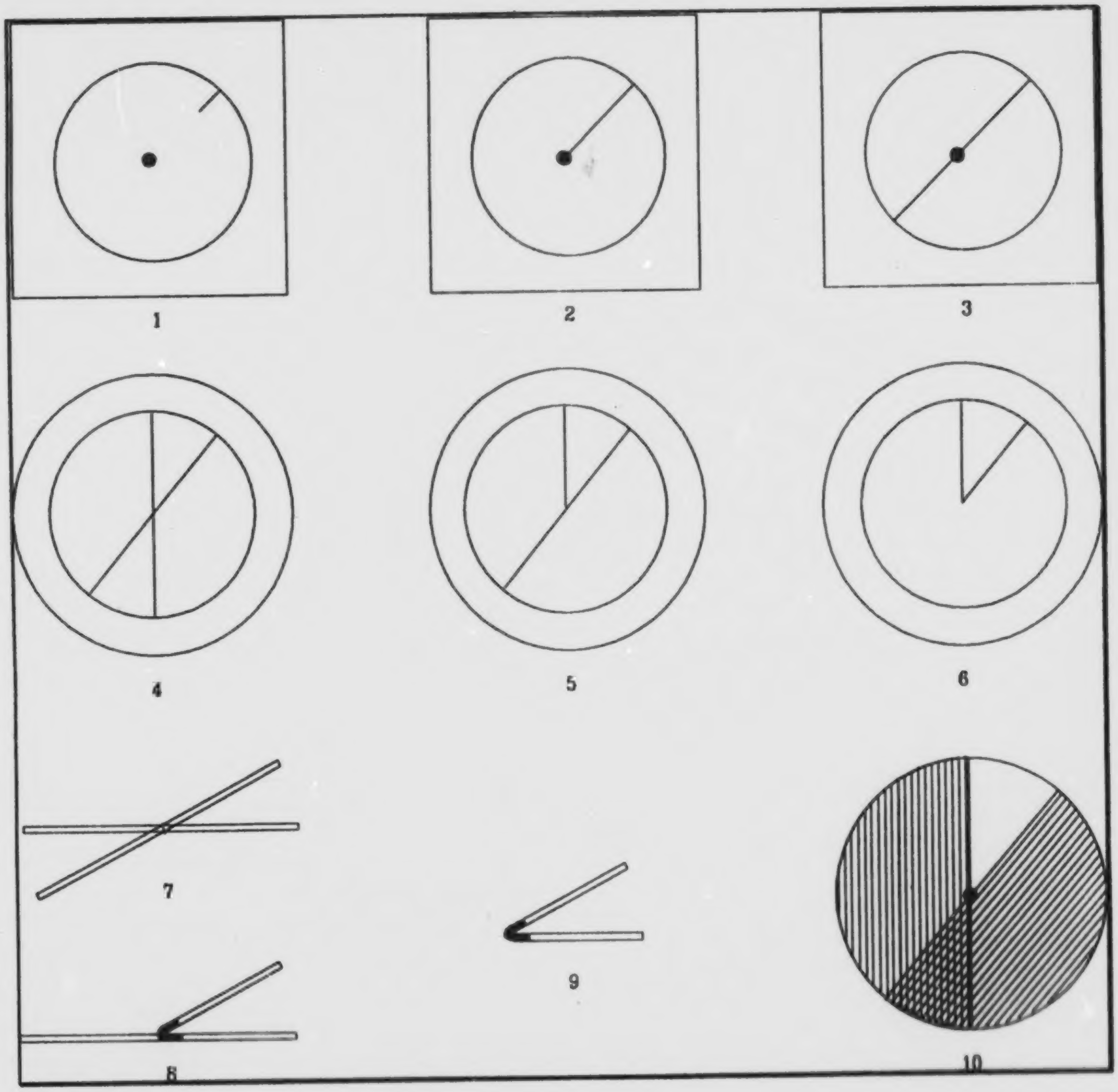

Figure 1 Abstract angle models. Models 1-3 were each made from a transparent plastic circle fastened at its centre to a cardboard square and rotatable about that point; the markings were all on the circles. Models 4-6 were each made from two concentric plastic circles slotted together so that they can rotate relative to each other; one line was marked on each circle. Models 7-9 were made of straws joined together either with thread (7) or a pipe cleaner (8,9). Model 10 was made from two plastic circles, each with one semicircle shaded black, fastened at and rotatable about their common centre.

For each context, twelve children were interviewed; their average age was 7.4 years. Students came from two Catholic schools in northern North Shore suburbs of Sydney.

RESULTS

Cricket

All 12 children were keen to try out what was certainly an unfamiliar cricket model, and they performed fairly well. Five set the batter correctly on the first or second attempt on both sides, whereas only 3 needed more than two attempts on both sides; 7 hit the ball back to the bowler at the first attempt.

All children found other examples of "things being hit", but only other ball games such as soccer and football.

Choice of abstract model suggested, on the other hand, that children has only just begun to abstract the angle implicit in the movement of the ball. Only 5 children modelled the rebound correctly with at least one model, but one of these selected only one model and a second did not select a correct model as the best. The 4 children whose best model correctly modelled the path chose models 5, 6 and 10 (twice). It is notable that, of these 4 children, only the one who had selected only one model had had any difficulty setting the batter in the introduction.

It is instructive to examine the various non-standard ways in which children modelled the path of the ball, as shown by how they used or rejected the models offered. Three methods were employed by more than one student:

- Four children at least once modelled a ball or a rolling action, or rejected a model because it could not roll or fell down. The path of the ball was shown by action on the cricket model rather than in the abstract model.
- Three children at least once used an abstract model to represent only one part of the path (bowler-batter or batter-fielder). Two children set models 7 or 9 to a straight line, one used the line in model 3, and one set models 6 and 9 to an angle of about 120°.
- Three children indicated on models 5 or 8 that the ball started at one end of the longer line, travelled to the batter at the other end, and then back to the centre and out to the fielder along the shorter line. Three of these children also rejected models 4 or 7 because half of one line would have no function; as one put it, *there aren't two fielders*.

Four other ingenious solutions were found to the problem of representing the path of the ball using the two lines in models 4-10, but none showed a single change of direction at the intersection of the two lines.

Children's drawings illustrated further difficulties in abstracting angles from cricket-like contexts. Only 6 of the 12 children drew the standard two line segments, 2 using broken lines. Of the others, 2 students drew only one position of the ball; both showed its path on their pictures with their fingers but could only draw its motion by depicting "air" behind it. Three students drew several different positions of the ball to indicate its movement, and one finished with an arrow pointing from the batter to the fielder. Only one student made a purely abstract drawing which dispensed with the players and the field.

Drawing appeared unrelated to other aspects of abstraction. Of the 6 children who made a standard drawing of the path of the ball, only 2 had chosen abstract models appropriately and only one had shown no difficulty in setting the batter initially.

Tiling

All 12 children seemed to understand completely the effect of the size of the corners of a tile on its fitting into a tiling pattern, even though all of them initially had some difficulty making a pattern with the 75° tiles.

All saw immediately that the proffered square would not fit into the space left by a damaged 75° tile, and all gave some valid reason. The most common global explanation was *it's a square not a diamond* or simply *it's the wrong shape*; but a few referred to the sides as *straight not slant*[ing], *too up* [sloping], or *over a bit*. In the only use of the word "angle" in the whole investigation, one student explained that *the diagonal line is at an angle*.

It was equally obvious to the children that a 60° tile would also not fit. Some explanations compared the length of the major axis of the two rhombi (*it's longer, bigger*) but most focussed on their minor axes (*skinnier, squashed-in, thinner, too much of a diamond, they don't go out the same*). A few comments (*too low, too high, slantier, too much pointed out*) referred to the sides.

All children also found words to compare the corners of the three tiles (*sharper, pointier, smaller, wider, spiky*). Some tested their sharpness with their fingers (one student looking intensely at the depth of the indentations on her skin), but this test did not discriminate as well as a visual test.

All children were also able to adjust the flexible rhombus to the shapes of the three given tiles and to demonstrate the sharpest tile the model would make; however, one objected that the ends were rounded and another pointed out that the rhombus formed by the inside edges of the sides would vanish when it was squashed up. Most could also imagine the sharpest possible tile, describing it variously as *really skinny, one bar of metal* or *like a bumpy stick*. One student objected that it would not be pointy any more because it would be a straight line.

Other examples of corners were easily found, including frequent mention of abstract shapes such as the triangle. However, several examples (*foil, paper, broken glass, sharp metal bits of doors*) suggested that children's idea of sharpness related to the fineness with which a corner was machined rather than to the inclination of its component edges - a point which reoccurs below.Of the 12 students, 8 indicated an appropriate modelling of tile corners using their best model (models 4, 6, 9 or 10 were each chosen twice). However, 4 of these 8 students also omitted at least one appropriate model (other than models 1-3, where the angle is not at all obvious) or used at least one model inappropriately.

There were two main categories of non-standard modelling, both very revealing.

- In model 9, 3 children were influenced by the shape of the pipe cleaner when the model was bent to show an acute angle. One student commented *it get's blunter each time you bend it* [i.e. the more you bend it], compared it to the sharp ends of the straw, and continued *it's only a tiny, winsy bit like it ... it's hard to get it up so high* [i.e. right into the corner]. Two of these children set model 9 inside the corner of the tile without aligning the straws with the edges of the tile.
- Five children set, on at least one model, the vertex of a tile corner at the intersection of two lines but only one edge against a line. Four of these children placed the corner at the intersection of a radius and the circumference in models 2, 3 or 10, and 2 children set it at the centre of models 6 or 10.

Children's drawings indicated other difficulties in abstracting the angle concept from corners. Nine draw conventional angles in the "hat" orientation, but 3 felt the need to complete the triangle. Only 3 children draw all three corners within 15° of their actual size. Two children drew all corners within a range of 10°; both of these had modelled only one edge of each tile corner and had not differentiated different tiles. All but one of the remaining 7 children correctly represented the order of size of the three corners; the median angles drawn by inaccurate children were 34°, 47° and 63° compared to the actual 60°, 75°, 90°.

It may be significant that the 3 accurate drawers all drew the right angled corner with one line vertical and that they were the only ones to do so.

Drawings also varied in the length of the arms (from 1 mm to 58 mm) but there was no obvious correlate of students' preferred lengths. Seven children draw angles which varied markedly in arm length, but in only 3 of these cases were the arm lengths in the same order as the angle sizes.

DISCUSSION

Children's attempts to represent the cricket and tiling contexts using the abstract angle models and in drawings revealed a great deal about their difficulties in abstracting the angle concept - both in seeing the common features and in ignoring the distracting features.

In the cricket context, the initial difficulty seems to be in representing the path of a ball by a straight line. The students in the present sample were well on their way. For although only 4 students correctly modelled the path, all but one of the remaining 8 tried to use lines on the model at least once to represent at least part of the path.

Also, whereas only 7 children drew line segments to represent the path of the ball, only 2 knew of no way to depict the movement.

But that is by no means the end of the story. The learner must recognise that the path only contains two line segments, that they must be joined at their ends at a point which represents the point of deflection, and that extensions of the line segments can be ignored. It is a long way to the level of understanding of the student who remarked that model 4 could represent the path of a cricket ball in 8 different ways. It may be noted that drawing does not focus attention on these abstractions, since children can represent the bowler, batter and fielder; then, as soon as they have learnt the convention of representing a moving object by a line, they automatically produce the correct configuration. It should therefore not be surprising that children who represent an angled path in a standard manner cannot necessarily represent it using an abstract physical model.

The tiling context revealed quite different problems. Here, the main difficulty for children seems to be in abstracting the two sides of the corner as the critical features, ignoring the exact configuration at their intersection. This difficulty might have been heightened in the present investigation by the finely cut corners of the tiles used; but we note that in another context investigated but not reported here (corners in a road), children had an exactly similar difficulty in ignoring the *curved* part of the road at the vertex of a corner. Children's tendency to focus on the vertex of a corner rather than on its sides confirms Davey and Pegg's (1991) finding that young children tend to think of a corner as a point.

The drawing task presented no difficulty representing the vertex, instead showing that most students were still in the process of abstracting the size attribute of a corner. We note that the physical models did not draw attention to size, since the arms could be easily adjusted to match the angle. Only 4 children spontaneously checked their (inaccurate) freehand drawings in a similar way by placing the corner of the tile on them; two then redrew the corners freehand (accurately) whereas two traced the corners from the tiles. It may be significant that the two who redrew freehand had both chosen appropriate best abstract models whereas the two who traced had not; the one student who represented all corners accurately at the first attempt had also chosen an appropriate best abstract model. The abstraction of angle as consisting of two sides would thus seem closely linked to the abstraction of the size of an angle.

CONCLUSION

The investigation of the cricket and tiling contexts has clearly shown several problems which children have in abstracting the angle concept. The other investigations (into turning, scissors, hills and bends) have shown some similar and some quite different problems, to be reported elsewhere.

The abstract models proved to be a valuable means of investigating children's angle concepts. Apart from models 1-3, which are only suitable for showing turning and then do not emphasize an angle, there was little difference between them; they were chosen with approximately equal frequency and were approximately equally often correct. We anticipate that the models could be useful aids in helping children to abstract the angle concept from various contexts and to link the various contexts together to form a general angle concept. It is our next aim to develop and test this proposition.

REFERENCES

Davey G. & Pegg J. (1991) *Angles on angles: students' perceptions.* Paper presented at the 14th Annual Conference of the Mathematics Education Research Group of Australasia, Perth.

Mitchelmore M.C. (1992) *Abstraction and generalisation are critical to conceptual change.* Paper presented to the Working Group on the Formation of Elementary Mathematical Concepts at the Primary Level, Seventh International Congress on Mathematical Education, Quebec.

Mitchelmore M.C. (1993). *Concrete contexts for abstracting angle concepts.* Short contribution to be presented at the 17th Annual International Conference on the Psychology of Mathematics Education, Tokyo.

Skemp R. (1971) *The psychology of learning mathematics.* Harmondsworth: Penguin.

White P. & Mitchelmore M.C. (1992). Abstract thinking in rates of change and derivative. In B. Southwell, B. Perry & K. Owens (Eds.), *Proceedings of the 15th Annual Conference of the Mathematics Education Group of Australasia,* 574-581.

ASSESSMENT AND REPORTING PRACTICES IN MATHEMATICS - AN INTRODUCTION TO A NATIONAL SAMPLE STUDY FOR THE 1991 REPORT ON SCHOOLING

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This project was commissioned by D.E.E.T. for the Australian Education Council's Working Party on an Annual National Report on Schooling and conducted during 1991-2. Its aim was to document school assessment and reporting practices across Australian schools. A substantial survey instrument was developed and used to profile assessment and reporting practices - influences, decisions and actions. In addition, a national network enabled access to individuals and schools with acknowledged good practices and a number of these were supported to prepare case studies of their work in the areas of assessment and reporting.

The survey instrument and results are discussed, along with some suggestions about influences on and connections between aspects of practice. Of particular note is the general reluctance to value assessments made informally and/or involving students' self-assessment in comparison with more traditional means. Areas of difference between primary and secondary teachers' responses are noted and discussed. The strong links between teaching and learning mathematics and, particularly, assessment are evident. Consideration of some of the case study material illustrates how a balance of assessment styles can be achieved and how such an approach is intimately linked to a contemporary teaching and learning program.

Much of the recent history of mathematics education in the compulsory years has been characterised by efforts to inform teachers about contemporary thinking in the discipline, for example - and importantly - through the publication of *A National Statement on Mathematics for Australian Schools,* A.E.C. (1991). Efforts have been made to encourage teachers to change their practices in ways consistent with increasing understanding of mathematics and how it is learnt. National projects such as the *Mathematics Curriculum and Teaching Program* (M.C.T.P.) as well as a multitude of state and locally based projects have focussed on improving the teaching and learning of mathematics through an emphasis on teacher professional development.

There have recently been substantial changes in the mathematics of the post-compulsory years in several states. Syllabus changes have focussed on the development and implementation of senior courses which provide for the needs of the more diverse group of students being retained at school. Often these courses have emphasised applications of mathematics. Both within these courses and generally in some states, non-traditional assessment techniques have been made mandatory (for example the V.C.E. and S.A.C.E. Stage 1 courses). These techniques, while remaining formal assessment events given the context of the needs of senior secondary schooling, represent a major shift in thinking and practice for many teachers.

The recognition of the need for schools and school systems to be more accountable for what they do is significant in considering the context of this study. The use of the Mathematics Profile to report on student attainment is premised on teachers having access to an assessment information base sufficient for them to make professional judgements. The nature of the profile outcomes indicates the need for a range of assessment practices to be employed to build this information base.

The purpose of the Sample Study was to document school assessment and reporting practices across Australian schools. This topic was agreed by the Australian Education Council (A.E.C.) at its 1990 meeting as one of national interest. The study was commissioned by the Department of Employment, Education and Training (D.E.E.T.) for the A.E.C. Working Party on an Annual National Report on Schooling. The selection of a topic of interest to be the subject of study in this way has been normal practice for the A.E.C. in recent years.

The study included both the government and non-government sectors of primary and secondary schooling, with the major focus on the presentation of a national view of assessment and reporting in mathematics as they operate at the school level. It operated under the direction of a Steering Committee comprising representatives of government and non-government educational authorities.

The full report of the Sample Study runs to some one hundred pages with appendices of a similar size. This paper presents the background and methodology of the study as well as some of the major findings.

THE SAMPLE STUDY DESIGN BRIEF

The study was in three stages:

Stage 1: Identification of the range of assessment and reporting practices operating in Australian schools and the development of a set of descriptors for documenting those practices.

Stage 2: Documenting the range of assessment and reporting practices in a small sample of schools identified by educational authorities.

Stage 3: Documenting the extent to which different assessment and reporting practices identified in Stage 1 are being used in government and non-government primary and secondary schools.

METHOD

Stage 1 - the descriptors

Some forty people from school, university and system backgrounds attended writing conferences (one or two days) in Melbourne, Sydney and Adelaide to develop the descriptors. On the basis of the 1988 study by Withers and Batten (1990), and because of the desire to report details of practices, a decision was made to avoid the use of generic assessment and reporting terms as mechanisms for describing practices. Table descriptors were developed to provide details. Major elements of assessment (and reporting), such as the purpose of the assessment, conditions under which the assessment took place, actions of those involved in the assessment process and how criteria for judging learning were established, were included in the descriptors.(see Appendix 1) These descriptor tables were the key means for gaining detailed information in the survey instrument.

Stage 2 - the survey

The teacher survey focussed on gaining details of teachers' assessment and reporting practices over two different periods of time, namely, the three to four week period immediately preceding the arrival of the survey in schools, and the whole of 1991. More detailed information was sought about practices in the shorter period. Five hundred and sixty-four teachers in years 2, 5, 9 and 12 in schools across Australia were surveyed. Approximately 80% of teachers responded.

The survey instrument for primary teachers was in six parts

1. Background information - personal contextual information
2. Teaching and learning - the style and focus of recent work in the class
3. Assessment - descriptor tables relating to Formal Assessment, Informal Assessment and Student Self-Assessment, as described in Clarke (1988).
4. Reporting - descriptor tables relating to Formal Reporting and Informal Reporting
5. A longer term view of assessment and reporting - a more general description of practices in 1991
6. Actual versus preferred practices - identification of desired changes, if any.

In addition, the secondary instrument contained an additional section entitled Different classes, Different approaches? which enabled respondents to identify whether and how they adjust the practices for different classes.

In addition to analysing the range of assessment and reporting practices, the study aimed to investigate differences between practices at different levels of schooling, factors affecting those practices, and the practices of various subgroups of teachers. These subgroups were organised on the basis of

- gender of teachers
- years of teaching experience
- secondary classes grouped on the basis of previous performance (or not)

- single-sex and mixed classes
- teacher attention to the development of positive attitudes to mathematics (or not)
- teaching and learning situations existing in the teachers' classrooms.

Stage 3 - Case study writing
The case studies were used to amplify information from the teacher surveys and to highlight successful practices in a range of different educational settings. Also, links were made to teaching and learning approaches in the classrooms. Factors influencing the introduction, development and operation of particular assessment and reporting practices were identified, together with the purpose and modes of assessment and reporting, and aspects of mathematics under consideration. A particular focus of some of the case studies was designated to be documentation of practices that had been demonstrated to be successful for girls.

Individuals and schools known to be using assessment and reporting practices that were seen to be successful for students and, possibly, innovative or associated with emerging trends in mathematics education were identified by systemic contact people, advisory group members and writing conference participants. Case study materials were received from thirty-five teachers in twenty-one schools (nine primary and twelve secondary) representing all sectors and seven states and territories.

Given time and resourcing constraints it was not possible to personally support case study writing except in a few instances. A writing framework (Appendix 2) was developed to assist teachers in the writing process and to act as a guide for their writing, but not to constrain it, in the context of working without outside support. The framework (or open-ended questionnaire) consisted of some major focussing questions with minor focussing questions down the side. The intention and major thrust of the case studies was apparent for all to see and the guidance provided by the framework was appreciated by many of the case study writers.

RESULTS

The principal findings of the study are, in summary:

1. Both primary and secondary teachers indicated the use of a considerable amount of formal and informal assessment. Primary teachers tended to value formal and informal assessment equally, whereas the clear preference of secondary teachers was formal assessment. Despite the high incidence of informal assessment reported by secondary teachers, it is clear that when it "really counts" they overwhelmingly use formal methods. For most teachers student self-assessment seems less important. There is, however, growing interest in that category of assessment, with primary teachers identifying it as one of the most important areas for future professional development.
2. The incidence of informal reporting in the survey period was very high, with nearly 90% of primary and 71% of secondary teachers indicating its use. Formal reporting was also indicated at a reasonably high level by both groups, probably reflecting the timing of the survey, which was conducted at the end of term 1, 1992. As was the case for assessment, the range of informal reporting practices used by primary teachers was more varied than the range used by secondary teachers.
3. For many students the formal assessment process involved them working independently, with little in the way of resources beyond their own knowledge, on tasks predominantly requiring only written responses. Even calculators were indicated to be available to students in less than 60% of cases in secondary, and 25% in primary, formal assessment events. As for computers and appropriate software, less than 8% of responses included them. However, some primary teachers described a richer assessment environment in which substantial resources were available to students during the assessment event.
4. There was evidence that different teaching and learning situations result in different assessment practices. It was clear that many teachers who have undertaken the kinds of pedagogical changes promoted in recent times have found ways of making their assessment practices somewhat consistent with their teaching practices. With formal assessment there were substantial differences between responses from teachers whose teaching practice included students having access to and use of resources, active involvement in their learning and working

collaboratively, and those who described more traditional teaching and learning situations. The group using the more contemporary practices also indicated a higher incidence of student self-assessment.

5. Differences were noted between the assessment and reporting practices of teachers with different years of experience. The more experienced teachers described a broader range of assessment practices than those of less experience, who tended to use more formal assessment and reporting. Although this is reassuring in one way, it does raise a question about the nature of pre-service teacher education in matters of mathematics assessment and reporting.
6. There was evidence that for some teachers student self-assessment in a mathematics context does not have mathematics learning as its major focus. In contrast to this, some of the case study teacher/writers identified the reflective process of student self-assessment as a powerful element of effective mathematics learning.
7. Teacher gender differences were observed in the occurrence of informal reporting, where women teachers reported its use with a relative frequency much higher than men did.
8. Student self-assessment occurred with much higher frequency in single-sex classes than in mixed classes. A very marginal difference also showed in the practice of informal reporting. With those exceptions, there were no apparent differences between the assessment and reporting practices of teachers of single-sex classes and those of teachers of mixed classes.
9. Secondary teachers of all-girl classes were more likely to take specific actions to support the development of positive attitudes to mathematics or appreciations of the nature and scope of mathematics than secondary teachers of other classes, but there were no qualitative differences in the comments they made about the nature of these actions.
10. Those teachers engaging in the practice of actively supporting the development of positive attitudes in their students towards mathematics or developing in them appreciations of the nature and scope of mathematics have an expanded set of assessment practices compared with those who do not. This was more evident for secondary teachers than for primary teachers.
11. Teachers involved in the case studies presented a very encouraging picture of rich and successful practices. The primary teachers described a wide range of informal assessment practices, including those that actively involved their students in the assessment process through various forms of self-assessment. They were clear about the outcomes for their students as a result of their assessment practices and why they valued what they were doing. Assessment was integrated with learning and part of daily classroom procedures. The teachers conveyed the feeling that they have never known more about what their children know than they do now. In the main, the secondary teachers described a wide range of successful assessment practices too, but their practices would be more aptly described as formal assessment practices.
12. A significant difference existed between secondary and primary teachers in the case studies in that although the primary teachers had well-established criteria for judging learning, there was little evidence of documentation of those criteria. That was not the case for secondary teachers, where criteria and organisational structures are well documented and clearly influencing assessment and reporting. One of the major aspects reported by most of the secondary teacher/writers was the sharing of assessment criteria with their students and the value they all placed on that practice.
13. There was some evidence to suggest that teachers had more influence over their assessment and reporting practices in the short term rather than the long term. In the latter, school policy was the overwhelming influence for secondary teachers, while for primary teachers a balance of school policy, parental expectations and their own knowledge prevailed.
14. Although there were differences between the year levels surveyed some 60% of teachers indicated a traditional approach to mathematics teaching. By comparison, very few involved in case studies responded in that way.

CONCLUSION

The study has highlighted the fact that much of assessment practice in mathematics is formal, remains in the control of the teacher, requires written responses and occurs without access to much in the way of resources. On the other hand, responses from teachers involved in case studies suggest that there is a growing cohort of teachers who have made a long term commitment to their own professional development and who, over time, have significantly changed their assessment and reporting practices, feel confident in what they are doing and are able to articulate the details of and rationale for their practice.

Teachers reported reasonably high levels of use of all assessment and reporting practices, suggesting that the need for teacher development lies not so much in the expansion of practices to include a wider range, but in expanding options within the range currently in place as well as valuing the information gained from that range. There is also evidence that teachers need support to develop strategies for student self-assessment.

A major challenge is to build on the progress that some teachers have made and, in so doing, shift the power of learning in the direction of the learner.. There is no doubt that to involve students more actively in their learning and in the assessment of their learning is likely to result in classrooms like those described in most of the case studies, but it is important that any increase of the range of assessment practices recognises the need for it to be focussed on the whole range of valued mathematical outcomes.

REFERENCES

Australian Education Council, *A National Statement on Mathematics for Australian Schools*, Curriculum Corporation of Australia, Carlton, Vic., 1991.

Clake, David, *Assessment Alternatives in Mathematics*< The Mathematics Curriculum and Teaching Program, CurriculumDevelopment Centre, Canberra, A.C.T., 1988.

Withers, Graeme, & Batten, Margaret, "Defining Types of Assessment" *in* Brian Low and Graeme Withers (eds), *Developments in School and Public Assessment*, Australian Education Review No. 31, A.C.E.R., Hawthorn, Vic., 1990.

Appendix 1 - Sample page from survey form showing descriptor table format

Example (Informal Assessment Descriptor):

On each line within the descriptor you may circle one or more words. If more than one word is circled *on a line* then the order of importance, relevance or frequency of use needs to be shown using numbers (1 for most important ..., 2 for next most important ..., and so on).

EXAMPLE ONLY

Purpose of the assessment	inform whom	[✓ student(s)] 1 parents & care givers [✓ teacher] 2 other teachers education system
	about	[✓ achievement] 1 [✓ possible actions] 2 attitudes behaviour
	to manage learning	[✓ provide feedback] 1 [✓ plan actions] 2 modify program
How criteria for judging learning were established	who decided them	student(s) [✓ students & teacher] teacher school consensus administration education system
	influences	[✓ personal knowledge] school agreed standards systemic curriculum documents explicit external standards
Conditions under which assessment occurred	who initiated assessment	[✓ student(s)] 2 [✓ teacher] 1 school external body
	assessed group	[✓ small group] 2 [✓ individual] 1 class
Actions taken by teachers		watch [✓ listen to discussions] 1 discuss [✓ conference] 2 [✓ question] 3 read mark
Interactions within the assessment (teacher with student)		[✓ negotiate] 1 encourage [✓ challenge] 2 assist give instructions
Consequent teacher actions	immediate	[✓ respond] 1 [✓ interpret information] 2 judge learning
	ongoing	[✓ report] 1 [✓ monitor] 4 modify program [✓ remediate] 3 [✓ extend] 2

To explain further, consider two separate lines from the above table

Actions taken by teachers	watch [✓ listen to discussions] 1 discuss [✓ conference] 2 [✓ question] 3 read mark
Interactions within the assessment (teacher with student)	[✓ negotiate] 1 encourage [✓ challenge] 2 assist give instructions

Explanation

The first line response means that the *most significant* **Actions taken by teachers** in the last 3-4 weeks within **informal assessment** were to

listen to discussions	1	(most important, relevant or frequent)
conference	2	(next most important, relevant or frequent)
question	3	(third most important, relevant or frequent)

while on the second line,

the response means that the *most significant* **Interactions within the assessment (teacher with student)** were to

negotiate	1	(most important, relevant or frequent)
challenge	2	(next most important, relevant or frequent)

When only one word is chosen on a line there is obviously no need to indicate a number because the order is self-evident.

Appendix 2 - Sample page from the Case Study writing framework

3. Briefly describe the assessment (and/or reporting) practice.

(You may care to provide a range of practices, if that best describes your situation.)

What are the key elements?

What are the main purposes of the assessment (and/or reporting)?

Does this practice consist of one mode of assessment (or reporting) or a range of modes?

Does it consist of planned events? Is it an integral part of the learning process?

What information is provided to students about the nature of assessment tasks? How?

What evidence is used to indicate the development of skills, processes and knowledge in mathematics?

How are the criteria for judging learning established? Who decides?

What sorts of communication are involved in this practice? Oral? Written?

What are the actions of the teacher and students in this practice?

Who is the main focus of this practice? Individuals or groups? Which specific groups?

4. What are the significant features of this practice?

What are the strengths? Highlights?

Do students become actively involved? Are they part of the decision-making process?

What informs the actions of teachers and students to influence future learning as a result of this practice?

What are the opportunities for self-assessment and peer assessment?

5. What aspects of mathematics are being assessed (or reported)?

See table on page 5 based on *A National Statement on Mathematics for Australian Schools*

Use the terms in the table to help describe the aspects of mathematics being assessed (or reported).

Include detailed content if that is appropriate.

NUDIST: A QUALITATIVE RESEARCH TOOL OR A MIRROR OF OUR OWN PEDAGOGICAL THEORIES?

JUDITH MOUSLEY, PETER SULLIVAN AND ANDREW WAYWOOD

Qualitative research produces a wealth of complex, interactive ideas which are difficult for a an educational inquirer to control. NUDIST, a qualitative data analysis tool, speeds both categorisation of data and statistical analysis of data, while maintaining an ability to re-visit the original material. This paper illustrates the use of NUDIST to synthesise ideas resulting from one open-response item from a questionnaire. The major issue raised is the subjective nature of decision-making about categories used to form a conceptual framework for data analysis using NUDIST.

RESEARCH PARADIGMS

Popkewitz (1984) claims that the notion of having different paradigms of educational research helps us to define our fields of work and ways of working as researchers. He states that:

> ... the concept of paradigm provides a way to consider a divergence in vision, custom and tradition. It enables us to consider science as having different sets of assumptions, commitments, procedures and theories of social affairs. (p. 35)

The paradigm in use for any educational research project impacts on relationships between researcher and subjects, on modes of data collection and methods of analysis, and on ways that findings can be summarised and integrated. It also affects the replicability of the study, an ability to apply findings in other settings, and ways that the study is likely to be reported. Even though research paradigms are general philosophical outlooks rather than systematically formulated doctrines (Candy, 1989), there are agreed sets of assumptions and common sets of procedures which form a basis for developments within each of the major paradigms. Educational inquirers who display an empirical-analytic stance, for instance, take a different role in the research process (in methodological, practical and political terms), conceive different research questions for different purposes, and use different data gathering and reporting methods from their colleagues working in the interpretive-humanistic paradigm.

Empirical research.

The use of scientific research methods in the study of pedagogy is based on their wide use and (generally unquestioned) acceptance in the study of causal relationships within the natural sciences. In the study of human sciences, there is some support for the ability of this paradigm to produce sets of findings about current behaviours and their causes, and hence to make some predictions about future behaviours.

With its emphasis on quantifiable observations which lend themselves to the application of mathematical tools for analysis, research methods within this paradigm are aimed at the collection of relatively "objective" data by researchers who are external to the human interaction. Scientific control of variables and rigour in data collection methods result in possibilities for replicating findings. The potential for generalising findings to the wider community is also considered important. The reliability of tests and of observation instruments thus become vital factors in this style of research as they are employed in order to describe, "measure" and compare social behaviours, attitudes and beliefs, or aspects of classroom interaction.

Unfortunately, statistical generalisations and notions of causality are not easy to apply to the complex, changing and multi-dimensional realities of human interaction within social institutions. Other research paradigms have therefore been developed for use in the humanities.

Qualitative research.

Hermeneutical educational inquiry is based on the gathering and interpretation of qualitative data. Such research aims to understand rather than to explain. The aim is to capture the individual rather than to generalise or predict.

The complexities and uniqueness of an event or the interactive beliefs of members of a sub-set of society at a particular time are recognised as "objects" worth studying, rather than replicable phenomena.

Because of its habit of producing holistic data which is specific to particular research sites, and as a result of its challenge to established research criteria such as "validity" and "reliability", qualitative research has been criticized for generally lacking rigour, objectivity and control of variables (see, for instance, Kerlinger, 1973). Forms of research which make up this paradigm are overtly affected by values, power relationships and subjective interpretations of both researchers and subjects: the role of human agency becomes more recognisable.

It is not the intention of this paper to outline the strengths and weaknesses of different research paradigms, or to detail their complementary natures, for these have been discussed widely elsewhere (see, for instance, Kerlinger, 1973; Lincoln & Guba, 1985; Smith & Heshusius, 1986). As Sowden and Keeves (1988) note, "hardline distinctions that are commonly made between quantitative and qualitative research methods are largely artificial" (p.514). The debate about which paradigm is more suitable for social research has set up an artificial dichotomy which has unfortunately delayed consideration of what each paradigm has to lend to the other and of where important "touchstones" can be developed further.

NUDIST: Creating a touchstone

Qualitative research produces a wealth of interactive, and perhaps iterative, ideas in the form of interview transcripts, field notes, responses to open-ended questionnaire items, statements in documents, and so on. These are complex and varied enough to be difficult for a researcher to control - to categorise, to compare verbally or statistically, to chart or graph, and to decide what to report or omit.

Another likely problem is that in reporting such data, raw material is lost as it is replaced with generalities, such as "30% of the respondents noted that ... " or "one group of children discussed _". Such reductionism fails to capture personal use of terms and ideas accurately, and doesn't make available to readers rich details or examples. Thus reduction to general categories involves loss of depth of meaning as well as the coalescence of discrete ideas. Sowden and Keeves (1988) raise this issue when they claim:

> ... while an increasing amount of empirical research using qualitative data is being carried out, _ it is being reported in such a way that the conclusions cannot be verified_ This gives rise to the anomolous situation that while in research the evidence is rich and detailed, the very richness and detail of the data collected prevent presentation in a coherent form that would lead to acceptance of the findings as a contribution to scholarly inquiry. (p. 525)

Readers must rely on accurate classification of raw data by authors and have no way of checking interpretations made or the appropriateness of orienting constructs. Replication of the processes by other researchers or looking anew at material gathered generally involves repetition of lengthy analysis and synthesis processes.

The computer software program NUDIST, a tool for organising and managing qualitative data, helps researchers develop a productive relationship between qualitative and quantitative research paradigms by enabling them to overcome some of these tendencies. It provides techniques for handling textual data which speed both categorisation and statistical analysis while maintaining the ability of the researcher or other interested persons to re-visit the original data.

Basically, NUDIST allows the management and organisation of data through indexing. Primary textual data is coded, to be retrieved according to emerging categorisations. Categories for indexing are certainly not pre-determined because, as one manual (Richards & Richards, 1990) notes:

> ... what the user does with the retrievals, using such software, is essentially offline: all the theory-building, the shaping of understanding and the reshaping of the data in accordance with the changing understanding, is done outside the computer. (p.6)

In addition to indexing, or using NUDIST as a tool for content analysis, it can be used for the building of theory. While the former use draws the researcher closer to the empirical paradigm, the latter emphasises the emerging nature of understandings and thus seems to move the research process into the heumeutic paradigm.

NUDIST makes it easier for qualitative data to be expressed in quantitative terms and then worked with statistically but this does not add validity or objectivity to qualitative research: the same work could be done more slowly by hand. However, the NUDIST manual claims that the program is more than a "code-and-retrieve" system in that it supports the emergence of theories.

> The indexing data base can be of any level of complexity - from the flat lists of codes necessary for most code-and-retrieve programs, to highly organised and complex tree-structured indexes of categories and sub-categories. This structuring allows indexing concepts to be organised and managed as theoretical systems, not just as labels. (p. 7)

This program is not a solution to the debate about the comparative merits of quantitative and qualitative research paradigms, where the real questions are about more fundamental issues of validity, reliability and objectivity, as well as the nature of truth and our ability to know it - and eventually about how research can be used for human betterment (Bates, 1980, Carr & Kemmis, 1983).

USING NUDIST IN A CURRENT RESEARCH PROJECT

The research study employed in this paper to illustrate the use of NUDIST involved the distribution of a questionnaire to mathematics educators and teachers. This was aimed at identifying common beliefs about characteristics of quality mathematics lessons. The first item was an open-response item:

> Throughout this survey we want you to imagine a mathematics lesson, at any year level, where the students are learning, for example, to estimate the mass of various objects, or to add fractions, or to record given information as a graph. Before turning the page, please write down the most important characteristics which a **quality** mathematics lesson on any of these concepts/skills would usually have.

Initially, all 125 responses to this question were typed. The NUDIST program produced a printout which identified the respondent's occupation, country, etc. For example, the response reproduced below (#2) was from a mathematics teacher studying a Graduate Diploma or Masters at Christ Campus.

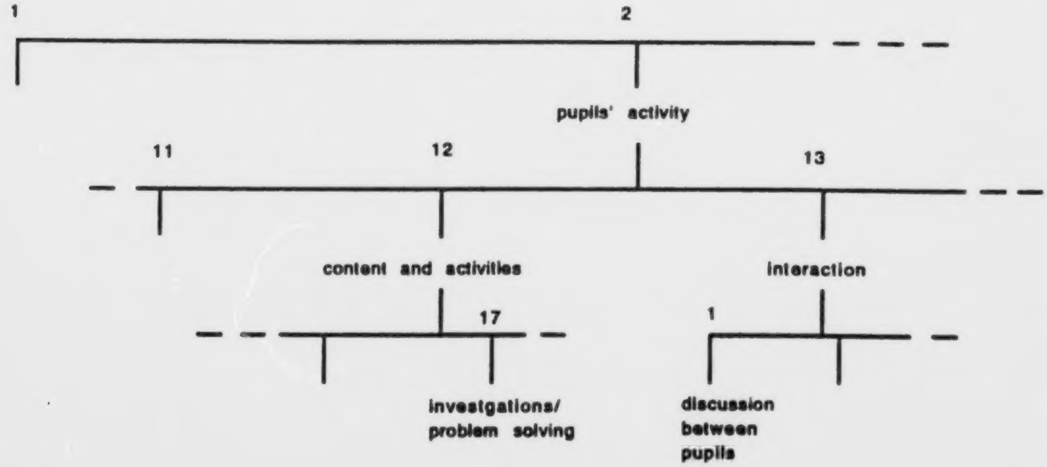
```
*2                                                              1
GRAD DIP/MASTERS - CC                                           2

* QUESTION 1                                                    3
The characteristics would include physical involvement, some    4
co-operative learning, discussion, problem solving and risk taking   5
```

Each phrase in each response was then categorised by hand under general categories and sub-categories, using a rudimentary taxonomy (verbal, then later numerical) which developed as we worked together on this task. Initially, classifications such as "teacher 's work (1)" and "pupil's work (2)" were created. Sets of subsidiary ideas were then categorised into nodes under these classifications. Using the same example, below, the first 2 relates to the classification "pupil action". The next two figures indicate a node (12 was "Content and activities", and 13 was "Interaction"). The remaining figures designate sub-nodes (e.g. 1 was "discussion between pupils", and 17 was "investigations/ problem-solving").

physical involvement	2	12	3		4	
co-operative learning	2	12	3		5	
discussion	2	13	1		6	
problem solving	2	12	17		7	
risk taking		2	12	8		8

A small section of the conceptual framework which developed would thus appear as:



Key words and common phrases were useful in deciding on how to classify phrases. For instance, "sharing ideas", "time for sharing perceptions", "sharing alternative solutions" and "listening to alternative solution strategies" seemed to be linked by common words as well as conceptually, so were all classified as *sharing strategies*. At times it was necessary to add further categories, or to split one as it became apparent that two distinctive groups of data were developing from one category.

This stage of the process was time-consuming but enlightening. There were some lively debates about possible interpretations of responses, and the meanings of jargon such as "non-threatening". We also discussed whether the structure arising was largely pre-determined either by our own pedagogical notions or by the wording of the original question. There was also concern about practicalities such as the effect of including a phrase in two different classifications or whether a term and an example of what was meant should be included as two entries.

Classifications were then typed against each phrase, and NUDIST was used to group phrases together under the taxonomy of headings created. The result was printed out. As an example of what was produced, two sub-nodes had the following amongst their data:

THINKING (2 12 18)

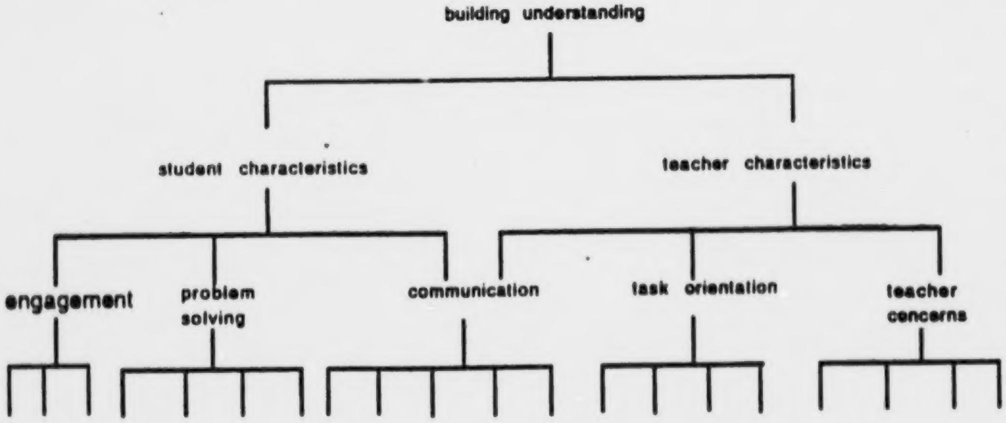
Aims to develop understanding opportunity for student thinking
Lesson requires thinking rather than repetition and mimicking
Promotes thinking
Lots of thinking by students about task
Speculating on how best to solve them and solving
Thought about the problem

CHALLENGING (2 12 9)

Challenging
Challenging task w/i reach
Challenging activities
Challenge
Should have ability for extension
Challenging but caters for individual differences

The collections of phrases were then re-examined to determine whether they formed a coherent set, and descriptors were moved if necessary.

On reviewing the data at this stage, it became clear that the headings used originally (teacher actions, content and activities, etc.) were not suitable for synthesising major features of quality lessons. Arranging sub-nodes around the main characteristics appearing led to the formation of six categories, tentatively called *Communication, Problem solving, Building understanding, Engagement, Task orientation* and *Teacher Qualities*. Thus a new taxonomy was formed:


building understanding
student characteristics
teacher characteristics
engagement
problem solving
communication
task orientation
teacher concerns

Strangely, this analytical exercise taught us as much about our own (as well as each others') concepts of quality lessons, and about the act of writing a questionnaire item as it did about the beliefs of our colleagues. We also learned a lot about the way different phrases are used in describing teaching and were reminded continually of the vagueness of most of the terms used. (What, for instance, is "real-world activity" and what are the reasons for it to be included in a mathematics lesson?) It is now hoped to use this data to engage people in further discussion about the appropriateness of the categories formed and about the possibilities of developing a more common language for describing teaching.

A most important feature of this analysis process is that the original data, as typed, is still accessible for easy comparison with new data - or for replication of the process (perhaps with new conceptual frameworks). While the final printout only showed our classifications at each level, each stage of the process (from raw data on) has been saved and is thus open to examination or re-working. With NUDIST, a log of changes is kept automatically. A history of what was done therefore forms the basis for keeping a record of, and retracing if desired, the research process.

THE ACT OF CLASSIFICATION

Categorisation has the benefit of focusing data and enabling it to be reduced to a state where it can be analysed in terms of quantity, discussed under broad headings or compared with other data. However, classification involves an orientation and rebuilding - and hence modification of - the original data. While the intention of the researcher might be to keep as close as possible to the intention of the contributor of a given phrase, two factors impinge on this process. First, the act of categorisation is by its very nature subjective. Selection of orienting categories depends on researchers' personal constructs of the task at hand. These, in turn, are shaped by perceived possibilities within the fields of research and educational theory and then by a developing notion of the overall "findings". Second, phrases and terms used to describe teaching and learning are themselves attempts to capture nebulous qualities and ways of behaving. These are used in various ways according to participants' prior experiences and understandings as well as their perceptions of audience. To place phrases within categories is to assume a meaning and perhaps to "bend" the data to fit with other data. As with most analysis of qualitative data, factors (such as inflection and body language) which would assist the accuracy of placement or summation are lost in the transcription process.

Problems related to the subjective nature of decision-making and interpretation of human beliefs and actions are not in any way specific to the use of NUDIST. Hermeneutic action is common to all research methods. Giddens (1984) notes that meaning is framed as individuals view the real world in terms of their personal understandings and that these interpretations are made in the light of their understandings of the theories, ideas and concepts developed by researchers in their fields. Further interpretation takes place as the ideas are published and take new form in the praxis of everyday social use. Such "moments" of decision-making are not so apparent in empirical research but are still present and attempts to control such factors bring their own set of limitations to research projects.

One group of questions raised in our discussions about the use of tools such as NUDIST related to how we, as educational researchers, make judgements regarding what information to seek, how to seek it, how to analyse it, then how to use it. We discussed how these decisions are influenced and limited by our own socio-cultural history as well as by our individual professional histories, personal understandings and current needs. Another group of issues raised was about the sharing of findings and what is considered by a researcher to be worth sharing. What is withheld from the reader? How do we share our findings with particular audiences, and is this done selectively? How accessible are the ideas presented to the consumers and producers of educational theory-in-use, i.e. teachers in classrooms? How can all of this fit into eight pages? There was also discussion about how we raise the issues of the problematic nature of judgements that have been made during the research process. At each stage of the process, decisions were being made about how to handle the data in its current form, but in most research reports, the nature of this decision-making is not presented as problematic.

The experience of thinking about the research processes undertaken has sensitised us to the workings of heumeneutics within educational research. The act of interpretation is ever new. Starting points and further developments, analysed and theory, conceiving and reconceiving, part and whole, what-is-seen and what-is-to-be-seen: the interplays between these are ongoing. All reflect a process of gaining understanding - the heumeneutic act. The researcher is involved in a process of representation and creating boundaries, but of expanding horizons of understanding. However, readers of published articles rarely see them as artistic creations: pictures of the educational researchers' experiences, intentions and growing conceptions. And the act of reading a research report in many ways parallels the creative acts of the researchers themselves.

REFERENCES

Bates, R. J. (1980). New developments in the new sociology of education. *British Journal of Sociology and Education 1* (1), 67-79.

Candy, P. C. (1989). Constructivism and the study of self-direction in adult learning. *Studies in the Education of Adults, 21* (2), 95-116.

Carr, W. & Kemmis, S. (1983). *Becoming critical: Knowing through action research.* Geelong: Deakin University Press.

Giddens, A. (1984). *The constitution of society.* Oxford: Polity Press.

Kerlinger, F. N. (1973). *Foundations of behavioural research: Educational and psychological inquiry (2nd edn.).* New York: Holt, Rinehart and Winston.

Lincoln, Y. & Guba, E. (1985). *Naturalistic inquiry.* Los Angeles: Sage.

Popkewitz, T. (1984). *Paradigm and ideology in educational research.* London: Falmer Press.

Richards, T. J. & Richards, L. (1990). *Manual for Mainframe NUDIST: A software system for qualitative data analysis on time-sharing computers.* Eltham: Replee.

Sowden, S. & Keeves, J. P. (1988). Analysis of evidence in humanistic studies. In J. P. Keeves (Ed.) *Educational research, methodology and measurement: An international handbook.* (pp. 513-526). Oxford: Pergamon Press.

Smith, J. K. & Heshusius, L. (1986). Closing down the conversation: The end of the qualitative/quantitative debate among educational inquirers. *Educational Researcher, 15* (1), 4-12.

6TH GRADERS UNDERSTANDING OF MULTIPLICATIVE STRUCTURES: A 5 YEAR FOLLOW UP STUDY

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An earlier study analysed responses by 70 young children to a variety of multiplication and division word problems at four interview stages in a 2-year longitudinal study (Mulligan, 1992). 75% of the children were able to solve the problems using a wide variety of counting, grouping and modelling strategies and the semantic structure of the problem strongly influenced solution process. A follow up study was conducted ***with the same sample*** *(n=45) in Grade 6, prior to entry to secondary school. Identical problem structures were used including problems involving decimals. A similar pattern of performance to the longitudinal study revealed a 60-85% success rate except for Sub-division, and poorer results for decimal problems. Informal methods used by children in the early stages of the study dominated their understanding of multiplicative structures in Grade 6. Evidence of children's lack of conceptual understanding of multiplication and division was revealed.*

INFORMAL MATHEMATICS

Research investigating young children's understanding of number concepts and problem-solving processes has revealed widespread use of informal or intuitive strategies developed prior to formal instruction (Carpenter and Moser 1984; Hughes, 1986; Steffe, Cobb and Richards, 1988; Steffe and Wood, 1990). Studies investigating solutions to addition and subtraction problems have indicated that children use a wide variety of modelling and counting strategies that reflect the semantic structure of the problem (Carpenter and Moser, 1984; De Corte and Verschaffel, 1987). These studies indicate that children possess considerable mathematical knowledge and skills prior to formal instruction that is developed from their informal experiences. However, when children experience formal instruction it cannot be assumed that their conceptualisations are linked with formal mathematical ideas or their own strategies match those encouraged in instruction (Hiebert, 1984; 1990). While the use of informal strategies often provides a meaningful way for solving problems initially, these strategies may be ineffectual for more complex mathematical tasks.

Many researchers have expressed concern over the widespread and continued use of informal methods well into secondary school (Bell, Swan and Taylor, 1981; Booth, 1984; Fischbein et al., 1985; Hart, 1981). These informal methods, consisting largely of counting, adding-on or building-up strategies, may be adequate for simple items but are not generalisable to more complex mathematical situations or operations, e.g. fractions or applications of algebra. Researchers investigating the strategies secondary school children used when solving a problem found "that many children are not using the proper mathematical methods taught them at school but rather are relying upon naive intuitive strategies" (Booth, 1981).

RESEARCH ON MULTIPLICATION AND DIVISION PROBLEMS

Over the past decade, researchers have analysed secondary student's solution processes to multiplication and division word problems based on differences in semantic structure, mathematical structure, size of quantities used, and student's intuitive models (Bell, Fischbein and Greer, 1984; Bell, Greer, Grimison and Mangan, 1989; Brown, 1981; De Corte, Verschaffel and Van Coillie, 1988; Fischbein et al., 1985; Nesher, 1988; Vergnaud, 1988). Various studies have shown that children's difficulties in solving problems are related to the size and complexity of numbers in the problem (Brown, 1981; Bell et al.,1981; Bell et al., 1984; Bell et al., 1989). When children were presented with problems having the same mathematical, semantic, and surface structure but differing in terms of the nature of the given numbers, they changed their minds or were unable to decide about the operation to be used.

The 'multiplier effect' has been identified where increases in problem difficulty occur when the multiplier changes from an integer to a decimal number, and to a decimal number less than 1. The choice of operation in solving a multiplication or division word problem could also be influenced by the notion 'multiplication makes bigger' and 'division makes smaller'. A 'competing-claims' theory (Bell et al., 1989), in relation to choice of operation in multiplication and division word problems suggests that children optimize the possibly competing claims of four factors: (i) numerical preferences, (ii) conformity with the lowest level of problem structure, (iii) numerical misconceptions, and (iv) ordering the quantities. Furthermore, hierarchies of difficulty in the word problems were found through to adulthood showing that students relied on methods causing "less cognitive strain before adopting harder ones" (p.447).

More recently, a growing number of studies on young children's solution strategies to multiplication and division problems have emerged (Anghileri, 1989; Boero, Ferrari and Ferrero, 1989; Brekke and Bell, 1992; Kouba, 1989; Mulligan, 1991; 1992; Murray, Olivier and Human, 1992; Steffe, 1988). These studies have provided complementary evidence that the semantic structure of the problem, an understanding of the problem context, and the development of counting, grouping and addition strategies influence solution process. Underlying intuitive models for multiplication and division may also be attributed to children's informal development of these processes and the difficulties experienced when problem-solving operations come into conflict with constraints of these models (Boero et al.,1989; Fischbein, 1989; Mulligan, 1991).

METHODOLOGY

A follow up study was conducted to investigate 6th Grade children's understanding of multiplication and division through word problems involving simple number facts, their solutions to problems involving decimals in measurement contexts, and their ability to generate written problems from symbolic forms. The study aims to further map the development of multiplication and division processes and identify key factors in this process. Comparisons between the children's performance, strategy use and intuitive models shown in Grades 2-3 and Grade 6 were analysed by individual profiles. Clinical interviews were conducted using identical procedures to those employed in the longitudinal study. The interview sample retained 45 of the 60 girls previously interviewed in Grade 3 in the longitudinal study (Mulligan, 1992). The girls were interviewed in December of Grade 6 prior to entry to secondary school and they were permitted to record their mathematical ideas on paper. Each interview was approximately one hour in duration.

INTERVIEW TASKS

The girls were required to initially solve ten multiplication and division word problems (Table 1) representing each problem structure used in the longitudinal study. Number size was increased to include more difficult combinations using number triples (8,9,72), (15,6,90), (7,8,56), (9,12,108), (12,3,36), (6,7,42), (15,5,75), (7,12,84), (24,6,4) respectively but these were not much more difficult than the large number combinations used in the previous interview in Grade 3.

Multiplication	**Division**
Repeated Addition There are 8 children in the library and 9 children are seated at each table. How many children are there altogether in the library?	Partition (Sharing) $72.00 was shared equally between 9 children. How much money did they get each ?
Rate If you need 15c to buy one sticker how much money do you need to buy 6 stickers?	Rate Peter bought 15 folders for $75.00. If each folder cost the same price how much did one folder cost? How much did 7 folders cost?
Factor John has 7 books on his shelf and Sue has 8 times as many. How many books does Sue have?	Factor Simone has 42 books on her shelf and this is 6 times as many as Lisa. How many books does Lisa have ?
Array There are 9 lines of children at assembly with 12 children in each line. How many children are there altogether ?	Quotition 84 cards were shared equally between the children. If they each had 12 cards, how many children were there?
Cartesian Product There are 12 different ice-cream flavours and 3 different sized cones. How many different choices of ice-cream cones could you make?	Sub-division I have 4 large chocolate bars to be shared evenly between 24 children. How much chocolate will each child get?

Four problems using multiplication of whole numbers and decimals in measurement contexts were included (Table 2). Additional tasks involving algorithms and children's writing of word problems to match symbolic forms were given, but are not reported in this paper.

Table 2: **Multiplication of Decimals Problems**

Repeated Addition (length) Susan used 3 epics of ribbon each 4.6 metres long. How much ribbon did she use altogether?	**Factor (mixture)** A painter mixes a special colour by using 3.2 times as much red as yellow paint. How much red paint should the painter use with 4 litres of yellow paint?
Rate (speed) A girl walked at an average speed of 4 kilometres per hour for 3.2 hours. How far did she walk?	**Enlargement (factor/size)** A photograph is made bigger (enlarged) 3 times its size. If the length of the original photograph is 5.3 cm, what will be the length of the big photograph?

DISCUSSION OF RESULTS

Performance across the ten problem structures indicated in Table 3 revealed a 60-85% success rate with few differences found between the performance on multiplication and division problems. There was a marked improvement in performance on the Rate, Cartesian Product and Factor problems in comparison with results in Grades 2 and 3. However, performance on the Sub-division problem was very poor possibly due to the increase in difficulty of the number combinations used where children were required to divide 4 chocolates by 24 to gain 6 equal parts. They were unable to simply halve and rehalve (or double) the quantities as for the earlier interview.

Performance across many problem structures was *generally lower than in Grade 3 and much lower than expected for children entering secondary school.* Although number combinations were slightly more difficult, basic number fact knowledge at this level would be required for coping with a range of multiplicative problems. A preliminary analysis of solution strategies revealed widespread use of known and derived facts but counting and grouping methods previously used in Grades 2-3 were also used. *Further analysis of individual profiles showed that 32 % of the sample were unable to solve half of the initial ten word problems.* In some cases *performance had decreased since the earlier interviews* with evidence that the children were relying on rote learned multiplication tables that caused frequent error and no estimation of the answer. It appears that many of these children had stopped analysing the problems and focussed on manipulating numerical combinations. Use of immature and inadequate additive procedures dominated responses for the majority of these children.

Table 3Percentage of Correct Responses by Problem Structure and Grade Level (Large Number Problems)

PROBLEM STRUCTURE	Grade		
	2* n = 70	3 n = 60	6 n = 45
Multiplication			
1. Repeated Addition	27	80	81
2. Rate	72	98	86
3. Factor	0	47	67
4. Array	39	78	76
5. Cartesian Product	1	10	64
Division			
6. Partition	34	83	79
7. Rate	51	85	79
8. Factor	0	10	60
9. Quotition	34	73	67
10. Sub-division	10	43	26
Multiplication of Decimals			
Repeated Addition (4.6 x 3)			62
Rate (4 x 3.2)			43
Factor (3.2 x 4)			50
Enlargement (3 x 5.3)			81

* March Interview

Results for the four problems involving multiplication of a decimal number >1 revealed common difficulties associated with influence of semantic structure particularly with the Rate and Factor problems. However, difficulties with operating on decimals were revealed but the full impact of the 'multiplier effect' was not possible without using decimals less than 1. Conceptual misunderstandings related to decimal notation and operations on decimals were revealed in many of the responses even where the solution was correct. Few children were able to provide reasonable estimates of the solutions.

IMPLICATIONS FOR TEACHING

The 5 Year follow up study has provided valuable insights into the development of multiplication and division processes and the difficulties experienced by a significant proportion of children. On the basis of children's responses, the use of additive and estimation strategies, especially efficient use of multiple and group counting, might be a more effective way, initially, of teaching multiplication and division. The identification of young children unable to use these strategies and solve simple word problems is essential so that the progression to effective use of known facts will follow.

Teaching programs could incorporate the development of informal strategies rather than focussing only on mastering number facts and computational skills too early, that may not relate to the child's strategy development. Teachers could facilitate more meaningful learning by establishing links between children's intuitive strategies and the formal teaching of addition, subtraction, multiplication and division. Perhaps the teaching of these processes in an integrated fashion, and based on the child's experience of a range of related problem situations might best reflect the natural development of these processes. The relative difficulty of different problem structures and number combinations has been more clearly identified and thus, teachers could expose children to these with a better understanding of the relative ease or difficulty which children may encounter.

The difficulties experienced with operating on decimals warrants teaching emphasis on representations of decimals, solving a wide range of problem situations, attention to misconceptions about 'multiplication makes bigger', 'division makes smaller' and efficient counting, number fact and estimation strategies. Teachers may need to identify which 'competing' factors influence children's responses to multiplication and division tasks.

REFERENCES

Anghileri, J. (1989). An investigation of young children's understanding of multiplication. *Educational Studies in Mathematics,* 20 (4), 367-385.

Bell, A., Fischbein, E., & Greer, B. (1984). Choice of operation in verbal arithmetic problems: the effects of number size, problem structure and context. *Educational Studies in Mathematics,* 15, 129-147.

Bell, A., Greer, B., Grimison L., & Mangan, C. (1989). Children's performance on multiplicative word problems : Elements of a descriptive theory. *Journal for Research in Mathematics Education,* 20 (5), 434 - 449.

Bell, A., Swan, M.& Taylor, G. (1981). Choice of operation in verbal problems with decimals. *Educational Studies in Mathematics,* 12, 399-420.

Boero, P., Ferrari, P., & Ferrero, E. (1989). Division problems: Meanings & procedures in the transition to a written algorithm. *For the Learning of Mathematics,* 9 (3), 17-25.

Booth, L. (1981). Child-methods in secondary mathematics. *Educational Studies in Mathematics,* 12, 29-41.

Booth, L. (1984) *Algebra: Children's strategies and Errors.* Windsor: NFER-Nelson

Brekke, G., and Bell, A.: 1992, 'Multiplicative structures at ages seven to eleven', In W. Geeslin and K. Graham (ed.), Proceedings of the Sixteenth Conference of the International Group for the Psychology of Mathematics Education, Program Committee of the 16th PME Conference, New Hampshire, USA, 1, 97-105.

Brown, M.L. (1981). Number operations. In K.M.Hart (Ed.), *Children Understanding Mathematics : 11-16.* London: John Murray.

Carpenter, T.P., & Moser, J.M. (1984). The acquisition of addition and subtraction concepts in grades one through three. *Journal for Research in Mathematics Education,* 15 (3), 179-203.

De Corte, E. & Verschaffel, L. (1987). The effect of semantic structure on first grader's strategies for solving addition and subtraction word problems. *Journal for Research in Mathematics Education,* 18 (5), 363-381.

De Corte, E., Verschaffel, L., & Van Coillie, V. (1988). Influence of problem structure, and response mode on children's solutions of multiplication word problems. *Journal of Mathematical Behaviour,* 7,97-216.

Fischbein, E., Deri, M., Nello, M.S., & Merino, M.S. (1985). The role of implicit models in solving verbal problems in multiplication and division. *Journal for Research in Mathematics Education,* 16 (1), 3-17.

Fischbein, E. (1989). Tacit Models and Mathematical Reasoning', *For the Learning of Mathematics,* 9, (2), 9-14.

Hart, K.M. (Ed.). (1981). *Children's understanding of mathematics: 11-16.* London: John Murray.

Hiebert, J. (1984). Children's mathematics learning: The struggle to link form and understanding. *The Elementary School Journal,* 5, 497-513.

Hiebert, J. (1990). The role of routine procedures in the development of mathematical competence. In T.J. Cooney & C.R. Hirsch (Eds.), *Teaching and learning mathematics in the 1990's,* 31-41, Reston, Virginia: N.C.T.M.

Hughes, M. (1986). *Children and number.* Oxford: Basil Blackwell.

Kouba, V.L. (1989). Children's solution strategies for equivalent set multiplication and division word problems. *Journal for Research in Mathematics Education,* 20 (2), 147-158.

Mulligan, J.T. (1991) The role of intuitive models in young children's solutions to multiplication and division word problems. Paper presented to the 15th Annual Conference of the Mathematics Education Research Group of Australasia, Perth.

Mulligan, J.T. (1992). Children's solutions to multiplication and division word problems: a longitudinal study, *Mathematics Education Research Journal*, 4, (1), 24-42

Murray, H., Olivier, A., and Human, P.: 1992, 'The development of young children's division strategies', In W. Geeslin and K. Graham (ed.), Proceedings of the Sixteenth Conference of the International Group for the Psychology of Mathematics Education, Program Committee of the 16 th Annual Conference, New Hampshire, USA, 2, 252-260.

Nesher, P. (1988). Multiplicative school word problems: Theoretical approaches and empirical findings. In J. Hiebert & M. Behr, (Eds.), *Number concepts and operations in the middle grades* , 2, 19-40, Hillsdale, N.J: Erlbaum.

Steffe, L. P. (1988). Children's construction of number sequences and multiplying schemes. In J. Hiebert & M. Behr, (Eds.), *Number concepts and operations in the middle grades,* 2, 119-140, Reston, Virginia: Erlbaum/ NCTM.

Steffe, L.P., Cobb, P., & Richards, V. (1988). *Construction of arithmetic meanings and strategies*. New York: Springer-Verlag.

Steffe, L.P., & Wood, T. (1990). *Transforming children's mathematics education: International Perspectives.* Hillsdale, New Jersey: Erlbaum.

Vergnaud, G. (1988). Multiplicative Structures. In J. Hiebert & M.Behr (Eds.), *Number concepts and operations in the middle grades.* 2, 141-161, Reston,Virginia: Erlbaum/N.C.T.M.

INTENSIVE MATHEMATICS INSTRUCTION FOR MATHEMATICS DISABLED STUDENTS : THE MATHEMATICS LEARNING CENTRE APPROACH

JOHN MUNRO

Access to effective learning opportunity for students who have mathematics learning disabilities is an ongoing concern for teachers, students, educational policy makers and the community at large. The types of opportunity offered range from the traditional "clinic" approach (for example, as described by Englehardt (1985), Irvin and Lynch-Brown (1988) and Scheer and Henniger (1982)) to the delivery of support within the student's classroom, often via teacher in-service. The Mathematics Learning Centre (MLC), developed in the former Melbourne College of Advanced Education lies between these two extremes in its approach to opportunity provision. This paper reports a recent evaluation of the effectiveness of the service delivery in the MLC. It examines the guiding model of mathematics learning on which the program is based and the evaluation procedures used to measure change.

Any educational practice makes assumptions about issues such as preferred outcomes, learner characteristics and the nature of learning learning process. These drive the implementation of the practice and influence its evaluation. The first assumption made by the model relates to the desirable outcomes of mathematics learning. The model identifies outcomes in the following areas: (1) mathematics content: conceptual and procedural knowledge, the use of formal mathematical symbolism and the automatization of parts of this knowledge and mathematical thinking, (2) a knowledge of how to learn mathematics, (3) attitudes towards mathematics, how it is learnt and one's self as a mathematics learner. These aspects are assumed to operate interactively in the mathematics learning. Students usually acquire outcomes (2) and (3) incidentally and spontaneously. In the present context these outcomes are targeted directly both in the teaching program and in its evaluation because, as will be discussed later, many mathematics underachievers have difficulty acquiring them in this way.

The second assumption relates to how mathematics ideas are learnt. The model assumes a constructivist view of mathematics learning (for example, Cobb, 1986; Steffe, 1990). The construction process involves student action and the investment of attention. Attention is more likely to be invested in learning when students have framed up for themselves purposes or challenges for learning. They are assumed to have a range of representational formats in which they can build these ideas. Their activity in the construction process can be described in part in terms of mathematics learning strategies; both mathematics-relevant reasoning strategies and the self-instruction strategies used to direct and manage the reasoning strategies. These strategies are used spontaneously and selectively. Regular successful use may lead to a strategy gradually becoming automatized. The ideas are built gradually through an hypothesis production, trialling and modification process. With an emphasis on "partial construction" the concept of failure is inappropriate. Errors are seen as signalling the need for further work on an idea. Belief in one's ability to learn is important; students who don't believe that they can learn or who believe that others don't expect them to learn are less likely to build.

These assumptions have implications both for the approach to teaching and for the evaluation of the MLC program. Attitudes about how matematics ideas are learnt, their use of mathematics learning strategies, their knowledge of what they can do when they find learning difficult are desired outcomes of the approach. Pupils are encouraged see how their purposes have been partially achieved through partial constructions. As discussed in the following section, many mathematics-disabled learners do not acquire these incidentally.

The learning characteristics of mathematics disabled students. Four approaches to the description of mathematics underachievement are discernible in the literature (Munro, 1987); the psychological descriptive approach, focussing on psychological difficulties that co-occur with mathematical difficulties, the error analysis approach focussing on the types of errors made by students, the neuropsychological approach that relates mathematics underachievement to neurological disorders and the information - processing approach, focussing on the relationship between mathematics difficulties and information processing. These approaches focus on students in isolation rather than on students in relation to their mathematics educational history. They rarely refer to the characteristics of the instruction to which the students have been exposed.

The present approach assumes that a student's mathematics performance at any time is a function of the interaction between the student 's preferred ways of learning and the assumptions made by the learning environment about how students learn. Students differ in their preferred ways of learning. Mathematics learning situations differ in the demands they make on how students learn. Mathematics underachievement occurs when there is a significant mis-match between how students prefer to learn and the demands made by the learning environment. Transitory mismatches may be expected as a regular component of any learning situation and are resolved as students develop new learning strategies and as the learning situation adapts to match student learning styles. The greater and more chronic the mis-match, the more likely is mathematics learning disability. The present approach assumes that mathematics underachievement can be more satisfactorily be reduced by focussing on the student-learning environment interaction than by examining intra-learner deficits.

A framework for analysing the types of mis-match that lead to mathematics underachievement is described in Munro (1992). In summary, frequently occurring assumptions that mathematics curriculum make about preferred ways of learning include the following

(1) The learning environment assumes that students can manipulate and process particular types of information in various ways. Mathematics disabled students may have difficulty processing quantitative data, visually-presented symbolic data or spatial symbolic data (Kosc, 1986), arithmetic operations mentally (Rourke & Strang, 1983) or related verbal information (Share, Moffitt & Silva, 1988). They may not be able to handle all of the information necessary for completing a mathematical task or to recall information from long-term memory.

(2) The learning environment assumes that students can use spontaneously a range of general-purpose learning strategies. Learning disabled students are characterized as non-strategic learners (Torgesen, 1980), less likely to activate spontaneously the range of strategies necessary for learning (for a review, see Flood and Lapp, 1990).

(3) The learning environment frequently assumes that in the acquisition of a idea, students can allocate most of their mental resources to building this idea, that is, they can manipulate subordinate ideas in a relatively attention-free way (Ackerman, Anhalt & Dykman, 1986). Mathematics underachievers have difficulty meeting this demand, for example, in automatizing "basic number facts" (Fleischman, Garrett & Shepard, 1982). The issue of automaticity can be applied to the use of mathematics learning strategies.

(4) The learning environment assumes that the students believe that they can learn mathematics and are motivated. Self-confidence in learning mathematics and achievement are moderately correlated. Self-confidence is determined in part by what students tell themselves about success and failure. Students who lack self-confidence frequently believe that success is beyond their control, failure is inevidable and that effort is useless because it will probably not lead to success (Kloosterman, 1988).

Thus, while the assumptions about learning made by mathematics curricula are valid for most students, they can be shown to be less valid for underachievers. Teachers working with mathematics-disabled students and the students themselves need to be aware of these assumptions and the steps that can be taken when they are not justified.

Who are the students for whom intensive mathematics instruction is most appropriate? The group of students who have had chronic difficulty benefiting from regular mathematics teaching comprises two sub-groups; those who can be assisted within their regular classroom by relatively minor curriculum modifications and those who don't know how to go about learning mathematics and who present as innumerate. These latter students need the opportunity to learn how to learn mathematics, to experiment and to take risks and to see themselves as able to learn mathematics. This "opportunity to re-learn" can best be provided in an intensive strategies context. In psychometric terms, students in the first sub-group may achieve at stanines 3 and 4 on normal mathematics tests and students in the second sub-group at stanines 1 and 2 (Pickering, Szaday & Duerdoth (1988).

The present investigation examines the effectiveness of the M L C model as a means of service delivery to mathematics-disabled learners in the second sub-group. It examines the extent to which disabled mathematics learners are assisted to integrate themselves gradually into regular classroom mathematics learning by helping them to: increase their mathematics knowledge, their repertoire and use of mathematics learning strategies and their perceptions of mathematics, how it is learnt and themselves as mathematics learners.

METHOD

Subjects : 76 students aged from 6 years to 14 years (median age ranged 10-5 years) are from grades 2 to 9. All were achieving at a mathematics level that was at least 2 years below their grade level and all met accepted mathematics disability criteria (Pickering, Szaday & Duerdoth, 1988).

Design : The effectiveness of the program is measured in terms of changes in mathematics knowledge, attitudes and mathematics learning strategies on an individual basis. The extensive use of normative - summative procedures is not seen as appropriate. Subjects differed on a range of criteria. The design focusses on the monitoring of change in learning behaviour and the reporting of change on several dimensions on an individual basis using mathematics tasks, behavioural checklists and subject reporting.

Materials : The test used to screen mathematics performance is the Diagnostic Mathematics Tasks (DMT) (Schleiger, 1980). Given the nature of students' level of performance, the DMT two grade levels below the grade level of a particular student was used initially to screen performance.

Procedure : The MLC teaching program is administered in several components;

(1) a 'teacher preparation' component in which teachers examine the nature, assessment and remediation of mathematics learning disabilities implemented in lecture-workshop context.

(2) a diagnostic - assessment component in which each teacher completes an initial evaluation of one student's mathematics learning needs by collating information from several sources: (a) the pupil's mathematics performance on reasonable mathematics tasks, error analysed in a clinical interview to identify the conditions under which the student can complete or self-correct tasks (b) the pupil's affective behaviours while working on mathematics tasks are monitored using a behaviour rating scale) (c) the pupil's mathematics learning in the regular classroom context, collated using a checklist to tap the classroom teacher's perception of the pupil's mathematical difficulties, (d) the pupil's mathematics learning at home and (e) general referral information, such as possible causes, correlates of the pupil's learning difficulty, learning difficulties and strengths in other areas and relevant family information.

(3) a planning component in which the student and teacher plan and negotiate an initial set of goals and program. The teacher's impression of the nature of the mathematics learning disability is discussed with the student, a consensus is reached about how the student can learn best and a program is negotiated. The focus here is "What would you like to learn in mathematics?" The teacher plans a teaching-learning program and discusses this with the group of Centre teachers, the pupil's classroom teacher and parents.

(4) a teaching component in which the teacher works with the student for 10 to 15 11/2 hour sessions under the supervision of a master-teacher skilled in the study of mathematics learning disabilities. To facilitate the pupil's gradual integration into regular mathematics lessons, the mathematics content studied where possible is that being taught in the student's regular class. -This frequently involves assisting the student to acquire prerequisite knowledge. Each session is reviewed and evaluated by the teacher. Pupils monitor their progress and to complete regular homework tasks. Throughout the program, communication with the pupil's parent and class teacher is maintained using a journal. Wherever possible the location of the program is gradually moved to the pupil's scho

(5) Each pupil's performance on each of the following criteria is re-assessed at session 11; (1) achievement, (based on the pupil's entry DMT and the proportion of set goals achieved, (2) frequency of use of each type of learning strategy, (3) attitude towards mathematics, (4) display of mathematics learning behaviours in the pupil's classroom and (5) parent perception of mathematics learning. A % gain score was calculated for each student (% gain score = change in score / total number of items x 100) and a median score for the group.

RESULTS AND DISCUSSION

The median % gain on each of the evaluation criteria across grade level and content area are shown in Table 1.

Table 1

Demonstrated gains in mathematics performance

Criterion	Assessment Procedure Used	Median % Gain
Achievement	1. Entry DMT.	58%
	2. Proportion of set goals achieved by Session 11	74%
Use of Learning Strategies	Frequency of using each type.	71%
Attitude Towards Mathematics	Frequency of positive-rated responses to attitude scale (42 item questionnaire).	83%
Performance in Child's Classroom	Frequency of positively rated behaviours (20 item behavioural rating scale).	68%
Parent Perception	Frequency of positively rated behaviours (using 24 item behavioural rating scales)	79%

These data show a substantial gain on all criteria. They indicate little, however, about the value of the program for individual students and the extent of individual change. As well, this analysis does not permit analysis of the precise factors that are most influential in bringing about the observed group changes. Individual reporting by students (several of whom had been involved unsuccessfully in earlier programs) identified most frequently the following as the most valuable aspects :

(1) the emphasis on 'having another go', being able to ask questions and to say "I don't know", being able to try things out without having to worry about avoiding errors,

(2) the emphasis on 'things to do' to learn matematics, for example, "telling yourself what to do / to use what you already know", using mental visualization and imagery,

(3) being able to learn things in different ways,

(4) being able to do things better in mathematics classes at school, being able to "show my classmates that I can learn maths",

(5) the emphasis on "knowing what works best for you",

(6) knowing what you can do to learn a difficult idea, knowing that although "something might be hard, you might know something about it a bit later", knowing what can make an idea hard to learn and possible things to do about it, and

(7) "planning my own program, seeing myself learning / making progress ", "understanding why I used to find maths hard to learn.

To examine the extent to which the gains made by session 11 were sustained six months later, 46 of the students were examined on a number of criteria. These criteria and outcomes were:

1) How valuable / effective / useful did the student judge the program ? A 20 item questionnaire with a 5-point rating scale for each item was used. The mean rating was between "good" and "very good".

2) The spontaneous use of relevant strategies when working through mathematics tasks was monitored by having students "thinking aloud"; 71% of the students used the appropriate strategies on at least 80% of possible occasions

3) The maintenance of a positive attitude towards mathematics; the median % of positive responses on the attitudinal questionnaire was 89% (range 82% - 90%).

5) Classroom teachers rating of pupil's current progress; (for example, whether the student continue to be engaged in mathematics lessons, takes learning risks spontaneously, attempts to solve problems without seeking assistance); 86% of the students were rated as making at least adequate progress .

6) Parent rating of the success or value of the program; 93% rated the program as having been of substantial benefit for their child and supported their ratings with a range of positive behavioural statements (for example, students attributing their current progress to their involvement in the program).

This evaluation supports the claim that the MLC program has been successful intervening in the mathematics learning of a group of students, who, at the beginning of the program, was seen as having severe mathematics learning difficulties. The gains in achievement, attitudes and mathematics learning strategies were shown to be sustained six months after the conclusion of the program. The individual gains made support the efficacy of the model.

The original aim in developing the MLC was to implement an evolving facility by which several purposes could be achieved simultaneously within a structure that had a sound mathematics learning base, that was optimally flexible and versatile, that could respond to changing needs and that was "resource-lean" (in other words, that made maximum use of existing resources). Several educational institutions in Victoria have adopted aspects of its model for the development of parallel facilities. The MLC continues to evolve. Parallel evaluations have examined its effectiveness as a teacher training facility. Future evaluations may be expected to target changing directions in the provision of services for mathematics disabled students.

REFERENCES

Ackerman, PT; Anhalt, JM & Dykman, RA. (1986) Arithmetic automatization failure in children with attention and reading disorders: Associations and sequela. Journal of Learning Disabilities, 19, 4, 222-232.

Cobb, P. (1986) Making mathematics: Children's learning and the constructivist tradition. Harvard Educational Review, 56, 3, 301-306.

Englehardt, JM. (1985) Math clinic purposes: organisation or values. Focus on Learning Problems in Mathematics. Spring, 7, 2, 41-63.

Fleischner, JE; Garnett, K. & Shepard, MJ. (1982) Proficiency in arithmetic basic facts computation of learning disabled and non-disabled children. Focus on learning problems in mathematics, 4, 47-56.

Flood, J & Lapp, D. (1990) Reading comprehension instruction for at-risk students: Research-based practices that can make a difference. Journal of Reading, 33, 7, 490-496.

Irvin, JL & Lynch-Brown, C. (1988) a national survey of US university reading clinics. Journal of Reading, February, 436-447.

Kloosterman, P. (1988) Self-confidence and motivation in mathematics. Journal of Educational Psychology, 80, 3, 345-351.

Kosc, L. (1986) Basic forms of dyscalculia. Focus on Learning Problems in Mathematics, 8, 3&4, 55-62.

Munro, J. (1992) Learning strategies : An approach to helping children learn mathematics. Working paper, Ministry of Education, Victoria.

Munro, J. (1987). Approaches to describing learning disabilities in mathematics. Success in learning mathematics, 2, 2, 18-24.

Pickering, D., Szaday, C., & Duerdoth, P (1988). One in eleven. Melbourne: VICCTRAC.

Rourke, B & Shang, J. (1983) Subtypes of reading and arithmetic disabilities: a neuropyschological analysis. In M Rutter (Ed.) Developmental neuropsychiatry. New York: Guilford Press.

Scheer, JK & Henniger, MT. (1982) Math Clinic: an ideal setting for parental involvement. Arithmetic Teacher, October, 48-51.

Schleiger, H.E. (1980). Diagnostic Mathematics Tasks (DMT) V C Press : Burwood.

Steffe, LP. (1990) Mathematics curriculum design: A constructivist perspective. In LP Steffe & T Woods (Eds). Transforming children's mathematics education: International perspectives. Hillsdale, NJ: L Erlbaum Association.

Torgesen, JK. (1980) Conceptual and educational implications of the use of efficient task strategies by learning disabled children. Journal of Learning Disabilities, 13, 364-371.

STRATEGY TRAINING AND MATHEMATICS LEARNING DISABILITIES

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This paper reports two investigations in which mathematics underachievers were taught to use cognitive and metacognitive strategies to facilitate information processing in two areas of mathematics learning: to read symbolic statements and to categorize their mathematics knowledge. One group of students learnt to use the relevant cognitive strategy while a matched group learnt, as well, associated metacognitive strategies (to evaluate the effectiveness of strategy use and to decide when they might use the strategy in the future). Both groups out-achieved a control group immediately after teaching. As well, the group taught both cognitive and metacognitive strategies were more likely to transfer the cognitive strategy to other areas of mathematics. The results are discussed in terms of models of human cognition and performance.

When solving a mathematics task, successful students engage spontaneously in various strategic activities (Cardelle-Elawar, 1992); they (1) sample the data defining the task in order to inform themselves of its nature, (2) relate the tasks to types that they have learnt previously, (3) plan a solution strategy and select from long term memory procedures that they apply to components of the data and (4) evaluate their solution and re-run some of these strategies if necessary. These four types of activities have been referred to as orientation, organization, execution and verification respectively (Cardelle-Elawar, 1992). They assume that students use their existing knowledge to make decisions about the nature of the task, about the data that may be relevant in classifying it and about the nature of the solution, whether it is reasonable, etc. Each area of activity involves two types of strategy; cognitive and metacognitive. The cognitive strategies involve manipulating ideas in various ways. The metacognitive strategies involve the planning, management and monitoring of cognitive strategy use (Haller, Child & Walberg, 1988). While the distinction between cognition and metacognition is frequently unclear , it permits a distinction between aspects of thinking during mathematics learning. Analyses of the mathematics performance of mathematics disabled students suggest difficulties in the spontaneous use of both types of strategies (Torgesen, 1980). Teaching them to use self-instruction strategies has led to improvement in solving arithmetic word problems (Montague & Bos 1986; Fleischner, Nusum & Marzola, 1987). Fewer studies have examined their use in algorithmic learning.

While they can frequently learn the steps comprising an algorithm, (that is, execution processes (Ackerman, Anhalt & Dykman, 1986)), these students have difficulty using this information on subsequent occasions. Arithmetic disabled students are less likely than their able peers to make spontaneously classificatory statements such "Oh, it's one of those" or "Is it like ...?" to make such statements. They are less likely to use their existing knowledge to organize the ideas that they are learning or to modify this knowledge in various ways. Effective algorithmic learning involves meaningful reading of symbolic statements, organizing the ideas learnt into semantic categories and using effective retrieval strategies. Error pattern analysis suggests difficulties comprehending and classifying tasks and selecting appropriate procedures. These students differ from their able peers in the features of tasks that they select to categorize them; they tend to use individual features while able students categorize using more general features and properties. They also have difficulty discriminating between relevant and irrelevant data. These difficulties may be associated with the use of search processes for locating and retrieving information in long term memory (Swanson & Rhine, 1985). These strategies are rarely taught directly. Most pupils learn them incidentally by trying out various "thinking actions".

One of the difficulties with studying strategy use is the extent to which it can be monitored. The technique of instructing pupils to "think aloud", that is to verbalize as they manipulate data, can interfere with learning. An alternative technique involves observing the effect of strategy teaching on subsequent performance. Teaching cognitive and metacognitive strategies as a way of ameliorating learning disabilities has been examined, particularly for word problem solution. The success of this teaching has been variable. The teaching of these strategies generally proceeds in the following sequence (Pearson & Dole, 1987); (1) the strategy to be learnt is demonstrated or explained, (the cognitive modelling phase), (2) teacher and pupils work together to apply the strategy (the guided practise phase), (3) pupils take more control of strategy use and make decisions about when and why they might use it in the future, (the overt self-guidance phase), (4) pupils independently use the strategy (the or faded self guidance phase) and (5) students apply the strategy in a range of contexts (the application phase). This phase is frequently omitted (Pearson and Dole , 1987). There are at least two possible reasons why some earlier strategy training has not been effective with learning disabled students; (1) it did not encourage students to evaluate the effectiveness of the teaching for themselves at an early learning stage (that is, develop metacognitive knowledge) and (2) it did not taken account of preferred ways of learning of individual students.

The present experiments compared learning under this teaching programme with one in which pupils evaluated the strategies that they are using and had the opportunity to accept or to reject use of them. Phase 1 was preceded by two others; one in which the need to learn new strategies as a way of overcoming existing difficulties was recognized by pupils and one in which each student selected the preferred reading strategy. The strategy was introduced 'something that some people do to make the task easier'. Pupils were invited to try it and then evaluate its value for themselves. In addition to the conventional activities at each phase (Pearson & Dole, 1987), they asked questions such as "Does it seem to work for me ? Why ? Does it help me when I do maths ? What can I do that I couldn't do earlier ? When will I use it in the future ?"

The effectiveness of strategy teaching was examined for two aspects of using one's existing knowledge in learning algorithms ; reading meaningfully symbolic statements and categorizing knowledge of mathematics procedures.

EXPERIMENT 1 : READING SYMBOLIC MATHEMATICS STATEMENTS

Meaningful reading involves constructing an impression of the intention coded in a written statement. Readers do this by making use of what they already know. Consider the mathematical statement $2x + 3 = 19$. The coded idea be represented mentally in different ways (Gardner, 1985), Some readers may prefer to represent ideas learnt in terms of visual imagery , some may demonstrate a verbal preference, tapping into one's linguistic knowledge and verbal coding systems, some may tap into their propositional logical knowledge and others may attempt to represent the ideas kinaesthetically. Once the statement has been represented, it is available for execution processes. In other words, students can use a range of cognitive strategies to read meaningfully a symbolic statement (they can visualize or verbalize it or represent it as a series of actions), particularly if it involves small numbers, that is, numbers that make comparatively low demand on available attentional resources for their representation (Munro, 1991). In this experiment the reading strategy taught to each student was linked with that student's preferred way of learning The learning preference was identified by providing students with a range of representational formats and having them select their preferred mode. The teaching was intended to increase students' awareness of how they could use these preferences in the context of mathematics. The present discussion did not examine issues associated with preferred ways of knowing, such as the stability of preference for a particular mode of representation.

In this experiment mathematics- disabled pupils selected their preferred representational format for mathematics and learnt to use the associated cognitive encoding strategy with mathematics tasks that they had found difficult. A second group, in addition, engaged in concurrent metacognitive activity. This design permitted an investigation of the comparative influences of cognitive and metacognitive knowledge on algorithmic learning. As well, differences in retention and transfer under the two conditions were monitored.

Method

Subjects. The subjects were 48 mathematics disabled grade 8 and 19 pupils who met accepted mathematics disability criteria (Pickering, Szaday & Duerdoth, 1988).

Experimental design. The pupils were arranged into groups of three, matched on mathematics task ability, general learning ability and grade level. The mathematics tasks to which the pupils applied the reading strategy were simple linear equations typical of middle secondary mathematics courses. All pupils had, within two weeks prior to the strategy teaching programme, completed this area of study and had achieved less than 10 % accuracy in equation solution. Two members of each group were allocated to two strategy learning groups and the third to a control group. Prior to the allocation, all pupils completed an intensive arithmetic computations unit on which they achieved at least 95 % competency on computations involving at least two operations. They were permitted to use a calculator. The pupils in the strategy learning groups learnt the strategies under one of two conditions; a conventional strategy learning condition (strategy learning; Pearson & Dole, 1987) and the modified condition that included a greater emphasis on self-evaluation of strategy effectiveness (strategy learning + evaluation). The training programmes were administered individually for each student over five or six thirty-minute sessions in the pupil's school. The pupils in the control group continued to solve typical linear equations for the five sessions.

Each pupil's task performance was measured before the beginning of the strategy learning programme, at the end of each session, at the conclusion of the programme and four months after the conclusion of the programme. A priori comparisons of means were made using linear contrast techniques (Howell, 1992). This provided the basis for comparing the maintenance and retention of the strategies. As well, the ability of students to describe the strategies that they had learnt was monitored at the beginning of sessions 2 to 6.

Procedure : The two teaching sequences described above were implemented. Students selected their preferred reading strategy by being cued to read a linear equation under each of the verbalize, visualize interpret as a actions conditions. The instructions for each training condition were as follows;

(1) for the verbalize condition the task was introduced by the experimenter as follows "One way in which I do these problems is to tell myself what they say. I listen to myself as I say it. I say this ($2x + 3 = 19$) as "two times a certain number add three is equal to nineteen. I try to say it the way I talk".

(2) for the visualize condition the task was introduced by the experimenter as follows "One way I do this is to make a picture in my mind of what the equation says. For $2x + 3 = 19$ the picture is two bags of bolts and three more bolts is equal to 19 bolts. How much is in each bag ? This is how I start off".

(3) for the action condition the task was introduced by the experimenter as follows "One way I do this is to think of the actions that it says. For $2x + 3 = 19$ I think of the actions. First I begin with a certain number, then I multiply it by two and add three. I end up with 19. What is the number ?

Following each condition the experimenter demonstrated the strategy for an equation and the student tried out the strategy with at least one equation. Students selected their preferred way of representing the mathematical statement and engaged individually in the relevant strategy teaching programme. The learning phases were spread over five or six 30 minute sessions. The students in the control group continued to solve linear equations in their regular classroom context for the six sessions.

Results

Mean task performance (proportion of correct task solutions) prior to teaching and at the end of each session are shown in Table 1. Both teaching conditions, but not the control condition, were associated with an improvement in accurate task solution ($p < .01$). Although the strategy learning + evaluation condition achieved a higher level of performance than the strategy learning condition for the final session, this difference was not significant ($p > .01$). The strategy learning + evaluation condition differed from the strategy learning condition in that it was associated with a more rapid learning of the strategy (that is, more rapid improvement to the ceiling level) and more sustained use of it than the strategy learning condition.

Table 1 : Mean task performance at the end of each session.

Condition	Pre-teach	Sess1	Sess2	Sess3	Sess4	Sess5	Sess6	Later
strategy	0.17	0.24	0.37	0.62	0.71	0.74	0.73	0.72
strategy + evaluation)	.18	0.21	0.54	0.76	0.87	0.94	0.93	0.95
control	0.20	0.28	0.33	0.26	0.38	0.34	0.23	0.37

The more rapid learning and use of the strategies under the teaching + evaluation condition is not surprising. When a strategy is observed to be assisting the solution of problems, it is likely be learnt more quickly. The focus on the effectiveness of a particular procedure is necessary for learning disabled pupils operating as non-spontaneous learners; these students are less likely to relate the actions taken and their value without focussing directly on the connection between them.

The retention and use of the strategies on a long-term basis was examined by monitoring strategy use sixteen weeks after the conclusion of the strategy learning both on solving equations and in the content learnt at that time. Both treatment groups demonstrated superior comprehension performance over the control group ($p < .01$). In terms of strategy transfer, when given a new task, the strategy learning + evaluation group was more likely than the strategy learning group to report using the particular reading strategy for new information ($p < .05$). Encouraging pupils to monitor and evaluate the use of strategies is linked with a greater preparedness to experiment with the strategies in unfamiliar contexts and to transfer their knowledge.

Experiment 2 : Categorization strategies

The second investigation examined the effect of increasing the accessibility of mathematics knowledge. In particular it examined the effects of assisting disabled students to organize their mathematics knowledge into functional categories and to remind themselves to access this knowledge. The tendency of mathematics-disabled students to classify tasks on individual perceptual characteristics rather than on more conceptual criteria has already been noted. These students are more likely to classify tasks such as

$$\begin{array}{r} 72 \\ -\ \underline{48} \end{array} \quad \text{and} \quad \begin{array}{r} 76 \\ -\ \underline{42} \end{array} \quad \text{or} \quad 1/6 + 1/6 = \quad \text{and}$$

$1/3 + 1/2 =$ as meriting the same procedures. In this experiment students, after mastering the steps in a mathematics procedure and applying the procedure in isolation to a criterion level of acceptable performance, learnt to discriminate between tasks that had the same operational symbol and that shared superficial features and to categorize them in terms of the procedure used to solve them. The study compared the effectiveness of cognitive strategy teaching with combined cognitive and metacognitive teaching. One group of students learnt to categorize instances of the tasks (the cognitive strategy group) while a second group learnt as well to instruct themselves to classify tasks on a subsequent occasion (the metacognitive group). When given a set of mixed computational tasks, the students asked themselves : "What does the task remind me of? What is it like that I have already learnt ? What did I do in this type of problem ?"

Method

The subjects were 30 third grade and 30 sixth grade mathematics-disabled students who met accepted mathematics disability criteria (Pickering, Szaday & Duerdoth, 1988).

Experimental design. The pupils were arranged into groups of three, matched on mathematics task ability, general learning ability and grade level. The mathematics tasks to which the pupils applied the classification strategy were either the subtraction of whole numbers tasks (third grade) or the addition of fractions tasks (sixth grade) described above. The design was similar to that used in Experiment 1. Prior to allocation, all pupils completed an arithmetic computations unit to at least 95 % competency on related computations. They were permitted to use a calculator. Two members of each group were allocated to two teaching groups and the third to a control group. One teaching group received classification teaching (the strategy group) while the second group received, in addition, self-instruction teaching (the metacognitive group). A priori comparisons of means were made using linear contrast techniques (Howell, 1992).

Procedure. The classification teaching was implemented on an individual basis as follows. Two instances of each type of task were written on a card and the student discussed differences between them. In the case of the subtraction tasks, the third grade students noted that both involved subtraction but for one type they needed to do something else before they subtracted; they had to "get it ready". They also discussed the 'getting ready' procedure. The four cards were sorted into two groups and the students suggested names for each (such as 'ready / not ready to take away'). The students sorted physically 10 tasks, written one per card into these categories. They were given a set of ten written tasks and categorized each by saying either "Ready / Not ready to take away". The self-instruction group, following classification teaching, were shown sets of randomly mixed tasks and learnt to ask themselves, for each task "What does this remind me of ? What is it like that I have already learnt ? " The learning phases took either five or six 30 minute sessions. The control group completed mixed sets of tasks and received corrective feedback. Task completion and classification ability were measured at the beginning and end of the strategy teaching program and four months after the conclusion of the program. As well, the nature of error patterns and the transfer of learning to other areas was monitored.

Results

Mean task performance (proportion of correct responses) of each group prior to teaching , at the end of teaching and four months later (later) are shown in Table 2.

Table 2 Mean task performance for each group

Performance	Strategy teaching group			Self-instruction group			Control group		
criterion	Start	End	Later	Start	End	Later	Start	End	Later
Task completion	0.18	0.43	0.62	0.14	0.95	0.94	0.08	0.13	0.17
Classifications	0.07	0.73	0.85	0.09	0.91	0.96	0.04	0.09	0.08

The three groups did not differ on either criterion prior to teaching (planned comparisons, $p > .05$). Both teaching conditions, but not the control condition, were associated with an improvement in accurate task solution, with the self-instruction condition achieving significance ($p < .01$). The retention and use of the strategies on a long-term basis was examined by monitoring the ability to solve mixed sets of tasks sixteen weeks after the conclusion of the strategy teaching. Both treatment groups demonstrated superior comprehension performance over the control group ($p < .01$) and the self-instruction group out-performed the strategy learning group ($p < .01$). In terms of strategy transfer, when given an unfamiliar type of mathematics task, the self-instruction group was more likely than the other groups to report attempting to categorize and to use what was already known. These findings support the importance of both cognitive and metacognitive strategy teaching. Teaching these students how to categorize was insufficient; it was necessary as well to teach self-instruction strategies that facilitate access to this knowledge.

Discussion

The findings of the present study support the claim that mathematics disabled students do not use spontaneously a range of cognitive and metacognitive strategies for processing mathematical data. The superior performance under the metacognitive condition indicates the need for both conditions in mathematics teaching. Why do mathematics-disabled students have difficulty learning to use these types of strategies, for example, to make use of what they already know ? Failure to use effective cognitive strategies has been attributed to inadequate metacognition (Cardelle-Elawar, 1992). Particular cognitive strategies are often used inflexibly and without selection according to the task at hand. Students may have access to strategies but don't use them spontaneously.

A possible explanation for the lack of spontaneous use of the most appropriate strategies and for the effectiveness of the type of strategy teaching described here may lie in the allocation of attention during task completion. These resources need to be invested in those processes not automatized. The allocation of attention and the executive component of metacognition are related. The importance of automatizing aspects of mathematics knowledge, so that these can be manipulated without the investment of mental attentional process for subsequent mathematics learning has been noted by several investigators (Ackerman, Anhalt & Dykman, 1986). Mathematics underachievers have difficulty meeting this demand particularly for the manipulation of "basic number facts" (Fleischman, Garrett & Shepard, 1982).

It is reasonable to expect that the issue of automaticity can be applied to strategy use during mathematics; any cognitive or metacognitive skill may demand attentional investment. The student who needs to invest these resources in processes that peers implement relatively automatically, may have proportionately less to allocate to 'building the new idea'. Increasing use of strategies is likely to lead to their attaining automaticity. Thus, pupils who continue to allocate a disproportionate amount of their attention to the manipulation of subordinate mathematical ideas may have less to allocate to the use of particular strategies and the opportunity to monitor their effectiveness. From this perspective, the effectiveness of the strategy teaching is not surprising; it teaches students to allocate their attention to essential aspects of data in a systematic way.

The effective allocation of attention can provide a basis for the restructuring of knowledge stored in long term memory. The students learning to read algebraic statements may have build 'templates' of these statements that they could apply to other instances. These templates may have permitted the students to process greater amounts of information at once. In a corresponding way, teaching these students to categorize what they already knew may have assisted them to build better-defined categories in verbal semantic memory and to access these. Rabinowitz's (1988) observations in relation to seeing strategies as isolated entities are relevant here; strategies are not used in isolation but rather are anchored in particular knowledge domains. The use of any strategy is related to the individual's ability to access related domain-specific knowledge, as well as students' perceptions of these strategies. Strategy use is higher when the strategy was being applied to conceptual knowledge that was easily accessible.

REFERENCES

Ackerman, PT; Anhalt, JM & Dykman, RA. (1986) Arithmetic automatization failure in children with attention and reading disorders. Journal of Learning Disabilities, 19, 4, 222-232.

Cardelle-Elawar, M. (1992). Effects of teaching metacognitive skills to students with low mathematics ability. Teaching and Teacher Education, 8, 2, 109-121.

Fleischner, JE; Garnett, K. & Shepard, MJ. (1982) Proficiency in arithmetic basic facts computation of learning disabled and non-disabled children. Focus on learning problems in mathematics, 4, 47-56.

Fleischner, J.E., Nusum, M.B. & Marzola, E.S. (1987). Devising an Instructional Program to Teach Arithmetic Problem-Solving Skills to Students with LD. Journal of Learning Disabilities, April, 214 - 217.

Gardner, H. (1985). Frames of Mind. New York : Basic Books

Haller, E.P., Child, D.A., and Walberg, H.J. (1988). Can comprehension be taught. Educational Researcher, 17, 5-8.

Howell, D.C. (1992). Statistical matehods for psychology.. Belmont, California : Duxbury.

Montague, M. & Bos, C.S. (1986). The Effect of Cognitive Strategy Training on Verbal Math Problem Solving Performance of Learning Disabled Adolescents. Journal of Learning Disabilities, January, 26 - 33.

Munro, J. (1991). The effectiveness of thinking strategy training for learning disabled adolescents. Paper presented at the Fifth International Conference on Thinking, Townsville, July 1992.

Pearson, P.D., & Dole, J.A. (1987). Explicit comprehension instruction : A review of research and a new conceptualization of instruction. The Elementary School Journal, 88, 2, 152-165.

Pickering, D., Szaday, C., & Duerdoth, P (1988). One in eleven. Melbourne: VICCTRAC.

Rabinowitz, M. (1988). On Teaching Cognitive Strategies: The Influence of Accessibility of Conceptual Knowledge. Contemporary Educational Psychology, 229 - 235.

Swanson, H.L. & Rhine, B. (1985). Strategy Transformations in Learning Disabled Children's Math Performance: Clues to the Development of Expertise. Journal of Learning Disabilities, December, 596 - 603.

Torgesen, JK. (1980) Conceptual and educational implications of the use of efficient task strategies by learning disabled children. Journal of Learning Disabilities, 13, 364-371.

PRODIGY: A SIMULATION SYSTEM FOR DIAGNOSIS AND REMEDIATION WITHIN THE DOMAIN OF COMMON FRACTIONS.

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This paper describes the on-going research and development activities of the PRODIGY project. The major aim of this project is to develop intelligent computer-based simulation systems which facilitate the development of high level diagnostic and remediation skills in the domain of common fractions. The development and the functioning of the present generation of PRODIGY simulation systems (PRODIGY1) is first described. The methods used in and results from the formative evaluations of the PRODIGY1 generation of simulation systems then are described. Some of the modifications which had their genesis in the results from the formative evaluations are briefly discussed. Three serious limitations of the PRODIGY1 simulations are identified and discussed. The paper concludes with a description of the architecture of next generation of PRODIGY simulations together with a discussion about how it is hypothesized that they will overcome the serious limitations of the PRODIGY1 simulations.

Research conducted in schools clearly indicates that most teachers either do not diagnose to obtain a clear view of their pupils' understandings and misconceptions or limit their attention to the product of their children's work rather than focussing on the processes and strategies employed by their children (Bennett, 1987). Because of this, many students are being presented with inappropriate lessons. The learning of mathematics thus is for many students a frustrating and unsuccessful experience. Unfortunately, little effective work is being done at present to ameliorate this lack of diagnostic skills in teachers.

In this paper, I describe the development and evaluation of the PRODIGY simulation system currently being done at the Queensland University of Technology's Centre for Mathematics and Science Education (CMSE). The major aim of this system is to provide a learning environment in which novice teachers may acquire expertise in the diagnosis and remediation of mathematical learning problems. As they are acquiring this expertise, it is envisaged that the novice teachers will develop higher levels of lesson structure knowledge (Leinhardt & Smith, 1986) than that achieved by current teacher education activities. It is also envisaged that the interactions with the system will facilitate the development of pedagogical content knowledge (Shulman, 1986) by helping novice teachers to better understand how particular topics, principles, strategies, and the like in mathematics are comprehended or typically misconstrued, are learned and likely to be forgotten.

PRESENT VERSION (PRODIGY1)

PRODIGY1 (Nason, 1991, 1993) is a simulation environment that lets teachers develop and practice skills in tutoring students who are having difficulty with the addition of fractions. With this system, the tutor first assesses the simulated student's knowledge by giving it some addition of fraction problems (e.g., 1/2 + 1/3) and observing how the answers to the given problems are generated. In order to gain further insights into how the simulated student generates answers to the problems, the tutor can ask the simulated student to explain how it got its answer for each problem. The simulated student responds to this request by presenting a written, sequential explanation of its procedures. When the tutor has decided what the student's misconceptions are, he can plan and administer a sequence of instructional activities. The simulated student may or may not overcome its misconceptions when it is given this instruction, so the tutor must again assess the student's knowledge. This cycle of diagnosis and remediation continues until the tutor is satisfied that the simulated student knows how to add fractions.

The PRODIGY simulation is based upon a set of real children. The simulation of each child was constructed in the following way:

1) In interview 1, each child was asked to carry out some addition of fractions items. As the subject carried out the computations, (s)he was encouraged to think aloud. Each session was videotaped. Following the problem solving session, a stimulated recall session was conducted with the aid of the videotape recording.
2) An initial production system model of the child's computational knowledge and an initial semantic network model of the child's intuitive and conceptual knowledge was generated from a detailed protocol analysis of the videotape transcript and the subject' stimulated recall.
3) As there was some ambiguities to be overcome before psychologically valid model could be generated, each child was recalled for further sessions in which other addition of fractions and fraction concept items were administered. The items administered depended on the information required by the investigator to complete the models. After each session, the production system and semantic network model were revised in the light of the new data produced. This observation-analysis-modification procedure was repeated until a stabilized model was produced. The primary purpose of this procedure was to construct a model for each child based on converging experimental evidence.

One of PRODIGY's three case studies is "Toni". Toni's bug is that whenever the two denonimators are different, it adds them to get the answer's denominator; then it cross multiplies the numerators and the denominators and adds these two products to generate the numerator of the answer (See Figure 1 below). Thus, if Toni is asked to add two fractions with different denominators, she always produces an incorrect answer. If they are unit fractions, it always produces an answer equivalent to one whole (e.g., 1/5 + 1/6 => 11/11). The diagnosis of this bug is complicated by Toni's tendency to perform the operations of generating denominator answer and numerator answer in random order. It is also further complicated by two other factors: Toni can correctly add fractions with common denominators and Toni's explanations of how an answer is generated are very convoluted and full of mathematical jargon. However, by observing Toni attempts at different types of addition of fraction problems and by carefully juxtaposing what they have observed against Toni's explanations, novice teachers soon can "get inside Toni's head" and discover the reasons for her errors. While doing this, they are simultaneously developing two crucial diagnostic skills: how to identify and ask the right questions and how to listen and observe while students are performing mathematical computations.

$$\frac{3}{4} + \frac{4}{5} = \frac{31}{9}$$

15 16
9

Figure 1 Toni

An expert teacher would recognize that Toni's misconceptions are deeply seated: Toni in fact does not understand that a fraction is a number and many other basic notions about fractions. The remediation thus would need to start at initiating level explorations of the basic notion of a fraction and then proceed through a sequence of initiating, abstracting and schematizing activities (Ashlock et al, 1980) on comparison of fractions, estimation with fractions, equivalence of fractions, and generating common denominators before Toni would be ready to learn how to add fractions with different denominators. Many novice teachers do not seem to realize this. Their remediations tend to focus on Toni's procedures not the underlying misconceptions. Their initial suggested remediations thus tend to have no effect at all on Toni's addition of fractions' procedures. Their remediations only have an overt effect on Toni's procedures when they address its basic misconceptions about fractions and have introduced correct procedural rules into Toni's repertoire of algorithmic processes. The important lesson novice teachers learn from this is that teachers need to focus on underlying misconceptions before attempting to remediate algorithmic procedures.

FORMATIVE EVALUATION OF PRODIGY1

While PRODIGY1 has been undergoing development, it has been formatively evaluated. Evaluation data has been collected from two different sources: teacher education students and expert educators.

Five groups of three undergraduate and two groups of two postgraduate teacher education students have been observed as they interact with the simulation system. These observations have focussed on their protocols and strategies for diagnosis, the speed of their diagnoses, the quality of their diagnoses and their suggested remediation programmes, and their ease in the use of the computer interface and its menus. As the groups proceed through the three case studies, particular emphasis has been placed on identifying qualitative changes that are occurring in the teacher education students' diagnostic strategies and in their general perceptions about the nature of and underlying causes of mathematical error patterns.

The evaluation data collected from the observations has been supplemented by data from post-simulation session interviews in which the groups were encouraged to discuss what they felt they had learnt from the simulation session, what new questions about diagnosis and remediation had emerged during the session and any difficulties they had with using the system. They also were invited to make suggestions about how they felt the system may be a more effective learning tool.

Expert educators also have been involved in the formative evaluation of the system. Experts in the field of multimedia education such as Prof. Joe Henderson (Dartmouth Interactive Media Laboratory) have provided comments about the computer-user interface and the effectiveness of the simulation aspects of the system; experts in the field of artificial intelligence in education such as Drs Stellan Ohlsson and Kurt VanLehn (LRDC, University of Pittsburgh) have provided comments about the cognitive modelling aspects of the system and mathematics education experts in Australia, Europe and the USA have provided comments about the mathematics education aspects of the system.

The formative evaluation data from the teacher education students and the expert educators have been used to identify how the system can be modified so that it may become a more effective teacher education tool. Many modifications thus have been made to the system during the past two years. For example, evaluation feedback from the teacher education students and from Prof. Joe Henderson indicated that the system would be much more effective if the users could gain access to the underlying thought processes of each case study and in particular the type of language the case study would use to describe what it had just done. The system thus was modified; users are now able to print out an explanation of how the case study generated its answer to the example it has just completed. Evaluation feedback from mathematics educators has resulted in greater emphasis being placed on the development of conceptual knowledge during the remediation phases of the simulation. Thus, in it present version, the error patterns of none of the three cases studies will be successfully overcome unless the users focus much of the remediation programme on overcoming the underlying misconceptions.

The formative evaluations have identified a number of limitations of the PRODIGY1 system which cannot be ameliorated by simple modifications to the PRODIGY1 system. Probably the most significant of these limitations is that PRODIGY1 only enables the user to directly evaluate the "virtual" students' knowledge of the addition of common fractions algorithmic knowledge. Although the information generated by the simulations may enable the users to make inferences about a case study's underlying conceptual knowledge, PRODIGY1 does not allow the user to directly evaluate the validity of these inferences. Another limitation is that it does not allow the user to assess the case study's understanding of these fraction concepts and processes at the concrete/oral and pictorial/oral levels of representation. One can thus only hypothesize the case study's knowledge at these non-written-symbolic levels. This information, however, is very important when one is planning and implementing a remediation programme. A third limitation of PRODIGY1 is that it does not enable the user to diagnose, begin initial instruction and then evaluate the effects of this initial instruction. In its present form, most of the remediation programme has to be completed before its effects become apparent while diagnosing. Because of these limitations, a second generation of PRODIGY simulations is currently being developed.

PRODIGY2: THE NEXT GENERATION

In the second generation of PRODIGY currently being developed, it is envisionaged that novice teachers will be able to investigate each case study's ability: to concretely/pictorially represent fractions; to represent fractions with oral and written representations; to translate from one form of representation to another; to compare fractions; to estimate the sum of two fractions; to generate equivalent fractions; and to generate common denominators. It also will enable the users to ascertain whether the case study has the notions that a fraction can represent being part of a whole or partitive division.

In order to include these new features in the next version of PRODIGY, a new production system model is currently being developed. The development of this new model is proceeding in two phases. In Phase 1, a semantic network of approximately 80 fraction notions has been produced. During the production of this semantic network, constant reference was made back to the semantic models generated earlier in this investigation. The information contained in this semantic network and the PRODIGY1 procedural knowledge production system is now being incorporated into a "hand-run" production system network. In Phase 2, this network will be translated into a computer-based production system which contains two parallel and complementary subsystems: a qualitatitive subsystem in which the number sense and estimation notions and procedures are located; and a quantitative subsystem in which the system's algorithmic procedures are located.

CONCLUSION

The PRODIGY-Mk2 architecture has implications for both education and A.I. Its architecture will easily be able to be adapted for simulating diagnosis and remediation of other mathematics topics such as operations with decimals and whole numbers and also many algebra topics. It thus has the potential to significantly affect the quality of mathematics education from early elementary to junior college levels. Its significance for A.I. is that this model integrates conceptual knowledge with procedural knowledge more directly than most previous A.I. simulation models. It also provides us with a means for exploring the learning relationship between both these types of knowledge. The major limitations of this simulation model seems to be that it can only simulate number and algebraic concepts and processes and not many important measurement and geometrical concepts and processes.

REFERENCES

Ashlock, R.B., Johnson, M.L., Wilson, J.W. & Jones, W.L. (1980). Guiding each child's learning of mathematics: a diagnostic approach to instruction. Charles E. Merrill, Columbus, Ohio.

Bennett, S.N. (1987). Teaching and learning mathematics in the primary school. Perspectives 34, 2.

Leinhardt, G. & Smith, D.A. (1986) Expertise in mathematics instruction: subject matter knowledge. Journal of Educational Psychology 77, 3, 247-271.

Nason, R.A. (1993). PRODIGY: Diagnosis and remediation in the domain of common fractions. Computers & Education: An International Journal 20, 1, 45-53.

Nason, R.A. (1991). PRODIGY: An intelligent computer-based system for developing teachers' expertise in the diagnosis and remediation of common errors in the domain of common fractions. Interactive Learning International 7, 3, 243-252.

Shulman, L.S. (1986). Paradigms and research programs in the study of teaching: A contempory perspective. In M.C. Wittrock (Ed.) Handbook of research on teaching. MacMillan: New York.

CHILDREN'S PERCEPTION OF MATHEMATICAL & MUSICAL PATTERNS

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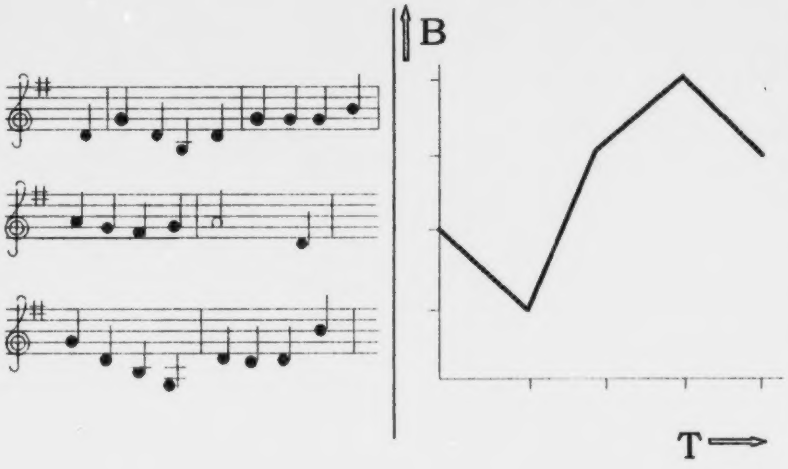
This paper focuses on the function of contour in the association of melodic and visual patterns, and investigates the role played by ability in mathematics and music in children's performance at matching melodies and music notation, and at matching melodies and graphs. Initially, tests in mathematics and music were administered to 101 children in Year 5. Factor analysis confirmed specific sub-scales in the tests and revealed a positive and significant correlation between results in the two tests. However this correlation was no greater than the correlations between other school subjects and mathematics and music. The melody matching tasks (four modality conditions altogether) revealed that format (conventional/non-conventional) was a significant main effect for graphs but not for music notation. Modality condition and level of contour complexity were main effects for both types of matching (melodies _ music notation and melodies _ graphs). Success at matching melodies and both types of visual contours was related to ability in music more than ability in mathematics.

Links between mathematics & music have been well documented for centuries. The Greek mathematician and philosopher, Pythagoras established that the pitch of musical notes was related to lengths of vibrating strings and organ pipes, and that simple mathematical ratios existed between resonating lengths for notes which harmonised. Since that time the study of acoustics and the production of musical sounds in instruments has progressed to the stage where the field is highly scientific, and mathematical modelling is very sophisticated. Not only are there people who study the mathematics and science of music, but also there exists anecdotal and research evidence indicating that there are those who are gifted in both mathematics and music. Browne (1987) has reported that the composition of some university orchestras is biased towards students majoring in science, engineering and mathematics fields (in contrast to arts and humanities). Among the top university orchestras in the USA is that of the Massachusetts Institute of Technology where some 80% of the members are students in these fields.

Educational research data on the "spin-off" effect of developmental music education programs, such as the program developed by the Hungarian composer and teacher Zoltan Kodaly, have shown that participating primary school children improve in academic areas of mathematics (number, geometry and problem solving), reading comprehension and spelling, and general learning ability (Bridges, 1979; Herbert, 1973; Gregory, 1988). Why does this happen, and what links are there, if any, between children's understanding and skill in the two subjects? This research project investigates these questions. Further, the project has been prompted by a perceived relationship between a musician's interpretation of music notation and a mathematicians interpretation of graphs. Both interpretation processes involve abstraction of visual contour from their particular conventional frameworks. The former then requires a translation to knowledge of musical sounds, while the latter requires a translation to knowledge of the mathematical context. For instance in Figure 1, (i) What does the tune sound like? and (ii) What does the graph tell us?

LITERATURE ON PERCEPTION OF MUSICAL AND VISUAL PATTERNS

The relationship between melodic and visual contour as perceived by children was studied by Morrongiello and Roes (1990). Five-year old and nine-year old children were asked to match visual contours to given nine-note melodies, and a number of significant factors emerged. Performance levels increased with age, but decreased with higher levels of contour complexity. Tonality (tonal versus atonal) was significant for nine-year olds but not for five-year olds, and musical training ensured better performance at the matching tasks.



: Samples of music notation and mathematical graphing.

The fundamental role of contour in melody recognition and melody discrimination has been well established. Dowling (1978) showed that the up-down contour of melodies plays the major role in recognition of musical stimuli. If the melodic contours are changed, subjects are able to recognise the melodies as being different (Bartlett & Dowling, 1980). Dyson & Watkins (1984) proved that contour changes were more easily identified if they occurred at contour reversal points. In a study of children's perception of melodies, Pick et al (1988) found that subjects were able to discern similarity between melodies that had the same contour but had changes in intervals between notes within the melodies. Cross-modal & intramodal matching of melodic & visual contours has been investigated by Balch & Muscatelli (1986). Key factors emerging from the study were rate of presentation, modality, and musical experience. For both low and high music experience groups, intramodal visual → visual recognition was best, followed by recognition in both cross-modal conditions (visual → auditory and auditory → visual) and last was the auditory →auditory condition. This result and the interaction between presentation rate and modality gives rise to the contour abstraction hypothesis which maintains that up-down contour abstraction for visual presentation is more efficient than for auditory presentation.

PURPOSE OF THE STUDY

The purpose of the study was to investigate children's performance at matching melodies with music notation and mathematical graphs in relation to their abilities in mathematics and music. The role of musical and mathematical training was to be further investigated by contrasting performance with conventional materials with performance with non-conventional materials. It was hypothesised that children with more ability in music would be more competent with the conventional notation, and that the use of non-conventional notation would reduce performance levels. Similarly, it was hypothesised that non-conventional graphs would reduce the performance levels of mathematically able children. The influence of contour complexity was to be studied by having materials of two levels of complexity, high and low. The study was designed to test the contour abstraction hypothesis by analysing differences in performance between the various modality conditions -melody → visual, visual → melody, visual → visual and melody → melody.

METHOD

Subjects: One hundred and one children in Year 5 (aged between 10 and 11) at two regular suburban state schools were used as subjects. All children underwent the set of tests in mathematics ability and music ability, but were split into two groups (50 and 51) for the two separate sets of matching tasks (melodies with music notation, and melodies with graphs).

Materials: The mathematics ability test was a written test which consisted of 25 multiple-choice items on three topics which reflected elements common to both disciplines, namely, number patterns, measurement of length and interpreting graphs. These topics are regarded also as essential components of a contemporary primary school mathematics program according to Australian Education Council (1990) and the National Council of Teachers of Mathematics (1989). The items were adapted from items included in standardised tests produced by the Australian Council for Educational Research (1974). All items of the mathematics test were completed in a class situation. Questions were also read out aloud to the children so that any with reading difficulties were not disadvantaged in this way. The music ability test was a composite written and aural test of 25 items based on items from the musicianship syllabus and practical aural tests produced by the Australian Music Examinations Board (1991). Some items were multiple choice response, the others short answer. Approximately half of the items in the test were completed in a class situation and the remainder were done on an individual basis. Items were read out aloud also, as with the mathematics test. Information was also collected about the children's music experience (instrumental and choral), and their levels of performance in other school subjects (language arts, social studies and science).

The matching task tests each consisted of 42 items requiring subjects to match (i) a 9-note melody with music notation or graph (auditory→visual), (ii) music notation or graph with a 9-note melody (visual→auditory), (iii) music notation or graph with music notation or graph (visual→visual), and (iv) melody with melody (auditory→auditory). Melodies were of low complexity (1 contour change) or high complexity (3 contour changes) and were played on a Sony portable audio cassette player placed adjacent to the subject. The melodies had been recorded directly from a Yamaha PSR7 keyboard using the familiar piano sound on the keyboard. Visual materials (samples of single line music notation or line graphs) were printed black on white paper, and were placed on the desk in front of the subject. Music notation in the matching task test was presented in conventional format (standard crotchets on a horizontal five-line staff, read from left to right), and also in non-conventional format (triangle shapes on a four-line vertical staff, read from top to bottom). Conventional graphs consisted of connected lines segments (none points) with up-down contour changes read from left to right. Graphs of the non-conventional variety however were to be read from top to bottom, with corresponding up-down contour of melody represented by left-right movement (low on the left and high on the right).

The experiment was a 2 x 3 x 2 factorial design (2 formats x 3 modality conditions x 2 complexity levels). The inclusion of items of the fourth modality condition (auditory→auditory) allowed comparison across all four conditions, giving a supplementary framework for analysis of results, namely 4 modality conditions x 2 complexity levels).

Design: 2 formats x 2 complexities x 3 modalities

Format	Complexity	Modality		
		Mel-Vis	Vis-Mel	Vis-Vis
Conv.	Low	Cell 1	Cell 3	Cell 5
	High	Cell 2	Cell 4	Cell 6
Non-conv.	Low	Cell 7	Cell 9	Cell 11
	High	Cell 8	Cell 10	Cell 12

Design: 2 complexities x 4 modalities

Format	Complexity	Modality			
		Mel-Vis	Vis-Mel	Vis-Vis	Mel-Mel
Conv.	Low	Cell 1	Cell 3	Cell 5	Cell 13
	High	Cell 2	Cell 4	Cell 6	Cell 14

In the matching task tests, the following procedure was carried out with each child. For the melody→visual modality condition, a 9-note melody was played on a cassette player to the child, and then a 9-note sample of music notation (or a 9-point connected line graph) was shown. The question was asked, "Is this music (or graph) the same as the melody you have just heard, or is it different?" In the visual→melody modality condition, a 9-note sample of music notation (or a graph) was shown to the child for approximately 5 seconds (long enough for the child to indicate that he/she had noted its features), and, after the page was turned, a 9-note melody was played. The child was asked, "Was that melody the same as the music (or graph) on the page before, or different?" In the visual→visual modality condition, a 9-note sample of music notation (or graph) was shown to the child, then the page was turned to reveal another sample of notation (or graph). The child was asked if the two samples of music notation (or the two graphs) were the same or different. In the melody→melody condition, two 9-note melodies were played consecutively and the child was asked if the two melodies were the same or different. At the end of the session the child was asked what strategy he/she used to remember the melodic and visual contours, and determined if they matched or not.

RESULTS

Ability Tests: Factor analysis of the mathematics ability test confirmed the three original, test sections as subscales: interpretation of graphs, measurement of length, and number patterns. However, the music test revealed three subscales musical performance, notation skills, and aural skills. These scales are process & skill related rather than topic related. There was a significant and positive correlation between mathematics and music scores (r = 0.42, p = 0.001), but this no more significant than the correlations between mathematics and music with other school subjects. See r values below.

	Music	Mathematics
Language	0.44	0.51
Social Studies	0.46	0.44
Science	0.45	0.51

Matching Tasks - Melodies and music notation: Analysis of the results revealed that format was not a significant main effect, indicating that overall there was no difference in performance between items in conventional and non-conventional format. However there were two significant main effects: modality and complexity. Children were more successful at intramodal tasks than cross-modal tasks: the order of success being (from high to low) visual→visual > visual→melody > melody→visual and melody→melody = visual→visual. Performance dropped significantly as complexity increased from low to high. An interaction between modality and complexity indicated that cross-modal task performance drops more as complexity increases than for intramodal task performance. There was a significant interaction between format, modality & complexity revealing that task performance depended significantly on the status of each of the three variables. There were some combinations which produced performance no better than chance level, for example conventional format, high complexity, melody→visual modality.

Music ability was a significant factor with performance at conventional format items and for melody→visual items. Further, there was a significant interaction between format, modality and modality. Music experience was a significant factor for low complexity items, and for test aggregate scores. Mathematics ability appeared to play little part in matching task performance.

Matching Tasks - Melodies and graphs: Contrary to the results for melodies and music notation, format was a significant main effect for matching melodies and graphs. Modality also was a significant main effect with intramodal tasks showing higher levels of performance than cross-modal tasks: the order of success being (from high to low) visual→visual > melody→melody > visual→melody > melody→visual. Complexity was another significant main effect, and it also had different effects in each of the modality conditions, as well as in each format, as indicated by significant interactions.

Mathematics ability was a significant factor in the performance of melody→visual items, but not for any other class of matching tasks (even though mathematical graphs were used as one of the matching halves). However musical ability was a significant factor for conventional items and for high complexity items. Further, music experience was a significant factor for conventional items, low complexity items, high complexity items, and for melody→visual items. Music experience also interacted significantly with format x complexity, modality x complexity, and with the test aggregate score. The importance of music ability was reinforced by the significance of aural scale scores for non-conventional items, low complexity items, the interaction of format x complexity, and for test aggregate scores.

Children's descriptions of strategy: The children generally agreed about how they remembered the melodic and visual contours, and determined whether they were the same or different. With the melody → visual items, they said that they could remember the up/down shape of the melody and the actual notes of the tune, along with the first and last notes. Some musical children could see the shape of the tune in their minds from hearing the melody. With the visual → melody items, again they could remember the shape (for example, "like a W"), kept a mental picture, and checked the melody note by note. Some musical children could hear the tune in head, just by looking at the visual contour. With the visual → visual items they again could remember the overall shape, in particular the beginning and the end, and could keep a mental picture of the contour. With the melody → melody items, they said that they could keep the tune in their minds, however the presentation of the second melody often wiped the memory of the first melody. Many children reported that the task was very difficult when there were "too many ups and downs".

Most children noted that the non-conventional format of music notation was harder to read than the conventional format because "it was opposite to normal way", but noted that the non-conventional graphs were no more difficult that the conventional graphs. Results showed that music notation format had more influence in the high music group than the low group. In the melodies and graphs test, results showed similar levels of performance for visual→visual tasks, but with cross-modal tasks, performance deteriorated with the introduction of non-conventional graph format.

DISCUSSION

The results have confirmed that musical ability is a significant factor in matching task performance for melodies and visual contours and that experience with conventional music notation is one reason for the tasks with music notation. The fact that mathematics ability is not a significant factor for graphs, may be due to the fact that children spend equivalent amounts of time doing mathematics but not for music, in which some children may learn two instruments and sing in the school choir but some just play the recorder for 15 minutes per week. The situation where music ability is a significant factor in both tests, especially with melody→visual items, points to musical children's superior accuracy at perceiving melodies and abstracting contour features. The importance of modality condition has been highlighted by the results, which show that children are more adept at intramodal tasks than cross-modal tasks, and that the contour abstraction hypothesis holds (but only for cross-modal tasks). An explanation for why visual→visual tasks were performed more accurately than melody→melody tasks with graphs and not music notation could be that music notation is more information dense than the comparable graph. One feature of the melodies and visual contours used in this study is the type of presentation, namely simultaneous versus sequential presentation. Melodies are by nature heard sequentially whereas music notation and graphs are presented as a whole. Hence, further research should investigate children's abstraction of contour from melodic and visual materials in relation to their simultaneous and sequential processing skills.

REFERENCES

Australian Council for Educational Research (1974). **ACER Mathematics Tests (AM Series).** Hawthorn, Victoria: Australian Council for Educational Research

Australian Education Council (1990). **National statement on mathematics for Australian schools.** Melbourne: Curriculum Corporation for AEC.

Australian Music Examinations Board (1991). **Manual of syllabuses.**

Balch, W. & Muscatelli, D. (1986). The interaction of modality condition and presentation rate in short-term contour recognition. **Perception & Psychophysics**. 40 (5), 351-358.

Bartlett, & Dowling (1980). Recognition of transposed melodies: A key distance effect in developmental perspective. **Journal of Experimental Psychology: Human Perception & Performance**, 6, 501-515.

Bridges, D. (1979). **Outcomes of a developmental music program.** Paper presented to the Fourth International Kodaly Symposium, Sydney, 1979.

Browne, M. (1987). **The intimate links between music and the lab.** in **Why music is basic**. Queensland Department of Education.

Dowling, W.J. (1978). Scale & contour: Two components of a theory of memory for melodies. **Psychological Review**, 85, 341-354.

Dyson, M. & Watkins, A. (1984). A figural approach to the role of melodic contour in melody recognition. **Perception & Psychophysics**. 35 (5), 477-488.

Gregory, A.S. (1988). **The effects of a musical instructional technique on a mathematical achievement of third grade students.** Doctoral Dissertation, University of Alabama.

Herbert, G. (1973). Cited in Bridges, D. (1979) **Outcomes of a developmental music program.** Paper presented to the Fourth International Kodaly Symposium, Sydney, 1979.

Morrongiello, B. & Roes, C. (1990). Developmental changes in children's perception of musical sequences: Effects of musical training. **Developmental Psychology,** 26 (5), 814-820.

National Council of Teachers of Mathematics (1989). **Curriculum & evaluation standards for school mathematics.** Reston, Virginia: NCTM.

Pick, A., Palmer, C., Hennessy, B., (1988) **Journal of Experimental Child Psychology**, Vol.45, pp.28-51.

TEACHING CHILDREN TO REPRESENT RECTANGULAR ARRAYS

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The development of young children's representations of a rectangular array as [perpendicular] intersecting sets of parallel lines is the focus of this paper. Drawing an array using lines, rather than drawing each rectangular area unit separately should assist children to represent the correct numerical structure and to perceive a row as a single, composite unit. Without a knowledge of the numerical structure of an array, it is unlikely that children will be able to apply repeated addition or multiplication skills to determine the number of elements in an array.

To draw a rectangular array, it seems necessary for children to grasp three properties of an array: that the area units must be congruent, that the units are aligned, and that each row has the same number of units. A teaching experiment was undertaken to determine the effect of stressing these features on children's array drawings. Emphasising that the units should be the same size did not help the children to draw the correct array structure, while stressing the alignment of units or that there is an equal number of units in each row, did seem to assist children to draw an array as lines.

Area is a particularly important topic in school mathematics. It is one of the most commonly used domains of measure in everyday life, and it is the basis for many models used by teachers and textbooks to explain numbers and operations with numbers (Hirstein, Lamb, & Osborne, 1978). This paper reports on one aspect of a study of the effects of children's perceptions of arrays on the development of their concepts of rectangular area measurement. The teaching experiment that is described in this paper arose from an analysis of the responses to a sequence of array drawing tasks given to 140 children in Years 1 to 4 of primary school.

Knowledge of array multiplication is a key prerequisite to understanding the formula for the area of a rectangle. An essential step in applying multiplication to arrays would seem to be a realisation that individual units in a row [column] can be grouped to form a new composite unit. To consider a row [column] as a single unit, however, is a fundamental change in a child's conception of what constitutes a unit. The change from children's constructions of single entities as units to their constructions of composite or multiple entities as units has been explored in depth by Steffe (1988; 1992) who believes that such a change is crucial to the development of multiplication. Hiebert (1988) noted that this change is not trivial because "a change in the nature of the unit is a change in the most basic entity of arithmetic" (p. 2).

Although arrays are an inherent part of multiplication, Outhred and Mitchelmore (1991) found that basic understandings of arrays were not well developed in children in Years 1 and 2 and some children in Years 3 and 4 seemed to have a limited knowledge of the essential features of array structure. Other researchers have reported that children may not identify array models with multiplication (Beattys and Mayer, 1989; English, 1982)
One aspect of children's knowledge of array structure would seem to be a perception of the array as groups of rows [or columns] that can represented by lines rather than as a collection of individual units. The use of lines may assist children to construct composite units as the lines emphasise the action of grouping individual units into a composite unit.

THE TEACHING EXPERIMENT

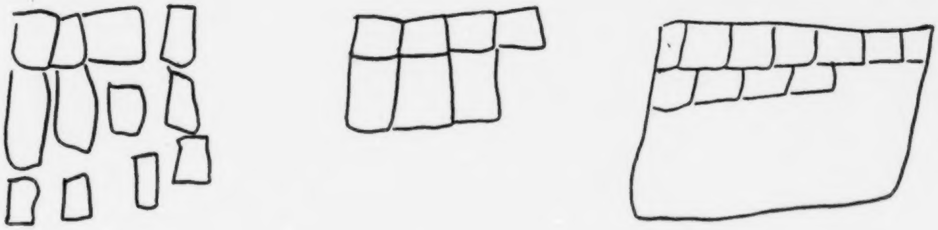
The results of the main study interviews indicated that children need to grasp three basic properties to draw an array. These properties are:

1 that the units must be congruent
2 that the units must be aligned both vertically and horizontally
3 that each row [column] has an equal number of units.

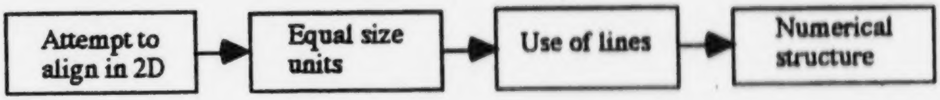
The first two properties are referred to as the *spatial structure* of an array, and the third property as the *numerical structure*. The aim of the study reported in this paper was to evaluate which of three instructional methods was most effective in teaching children to draw arrays using lines. Each method emphasised one of the above properties of an array and only square units were considered. The three properties seemed to be independent as an analysis of the interview data provided examples of children's drawings in which each property occurred in isolation (see Figure 1). The properties were also found to occur in combinations of two categories without the third. For example, the squares might have been drawn so that they were aligned vertically and horizontally and they were approximately the same size, but each row might not have the same number of squares.

If rows of congruent units [squares] are drawn without gaps, it follows that the columns so formed must be aligned, whereas vertical and horizontal alignment of squares does not necessarily imply congruence. Thus drawing children's attention to the size of the squares should correct any alignment errors and ensure that the numerical structure is correct, whereas alerting children to the alignment of the squares might not result in congruent squares being drawn. Similarly, pointing out to children that there is an equal number of squares in each row [column] does not subsume the properties of alignment and congruence.

Figure 1 Examples of the three properties of arrays occurring independently of each other



Therefore, the hypothesis to be tested is that for children who have begun to connect units in two dimensions (2D), the three teaching approaches will be ordered from most (1) to least effective (3); that is, the hypothesised development of the array structure is:



This hypothesis assumes that in general, children have to master the geometrical structure of the array before the numerical structure. This assumption may not be warranted when the array is small or when a concrete model is present for children to copy.

SELECTION OF THE SAMPLE

A group pre-test was used to select children who were at a particular level in their development of the array concept. The pre-test was given to two class groups of children in each of Years 1 and 2 in two schools; it consisted of two tasks. The first task was to make a 3x4 array using 4 cm cardboard squares and then to draw the array. In the second task (an indicated grid), marks were shown on the sides of a 5 cm x 8 cm rectangle to indicate the number of 1 cm squares that would fit along each side; *the child was asked to draw the squares to cover the rectangle*. A 1 cm square was shown next to the rectangle to indicate the size of the squares.

The model task was a good introductory activity to explain the concept of covering to the children. A large rectangle on a sheet of blue cardboard was covered with stick-on yellow squares while the children watched. Then the cardboard was turned over to show the completed model with the squares taped down to ensure that they were carefully aligned. The children were asked to draw the squares so that their drawings looked exactly the same as the model. Next the children were asked to complete the indicated grid task. An indicated grid drawn on a large sheet of cardboard was used to illustrate the instructions to the children.

Children who attempted to align the squares in two dimensions but who had not yet mastered this skill were chosen to be part of the sample. These children were allocated to the three groups so that the groups were balanced for year level, gender and for similarities among the children's pre-test drawings on the indicated grid task. Two of the groups consisted of 15 children, the other of 16 children.

THE TEACHING SESSION

One individual teaching session was given to each child and the indicated grid task was presented as a post-test at the end of the session. The same procedure was followed for each teaching method. The first teaching activity was a variation of the model task; the child made an array by covering a rectangle with cardboard squares. A transparent overlay showing the array drawn as two sets of lines was placed over the squares to emphasise the array structure; the child's attention was drawn to the constant size of the squares, to their alignment, or to the number of squares in each row [column] depending on the instructional group to which the child belonged. This initial activity was repeated with different rectangles and squares of different sizes. Next the child was shown a set of pictures of arrays. There were three different sets of pictures and in each set, a single feature was stressed. The pictures exemplified some of the different types of children's drawings (see below). The child was asked to suggest how the drawings could be improved. If the child did not mention the specific property, the teacher gave a non-committal response, while comments that concerned the relevant feature were discussed. After this task the child made a large array (5 x 6) that s/he was asked to copy. A large array was used to elicit drawing errors. The differences between each child's drawing and the model were pointed out, again stressing the appropriate feature. A similar procedure was followed for a partially completed array (squares were shown on two adjacent sides of a 6 x 7 rectangle); the child was asked to complete the drawing. Any difficulties that involved the relevant property were then discussed. Three numerical tasks were given after the post-test to see if an emphasis on the numerical structure of arrays assisted children to use the row or columns structure to systematise their counting.

RESULTS

The results of the earlier study (Outhred & Mitchelmore, 1991) found that children's drawings of arrays could be classified on two different (but related) dimensions, spatial and numerical structure. The spatial categories that were used were:

1. the child did not attempt to align the squares in two dimensions;
2. the child attempted to connect the squares (drawn individually) in two dimensions but experienced difficulties with alignment;
3. individual squares were drawn, that were reasonably well aligned;
4. the child made some use of lines in drawing the array. Several alternative methods were observed: lines might be drawn in one dimension only (that is, either horizontally or vertically, but not both); some squares might be drawn individually and the remainder of the array drawn as horizontal and vertical lines; or the array might be drawn as two (perpendicular) sets of parallel lines.

The numerical categories were:

1. there were not an equal number of elements in each row or column;
2. equal numbers of elements were drawn in each row [column] but the number of elements was incorrect (i.e., the array was not 5x8);

3 the array was numerically correct.

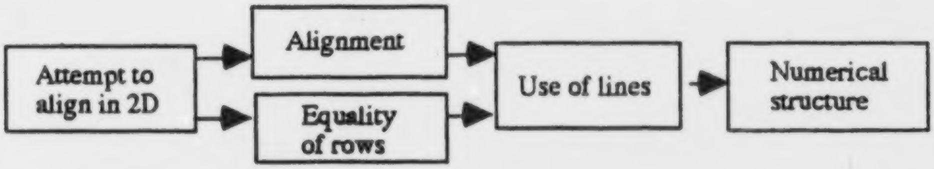
The children were selected for the sample because they had been given a spatial score of 2 and a numerical score of 1 for the indicated-grid pre-test task. The spatial and numerical category scores were used to analyse the differences among the three groups. The post-test spatial and numerical category scores on the indicated grid task are given in Table 1. A Kruskal-Wallis non-parametric test indicated that there are significant differences among the three groups (p=.009) for the spatial category scores but not for the numerical category scores (p=.09).

Table 1 Frequency distribution of the spatial and numeric category scores for the indicated grid post-test task.

	Spatial category			**Numerical category**		
Teaching group	**2**	**3**	**4**	**1**	**2**	**3**
Size	11	3	1	10	1	4
Alignment	4	2	10	4	2	10
Equality of rows	8	1	6	7	1	7

It can be seen from Table 1 that the hypothesised model for the development of the array structure is not supported as teaching children that the squares must be equal in size was the least effective strategy. Teaching children that "the squares must be underneath each other so that the edges meet each other" or emphasising that the number of squares in each row must be the same seemed to be similarly effective teaching strategies to assist children to develop the strategy of representing squares by drawing lines. Why might this be so? It was not the result that was expected. A possible explanation might be that size is too global a description to help the children draw congruent squares. To use information about size children must derive the essential drawing strategy, that is, squares that are equal in size will necessarily be aligned. In contrast, the teaching procedures for both alignment and number emphasised that the squares should be drawn underneath each other. While this was made explicit when stressing alignment, it was implicit in the idea of an equal number in each row, as children are first taught to make equal groups by one-to-one matching.

Does the evidence from the teaching experiment support the conjecture in the hypothesised model that it is the development of lines that leads to a numerically correct structure? The data from the pre- and post test tasks did seem to support this conjecture. It is not imperative that children use lines before they draw the correct numerical structure for the indicated grid task, but only four of the 21 children in numerical category 3 constructed an array without drawing lines. While these four children drew the correct numerical structure by drawing arrays of individual squares, usually this strategy resulted in alignment problems and thus an incorrect numerical structure. The results in Table 1 indicate that the model for the development of array structure for the indicated grid task should be represented as:



Correct numerical structures were achieved quite differently for the two tasks that were given in the pre-test. When copying from a model, drawing individual squares was the most commonly used and successful strategy as children found it difficult to draw an array of lines when the unit subdivisions were not shown. The children who attempted to draw lines usually drew equal rows but they had difficulty in subdividing the rectangle into the correct number of squares. For the indicated grid task an individual-square strategy was rarely successful, perhaps because the array to be drawn was larger, the size of the squares was constrained by size of the rectangle, and the children had to visualise the array structure.

WHAT DO CHILDREN NOTICE ABOUT INCORRECT DRAWINGS

> That's kind of nearly, just like that, but it's got little gaps there, it would be if you had joined those up. This one is a little bit too big, they've all got to be the same size. It's easy for me.

The responses to the task in which children were asked to comment on other children's drawings were examined in more detail by analysing the audio tapes of the teaching sessions. This was done to determine the features that children commonly noticed, and whether children mentioned the property that was emphasised in the teaching session. Children's comments about the three arrays were classified into four categories, size, alignment, number and gaps. The 'gaps' category was included as children frequently commented on spaces between the squares.

The feature that most children noticed was the comparative sizes of the squares, whether this feature had been stressed in the teaching session or not. The presence of gaps was also commonly noticed by most children, except for children in the group in which size had been emphasised. That the squares were aligned and that the number of squares in each row [column] was the same were only commented on by children who had been taught about these features. The pre-occupation with size might account for the frequency with which children added extra rows, columns or squares to their drawings. The urge to make the "squares" look the same size seemed to over-ride any other considerations. Thus children who had tried to align rows with each other but who could not keep the columns vertical (the squares in the bottom rows were smaller) would add a few extra squares to the last few rows because "it [the drawing] doesn't look right".

DISCUSSION

The results of this experiment indicated that teaching children that the squares in an array were all the same size was not the most effective method to help the children to perceive that rows and columns could be represented by lines and to draw a numerically correct array. This result was not expected because the property that the squares are the same size logically subsumed the other two properties of an array, alignment of the squares and the equality of rows [columns]. Teaching children that the squares are aligned or that there are the same number of squares in each row [column], however, both seemed to be effective methods of moving children from drawing squares individually to representing groups of squares using lines.

Children's responses to the task in which they were asked to comment on other children's drawings suggested that the two key features that assisted children to draw lines were not noticed by children unless these features had been specifically taught. The need to point out the alignment of the squares and the numerical structure of arrays to children may explain the effectiveness of emphasising these two properties in helping children to realise that rows and columns could be represented by lines. These two methods also gave more specific guidelines for drawing an array accurately as individual squares than teaching children that the squares should be the same size. An accurate representation of an array as individual squares seemed to be an important step in shifting to a line strategy.

Therefore, although some drawing techniques must be developed, the most important skill in representing an array seems to be based on an understanding of a fundamental property of rectangular arrays in general: the elements of an array are collinear in two directions. It is this property that allows the partitioning of the array into equal rows and columns which may be a first step in perceiving that these groups of single units may be reconstituted as composite units.

REFERENCES

Anghileri, J. (1991). The language of multiplication and division. In K. Durkin & B. Shire (Eds.), Language in mathematical education: research and practice (pp. 95-104). Milton Keynes: Open University Press.

Beattys, C., & Maher, C. (1989). Exploring children's perceptions of multiplication through pictorial representations. In G. Vergnaud, J. Rogalski, & M. Artique (Ed.), Thirteenth International Conference for the Psychology of Mathematics Education, 1 (pp. 109-116). Paris:

English, L. (1982). An investigation of Year two children's interpretation of multiplication when presented as both a set and an array model. In Mathematics Education Research Group of Australasia, (pp. 31-47). Melbourne:

Hiebert, J., & Behr, M. (1988). Introduction: capturing the major themes. In J. Hiebert & M. Behr (Eds.), Number Concepts and Operations in the Middle Grades (pp. 2-18). Reston: National Council of Teachers of Mathematics.

Hirstein, J., Lamb, C., & Osborne, A. (1978). Student misconceptions about area measure. The Arithmetic Teacher, March, 10-16.

Outhred, L., & Mitchelmore, M. (1991). The transition from concrete to pictorial representation of area. In Mathematics Education Research Group of Australasia Conference, Perth.:

Steffe, L. (1988). Children's construction of number sequences and multiplying schemes. In J. Hiebert & M. Behr (Eds.), Number Concepts and Operations in the Middle Grades (pp. 117-140). Reston, Virginia: National Council of Teachers of Mathematics.

Steffe, L. (1992). Schemes of action and operation involving composite units. Learning and Individual Differences, 4(3), 259-309.

COOPERATIVE, COMPETITIVE, AND INDIVIDUALISED LEARNING PREFERENCES: MATHS TEACHERS ARE DIFFERENT - WHY?

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The benefits and disadvantages of cooperative, competitive, and individualistic learning in school subjects has been considerably discussed in recent years. A coordinated set of three instruments (the "Learning Preference Scales") is now available for use by teachers and researchers in investigating the preferences of students, teachers, and parents. ***The Learning Preference Scale - Teachers*** *was administered to large samples of primary and secondary teachers in Sydney (N=619) and Minneapolis (N=342), and secondary teachers in the English Midlands (N=278). Differences among teaching subjects and between sexes are discussed. In all three locations, the learning preferences of Mathematics teachers were strongly oriented to competitive learning. Discussion is focussed on the pedagogical epistemology of mathematics teaching and learning, and on the belief systems of teachers.*

It has been recognised that the belief systems which teachers construct about the nature of their subject and the effective learning of it are powerful influences on classroom practice (Thompson, 1992). It has been proposed that teachers of Mathematics hold at least four "dominant and distinct views of how mathematics should be taught" ranging from a constructivist learner-focused view to content-focused and classroom-focused views (Kuhs and Ball, 1986). It can be argued that a constructivist position would favour problem-solving, social engagement, and self-awareness in the process of active investigation, whereas several of the other positions would emphasise drill-and-recitation, computational procedures, and performance measured against standard indicators. In harmony with the constructivist position, it has been argued that "the use of small groups will lead to more meaningful assignments and less time spent on needless review and individualized seatwork" (Good, Mulryan, and McCaslin, 1992, p. 167). A recent issue of **Cooperative Learning** featured this theme (Davidson, 1989). This study is an analysis of data on the cooperative, competitive, and individualised learning preferences of teachers, in which the preferences of Mathematics teachers are selected for special attention. The findings are germane to speculation about a prevailing epistemology which may inform the belief systems of many Mathematics teachers, and which may directly influence notions of "good practice" in large numbers of Mathematics classrooms.

METHODS

Instrument

Preferences for cooperative, competitive, and individualised learning modes were obtained by means of the **Learning Preference Scale** (Owens, Barnes, and Straton, 1990). There are 33 items, brief statements about a feature of learning by cooperating with others, by competing with others, or by working alone. Items referring to each of these learning modes are content-matched in 10 groups, and one additional group contains unmatched items. Each content group, therefore, contains three matched cooperative, competitive, and individualised items, and each preference subscale in the LPST, therefore, is composed of 11 items. Teachers respond to each item by indicating how "true" or how "false" the statement is for them. A four-point Likert response scale is used, and numerical values are assigned to the answers on a 4-3-2-1 basis, with 4 representing the strongest preference. For six items expressed in negative phrasing, the scoring is reversed. Three main subscale scores (minimum 11, maximum 44) are calculated for each teacher, indicating strength of preference for Cooperative, Competitive, and Individualised learning situations.

The version of the LPST used in England was identical to the Australian edition. The version used in Minneapolis was an "American Revision" in which three one-word alterations were made (e.g., "grades" substituted for "marks" in reference to assessment). These changes were minor matters of idiom rather than major ones of substance. A complete handbook including the LPST and its two companion scales for students and parents is now available (Owens and Barnes, 1992). Data from both England and the United States are included.

Sample

The Australian (Sydney) sample of teachers was drawn from more than 30 schools and in-service courses in the metropolitan area in the early 1980's. A total of 619 teachers completed the LPST, three quarters of whom were secondary teachers. Although a wide range of schools were sampled (single-sex, coeducational, government, Catholic), data were collected only from staff who voluntarily completed the scale. In some schools the entire staff participated; in others, fewer than half returned completed forms. Participants from primary schools were predominantly female, whereas secondary teachers were principally male; overall there were slightly more females in the sample. More than 80% had greater than three years of teaching experience (Table 1).

The American (Minneapolis) sample of teachers was drawn from nine schools in two suburban school districts in a major midwestern city in the United States in the early 1980's. A total of 342 teachers completed the LPST, two thirds of whom were secondary teachers. A linked set of schools (two elementary schools-one junior high school-one senior high school) was chosen in each district, with the addition of one more elementary school and an n-service group. All schools were public and coeducational. As in Australia, participation was voluntary, elementary teachers were predominantly female, and secondary teachers were principally male. Again, there were more females than males in the overall sample. More than 80% had greater than ten years of teaching experience (Table 1).

Insert Table 1

The English (Midlands) sample of teachers was drawn from 13 secondary schools and one in-service group in six Midlands counties in 1991. A total of 278 teachers from coeducational government and Catholic schools completed the LPST. There were more males than females in the sample, and participation was voluntary. No information on teaching experience was gathered from these teachers (Table 1).

Procedures

The teacher data in Sydney were gathered by the author with major assistance from collaborating teachers in a number of schools. In Minneapolis, the data were gathered by the author as a result of a direct personal appeals to school staffs, backed up by administrative reminders. In England, direct personal appeals assisted by collaborating teachers enabled collection of the data. Participation in each situation was voluntary and confidentiality was assured.

Results

A two-way analysis of variance was carried out for each of the three LPST scores with the three batches of data. The independent variables were Sex (2) and Teaching Subject (4 secondary only - England, 5 primary and secondary - Sydney, 5 elementary and secondary - Minneapolis). It is important to note that the Maths and Science teachers were combined in the data coding in Sydney and Minneapolis, while they were coded into separate categories in the England data. The main and interaction effects from the analyses of variance in the three separate locations are presented in Table 2.

Insert Table 2

Significant differences in learning preferences between male teachers and female teachers were found only for competitive learning in Sydney and Minneapolis. Males were more inclined to competitiveness than females in both locations. The same difference was noted in England but this did not reach significance.

Differences between teaching subject groups required additional analyses using standard tests of significance. The Scheffe Procedure was selected to follow a oneway analysis of variance of the data for each testing location. In several instances this conservative test of significance failed to clarify differences evident from the overall analysis of variance. In Sydney (Table 3A) this further analysis revealed that Infants and Primary teachers had greater preference for cooperative learning than secondary Maths/Science, and to some extent, Humanities teachers.

Insert Table 3A

Amongst secondary teachers, Maths/Science teachers and Social Science teachers expressed greater preference for competitive learning than Humanities teachers. Finally, Infants teachers had stronger inclinations toward individualised learning than secondary Maths/Science teachers.

In Minneapolis (Table 3B) there were no apparent significant differences amongst the groups of teachers for cooperative•learning despite markedly low scores from Maths/Science teachers, nor for individualised learning despite low scores from both Maths/Science and Humanities teachers.

Insert Table 3B

For competitive learning, Maths/Science teachers expressed stronger preferences than all Elementary teachers, and Humanities teachers had stronger preferences than Upper Elementary teachers.

In England (Table 3C) the only difference to reach significance was that Maths teachers expressed a stronger preference for competitive learning than Humanities teachers.

Insert Table 3C

When the three locations were combined into a single database, very strong differences became evident in preferences for learning (Table 4).

Insert Table 4 and Table 5

Sydney teachers are noticeably more competitvely and individualistically inclined, and less cooperatively inclined than•the teachers from Minneapolis or from the England Midlands Counties (Table 5).

DISCUSSION

The picture that emerges from this analysis is that secondary teachers in Sydney schools seem to function in a predominantly competitive-individualistic ethos, if learning preferences are any guide to teaching practices. These data from teachers are closely paralleled by similar data from students both in Sydney and in Perth (Owens, Nolan, and McKinnon, 1992). Both at primary and secondary level, students express strong preferences for competitive and individualistic learning in comparison with students from the same English and American schools used in this study. This is true particularly for the boys. Moreover, students from New Zealand schools seem also to be strongly inclined this way (Nolan, McKinnon, and Owens, 1992), leading one to postulate a stereotypically male, combative and confrontive "Antipodean mind-set". Teachers and students of this persuasion reinforcing each other's common values and practices create a distinct, and not necessarily healthy, web of expectations for schooling. It is, to say the least, a far cry from the conception of "humane learning communities" cogently advanced by Campbell and Robinson(1979).

Into this general ethos teachers bring the concerns of their particular discipline and the customs of subject matter teaching. The data from this study seem to show that both in Australia and overseas, teachers of mathematics are strongly inclined toward the use of competition in learning, and that they may look with some scepticism at the possibilities of small-group cooperative learning. This set of findings lends some credence to the conception of "teacher epistemologies" which has been advanced to explain characteristic ways in which teachers of different subject matter approach their tasks (Young, 1981, 1992). Such an epistemology for mathematics teachers, insofar as it is evident in preferences for learning, appears to contain the desire to demonstrate superiority, the need to strive to be best, the selection of tasks which lead to ranking of performance, tidiness of procedure, and direct comparison of results. An obvious question is "how does it happen?". To what extent are persons of this existing persuasion drawn to the notion of teaching mathematics, and to what extent are they trained to see the importance, indeed inevitability, of these aspects of teaching as they experience the combined power of discipline-based teacher education and professional socialisation?

REFERENCES

Campbell, W. and Robinson, N. (1979). What Australian society expects of its schools, teachers, and teaching. St Lucia: University of Queensland, Department of Education.

Davidson, N. (Ed.) (1989). Cooperative learning and mathematics (Special Issue). Cooperative Learning, 10(2).

Good, T., Mulryan, C., and McCaslin, M. (1992). Grouping for instruction in mathematics: A call for programmatic research on small-group processes. In D. Grouws (Ed.), Handbook of research on mathematics teaching and learning (pp. 165-196). NY: Macmillan.

Kuhs, T. and Ball, D. (1986). Approaches to teaching mathematics: Mapping the domains of knowledge, skills, and dispositions. East Lansing: Michigan State University, Center on Teacher Education.

Nolan, P., McKinnon, D., and Owens, L. (1992, November). The learning preferences of New Zealand students. Paper presented at the Joint Conference of the Australian Association for Research in Education and the New Zealand Association for Research in Education, Geelong.

Owens, L. and Barnes, J. (1992). Learning Preference Scales - Handbook and Test Master Set: Teachers, Students, Parents. Hawthorn: Australian Council for Educational Research.

Owens, L., Barnes, J., and Straton, R. (1990). Learning preference scale - teachers. Sydney: University of Sydney, School of Teaching and Curriculum Studies.

Owens, L., Nolan, P., and McKinnon, D. (1992, November). A comparison of the learning mode preferences of students in four countries -Australia, New Zealand, England, USA. Paper presented at the Joint Conference of the Australian Association for Research in Education and the New Zealand Association for Research in Education, Geelong.

Thompson, A. (1992). Teachers' beliefs and conceptions: A synthesis of the research. In D. Grouws (Ed.), Handbook of research on mathematics teaching and learning (pp. 127-146). NY: Macmillan.

Young, R. (1981). A study of teacher epistemologies. Australian Journal of Education, 25, 194-208.

Young, R. (1992). Critical theory and classroom talk. Clevedon: Multilingual Matters.

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Table 1. Background Information For the Samples of Teachers in Three Countries Who Completed the Learning Preference Scale-Teachers

Location	N[a]	Sex	Experience	Subject/Level
Australia: Sydney	619	M: 281 F: 335	< 3 yrs: 127 > 3 yrs: 490	Infants: 64 Primary: 85 Humanities: 186 Social Science: 98 Science/Maths: 132 Industrial Arts: 19 Home Science: 14 Physical Educ: 6
USA: Minneapolis	342	M: 148 F: 188	< 10 yrs: 55 > 10 yrs: 276	Lower Elem: 28 Upper Elem: 89 Humanities: 67 Social Science: 35 Science/Maths: 48 Industrial Arts: 9 Home Science: 11 Physical Educ: 11 Special Educ: 23 Other: 7
England: Midlands	278	M: 141 F: 125	no information gathered	Humanities: 95 Social Science: 26 Science: 45 Maths: 40 Craft/Technolog y: 9 Home Science: 7 Physical Educ: 18 Special Needs: 13 Other: 1

[a] Some of the teachers in each sample returned incomplete information.

Table 2. Teaching Subject x Sex of Teacher Analyses of Variance of the Cooperative, Competitive, and Individualised Subscale Scores (LPST)

Source of Variance		Cooperation		Competition		Individualisation	
	df	MS	F	MS	F	MS	F
Australia: Sydney			N = 564				
Subject	4	73	6.4***	48	2.8*	28	2.2
Sex	1	4.7	<1	332	19***	3.1	<1
Year x Sub	3	6.4	<1	18.5	1.1	5.3	<1
England: Midlands			N = 205				
Subject	3	14.7	<1	111	3.7*	1.5	<1
Sex	1	25.2	1.2	21	<1	1.2	<1
Year x Sub	3	9.9	<1	10.7	<1	13.9	<1
USA: Minneapolis			N = 267				
Subject	4	52.7	3.3*	77	3.1*	33.9	2.5*
Sex	1	3.6	<1	195	7.8**	1.3	<1
Year x Sub	4	4.4	<1	41.4	1.6	8.8	<1

** $p < .001$

Table 3A. Comparisons of the Learning Mode Preferences of Primary and Secondary Teachers in Sydney Using a Oneway ANOVA with the Scheffe Procedure (alpha=.05).

LPST Subscale Score	Higher Mean	Differs significantly from	Lower Mean
Cooperative Learning Preference	Infants	<--------------->	Maths/Science
		and	
	Primary	<--------------->	Maths/Science Humanities
Competitive Learning Preference	Maths/Science Social Science	<--------------->	Humanities
Individualised Learning Preference	Infants	<--------------->	Maths/Science

Table 3B. Comparisons of the Learning Mode Preferences of Primary and Secondary Teachers in Minneapolis Using a Oneway ANOVA with the Scheffe Procedure (alpha=.05).

LPST Subscale Score	Higher Mean	Differs significantly from	Lower Mean
Cooperative Learning Peference		No Significant Differences	
Competitive Learning Preference	Maths/Science	<--------------->	All Elementary
		and	
	Humanities	<--------------->	Upper Elementary
Individualised Learning Preference		No Significant Differences	

Table 3C. Comparisons of the Learning Mode Preferences of Secondary Teachers in England Midlands Counties Using a Oneway ANOVA with the Scheffe Procedure (alpha=.05).

LPST Subscale Score	Higher Mean	Differs significantly from	Lower Mean
Cooperative Learning Preference		No Significant Differences	
Competitive Learning Preference	Maths	<--------------->	Humanities
Individualised Learning Preference		No Significant Differences	

Table 4. Oneway Analysis of Variance by Location of the Cooperative, Competitive, and Individualised Subscale Scores (LPST)

Source of Variance	df	Cooperation		Competition		Individualisation	
		MS	F	MS	F	MS	F
Testing Location: Sydney, English Midlands Counties, Minneapolis N = 1238							
Location	2	359	23.9***	1496	61.4***	393	29.4***

** p < .001

Table 5. Comparisons of the Learning Mode Preferences of Teachers in Sydney, English Midlands Counties, and Minneapolis Using a Oneway ANOVA with the Scheffe Procedure (alpha=.05).

LPST Subscale Score	Higher Mean	Differs significantly from	Lower Mean
Cooperative Learning Preference	English Midlands Minneapolis	<---------------->	Sydney
Competitve Learning Preference	Sydney	<---------------->	Minneapolis English Midlands
Individualised Learning Preference	Sydney	<---------------->	Minneapolis English Midlands

THE CONCEPT OF FAIRNESS IN SIMPLE CAMES OF CHANCE

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This study examines the mathematical concepts of "fairness" and "expectation" in probabilistic situations. The subjects were 40 high school students in Semester I, Year 11, Maths in Society classes in three Queensland high schools. Twenty "gamblers" were identified by questionnaire and subsequent interview. A control group of similarly achieving "non-gamblers" was selected. The research compares the ability of each group to contract a working definition of the concept of mathematical expectation and to use this concept in determining the fairness of a number of games of chance.

This study examines the mathematical concept of "fairness" as it applies to simple single-event games of chance involving coins, dice and cards.

In the determination of fairness, two aspects are examined:

- -the misuse of an heuristic of "representativeness" or "availability"
- -the use of an intuitive understanding of the concept of "expectation".

Misconceptions in probabalistic reasoning involving the use of "representativeness" and "availability" heuristics have been well documented by researchers including Shaughnessey (1977, 1981, 1983), Scholtz (1986), Tversky and Kahneman (1982), and Peard (1991a, 1991b, 1991c).

The use of "representativeness" to determine the fairness of a coin or game is illustrated when in situations the subject takes a short term sequence of events as being "representative" of the long term situation and erroneously concludes bais or unfairness.

"Availability" is used to come to the same conclusion by reasoning that such short term sequences are not readily recalled. More "balanced" results are more readily "available".

Bright, Harvey and Wheeler (1981) in a study of fair and unfair games claim that "fairness" is best described by calling attention to an intuitive understanding of "unfairness". In refering to students in years 4-8 they claim that "Helping students recognize when a situation is fair or unfair is a reasonable expectation of the school curriculum." (p.50). Research by Anderson and Pegg (1988) also reported difficulties primary school pupils encountered with the determination of fairness.

The mathematical concept of fairness, as opposed to a merely intuitive understanding, relies on the concept of "expectation". A game is "fair" if all participants have equal mathematical expectation. This in turn requires an understanding of mathematical expectation which is defined as the product of probability and return.

These concepts are clearly beyond the elementary level but require the application of only basic probabilistic reasoning. For simple games involving only two players, one need only determine the probabilities for each to win and then calculate the required amounts for each to be a constant product (or inverse proportion). This constitutes an effective concept of equal mathematical expectation for both players.

Bright et. al. note:

In complex situations it may be difficult to determine mathematically whether a situation is fair. (p 50)

Lovitt and Clark (1988) questioned whether pupils about to leave school had realistic ideas about the outcomes of gambling and concluded that "there is a huge gap between perception and reality" (p.77) in which pupils demonstrated misconceptions of the concept of expectation. Although they did not refer to any heuristic involved in arriving at these misconceptions, it would appear that an "availability" heuristic was in fact involved.

The inclusion of basic probability and its applications in the general school mathematics curriculum, both elementary and secondary, has been a relatively recent development. Pereria and Swift (1981) writing in the N.C.T.M. Yearbook made a strong arguement for probability to be part of every students education. Since then considerable progress has been made world wide as is evidenced by the N.C.T.M. statement of Standards in the

U.S.A. and the inclusion of "Chance and Data " in the Australian National Statement which makes specific reference to "fairness" and "expectation".

In Band B (upper primary) only an intuitive notion of fairness is expected. Possible activities include:

> *"Make non-numerical predictions about about equally likely events such as those involved in rolling a fair die and compare predictions with results of experiments. (p.170)*

Reference to expectation first occurs in Band C (lower secondary) where possible activities include:

> *"Investigate uses of probability in insurance... Study common games of chance to find the expected return....note that statements of odds which appear in gambling contexts reflect statements of subjective probability as well as statements of return on money invested....return on a win may be high but the chance of that win is correspondingly small." (p.175).*

In Band D (upper secondary) possible activities include:

> *"Devise, play and analyse a variety of "fair" and "unfair" games. Calculate and interpret expected values..."(p.182).*

However numerous difficulties with the implementation of such programs have been reported. In Australia, teacher unfamilarity with much of the content is recognised. See for example Peard (1987). Pedagogical problems with the teaching of probability are also well documented. See
Garfield and Ahlgren (1986, 1988), Kapadia (1984), Brown (1988), Pegg (1988), Green (1982, 1986), del Mas and Bart (1989).

Thus it is reasonable to assume that at the present time very few students will have had formal instruction in the topics of fairness and expectation prior to the Senior Secondary grades and that only some will gain this knowledge in these years.

OBJECTIVES:

The subjects in this study were 40 high school students in Semester I, Year 11, Maths in Society classes in three Queensland high schools. Two of these schools were in a lower socio-economic region, close to horse racing,dog racing and trotting tracks. Many senior students in these schools followed the races.

The study is part of a larger study investigating the construction of various probabilistic concepts within a social context by students whose background includes a familiarity with the phenonomen of gambling, particularly in relation to "track" betting. These are subsequently refered to as "gamblers". Interview questions established that all of these subjects were familiar with betting in track situations, the use of "odds", and methods of calculating payouts.

The objectives of the study were to determine:

(1) The pupils ability to recognise fairness in simple games of chance.

(2) Whether or not an heuristic was misused in incorrect identification.

(3) Whether or not there was any difference in this ability between "gamblers" and "non-gamblers"

(4) The ability of the students to recognise or construct a concept of expectation in simple games of chance in which players have unequal chances.

(5) The ability to use the concept of expectation in determining fairness.

(6) Whether or not these abilities were related to: social background (gambling), school achievement, gender.

METHODOLOGY:

The "gamblers" were identified by questionnaire administered with the help of either the regular classroom teacher or a special needs teacher. A subsequent interview was given to validate responses. Only those indicating a "great deal" of interest in at least one form of track racing were considered as "gamblers". A control group of "non-gamblers" was selected from those responding negatively to all forms of gambling and games of chance.

All schools were coeducational and an approximately equal number of male and females responded positively to interest in gambling. Thus a balance of subjects by gender was easily obtained. A balance of subjects by achievement was also obtained.

The research methodology employed was that of the structured clinical interview as described by Romberg and Uprichard (1977).

The interview asked open-ended questions relating to:

Category 1 - Representativeness and Fairness.

- the subjects' ability to recognise when a simple game of chance is "fair" and whether or not a heuristic of representativeness or availability was used in the decision making.

Questions:

The first questions asked were of the type:

1. (a) "You and I play a game of chance in which a coin is tossed. Heads I win, Tails you win. Of the last 15 people who played this game with me 10 lost. Is this a fair game?"

Similar questions relating to rolling a single die and drawing cards from a deck followed.

These questions are similar to those asked by Shaughnessey (1981). He reported a high incidence of the use of availability to conclude that the coin tossing game was not fair.

Those believing the games to be unfair do so by either using the short term results, for example, of 15 tosses to be "representative" of the long term probability of the coin or reply that they expect the next person to lose since "people tend to lose at this type of game" (availability).

Thus the next questions asked in this study were:

(b) "Is the coin/die/card game fair?"

(c) " Why?" or "Why not ?", depending on response.

Follow-up questions in the structured interview were of the type:

To those who responded affirmatively to (b)

(d) "How many tosses would you need to conclude that the coin was unfair?"

Those who recognised that a very long run was required before bais could be suspected were considered to be free of the misuse of the representativeness heuristic.

2. (a) "You and I play a game of chance which involves throwing a single die. We each bet $1, winner takes the $2. If the numbers are 1, 2, 3 I win, if they are 4, 5, 6 you win. Is this a fair game?"

(b) "If we change the rules so that if they are 1, 2, 3, 4 I win, 5, 6, you win. Is this a fair game now?"

(c) "Why or why not?"

(d) "Can we change the amounts each player puts in to make this game fair?"

This last question then leads in to the concept of "expectation"

Category 2 - Expectation and Fairness.

Questions: (following from above)

3. (a) "Since I have the better chance of winning can we make the game fair by increasing the amount I put in?"

Those who responded negatively to this were considered to have no concept of expectation. Typical responses were:

" You will still have a better chance than me and that's not a fair game."

To those who responded affirmatively:

(b) "How much should I put in?"

To demonstrate a basic understanding that expectations can be made equal, it was not required that the subject use formal mathematical language. A typical response was:

"Well you have four chances to my two, that's twice as many. So if you put in twice as much, that would

be fair."

The extent of understanding was investigated further:

(c) "What if I chose five numbers and left you with only one? How much should I put in now?"

For those who were able to answer this correctly different situations were then investigated.

e.g.

(d) "If we draw cards from a deck and I choose any Ace leaving you the rest, how much more than me should you put in to make the game fair?"

(e) "If I choose just one card such as the Ace of Spades,how much now?"

(f) "If I choose the 16 "coloured cards" - ace, king ,queen,jack of each suit, leaving you the 36 remaining cards and I put in $1, how much should you put in?"

Those who were able to demonstrate consistently in all of these situations that "fairness" can be established by each contributing an amount in inverse relationship to the probability (i.e. an equal product of probability and return or equal expectation) were considered to have a complete understanding of the basic concept.

An exact answer to the last question was required for this. It was not sufficient to reason along the lines (as did some):

"I have more than twice your chances so I should put in more than twice as much."

A "complete" understanding required reasoning that resulted in the calculation of 36/16 x $1 = $2.25.

RESULTS AND ANALYSIS OF DATA:

From the responses to these questions subjects were classified:

Category 1 - Representativeness and Fairness

(1) Recognises a fair simple game

33 of the 40 were able to recognise that in all situations the game/coin/die were fair and that deviations were not unreasonable.

(2) Uses an heuristic to misjudge a fair game

5 of the 40 were classified in this category.

Of these 3 responded using the "representativeness" heuristic and 2 using an "availability" heuristic.

2 reponded that they were unable to make a decision.

None of the 5 used the heuristic in questions of the type of 1(e),(f)-very short sequences.

5 of the 7 were non-gamblers but due to the small size of this category no test of significance was performed.

Rather, we note that the majority of both gamblers and non-gamblers were able to recognise that the situation itself was in fact fair.

(3) Free of the "representativeness" misconception

(correct response to Q.1(d))

Of the 33 who recognised fairness 23 were able to conclude correctly that a much longer sequence than that given would be required to infer bias or unfairness.The others were unsure or undecided.

Category 2 - Expectation and Fairness

(1) No knowledge of mathematical expectation.

These subjects were unable to answer Q3(a) correctly and would tend to reason: "A game can only be fair if each player has the same chance of winning" Two "non-gamblers" admitted to having no basis on which to make decisions of fairness.

Total: 21 Gamblers : 8 Non gamblers: 13

(2) Some intuitive knowledge of the use of expectation in determining fairness. These subjests answered questions 3 (b) and (c) correctly but were unable to answer all of the more complex questions 3 (d) - (f)

Total: 13 Gamblers :10 Non gamblers: 3

(3) A thorough knowledge of the basic concept of mathematical expectation as demonstrated by their responses to all parts of question 3

Total: 6 Gamblers : 5 Non gamblers: 1

The Null hypothesis

Ho: "There is no difference between the gamblers and the non-gamblers in their knowledge of mathematical expectation." was tested using a Chi-squared test of statistical significance and rejected at the 5% level.

Table 1

	Observed				Expected Under Ho			
	None	Some	Thorough		None	Some	Thorough	
Non-Gamblers	13	3	1	18	9.45	5.85	2.7	18
Gamblers	8	10	5	22	11.55	7.15	3.3	22
Total	21	13	6	40	21	13	6	40

An analysis of this ability by achievement:

Table 2

	Observed			
	None	Some	Thorough	
HA	8	8	3	19
LA	13	5	3	21
Total	21	13	6	40

Table 3

	Oberved			
	None	Some	Thorough	
Male	8	7	3	18
Female	13	6	3	22
Total	21	13	6	40

In both of these the Null hypothesis cannot be rejected and we conclude that this knowledge is not related to either school achievement or gender.

IMPLICATIONS:

Category 1

Since the misuse of an heuristic to conclude unfairness was not common amongst either group we cannot compare groups. These misconceptions were not as frequent as is reported in the literature. Shaughnessey (1981), for example, found the misuse of availability to imply unfairness widespread even amongst college entrants. Tversky and Kahneman (1982) noted that "misconceptions are not limited to naive subjects" (p.5). However Kapadai (1984) has questioned much of this research and suggests that some of the misconceptions may actually refer to misinterpretation of the question. The results of this study which were obtained from a structured clinical interview rather than questionnaire or test items would seem to support Kapadia in this.

Category 2

The fact that the gamblers were significantly better at using expectation to determine fairness has a number of important implications.

First, the concept is not part of the regular school curriculum - they do not use the term "expectation" but construct what is essentially an equivalent procedure.

Or as Davis (1989) says "as intelligent responses to their environment"(p.32).

Second, since all of the gamblers were familiar with track betting, the use of "odds" in betting situations and the calculation of resulting payouts, it is hypothesised from the results of this study that this mathematical knowledge may be attributed to the prevalence of gambling within the social background of this group.

The fact that this ability did not relate to school achievement or gender would tend to give support to the hypothesis.

As such, the knowledge may be considered as a form of "ethnomathematics" as defined by D'Ambrosio (1985):

> *..mathematics which is practised among identifiable cultural groups (whose) identity depends largely on focuses of interest and motivation. (p. 45)*

This has implications for the classroom teacher. As Clements (1988) says "It needs to be remembered that often in Australia there are unique factors influencing how children learn mathematics." (p.5)

With the concepts of fairness and expectation now specifically within the curriculum, the teacher must be aware of the knowledge that pupils bring with them to the classroom.

REFERENCES

Anderson, P., & Pegg, J. (1988). Informal probability - An investigative teaching approach. Mathematical interfaces. *The Proceedings oth the 12th Biennial Conference of the Australian Association of Mathematics Teachers*, (pp. 92-95). Newcastle.

Bishop, A. J. (1988). *Mathematical enculturation: A cultural perspective on mathematics education*. Dordrecht: Kluwer.

Bright, G., Harvey, I., & Wheeler, M. (1981). Fair games and unfair games. In A. P. Shulte (Ed.), *Teaching Statistics and Probability*. Reston VA: NCTM.

Brown, C., Carpenter, T., Kouba, V., Lindquist, M., Silver, E., & Swalford, J. (1988). Secondary school results for the fourth MAEP maths assessment. *Mathematics Teachers, 81*(4), 241-248.

Clements, M. A. (1988). Looking into the past as a guide to the future. *Vinculum, 24*(4), 3-6.

D'ambrosio, U. (1985). Ethnomathematics and its place in the history and pedagogy of mathematics. *For the Learning of Mathematics, 5*(1), 44-48.

Davis, G. E. (1989). Attainment of rational number knowledge: A study on the constructive evolution of mathematics. In N. F. Elleston & M. A. Clements (Eds.), *School Mathematics The Challenge for Change*, (pp. 31-45). Geelong: Deakin University.

del Mar, R. C., & Bart, W. M. (1989). The role of an evaluation exercise in the resolution of misconceptions of probability. *Focus on Learning Problems in Mathematics, 11*(3), 39-53.

Garfield, J., & Ahlgren, A. (1986, August). A study of difficulties in learning probability and statistics. *Abstract of Invited Papers. In the Second International Conference on the Teaching of Statistics*. Victoria: University of Victoria, Canada, 4.

Garfield, J., & Ahlgren, A. (1988). Difficulties in learning basic concepts in probability and statistics: Implications for research. *Journal for Research in Mathematics Education, 19*(1), 44-59.

Green, D. (1982, August). A survey of probability concepts in 3000 pupils aged 11 - 16 years. *Proceedings of the First International Conference on Teaching Staistics*. University of Shaffield. 776-782.

Green, D. (1986, August). Children's understanding of randomness. *Abstract of Invited Papers. The Second International Conference on the Teaching of Statistics*. Victoria: University of Victoria, Canada, 5.

Kapadia, R. (1984). Probability - The subjective approach. *Mathematics Teaching, 108*, 46-49.

Kilpatrick, J. (1987). What constructivism might be in mathematics education, In J. C. Bergerson (Ed.), *proceedings of the Eleventh International Conference on the Psychology of Mathematics Education, (Vol. 1, pp. 3-27)*, Montreal: PME.

Lovitt, C., & Clark, D. (1988). *MCTP Activities Book* (Vol. 1), Woden: Curriculum Development Centre.

National Statement on Mathematics for Australian Schools (1990). Curriculum Corporation. Australian Education Council.

Peard, R. (1987). Teaching statistics in Queensland - qualifications and attitudes of teachers. *Teaching Mathematics, 12*(4), 13-18.

Peard, R. (1991a). Misconceptiona in probabilistic Reasoning. *Proceedings of the AARE Conference*, Surfers Paradise, November 1991.

Peard, R. (1991b). Misconception in probability: Mathematics ideas. *Proceedings of the 1991 Mathemathic Association of Victoria Comference*. La Trobe, Melbourne, December.

Peard, R. (1991c). *Misconception in probability* (Tech. Rep. No. 3). Brisbane, Queensland University of Technology: Centre for Mathematics and Science Education.

Pegg, J. (1988). *The chances are ....* Armidale, NSW: New England Mathematical Association.

Pereira-Mendoza, L., & Swift, J. (1981). Why teach statistics and probability - A rationale. In A. P. Shulte (Ed.), *Teaching Statistics and Probability* (pp. 1-7). Reston, VA: National Council of Teachers of Mathematics.

Romberg, T., & Uprichars, A. E. (1977). The nature of the clinical investigation. *Clinical Investigation in Mathematics Education*. Michigan: Research Council for Diagostic and precriptive Mathematics, 1-14.

Shaughnessy, J. M. (1977). Misconceptions of probability: An experiment with a small-group, activity-based, model building approach to introductory probability at the college level. *Educational Studies in Mathematics, 8*, 295-316.

Shaughnessy, J. M. (1981). Misconceptions of probability: From systematics errors to systematic experiemtns and decisions. In A. P. Shulte (Ed.), *Teaching Statistics and Probability* (pp. 90-100). Reston, Va: National Council of Teachers of Mathematics.

Shaughnessy, J. M. (1983). The psychology of inference and the teaching of probability and statistics. In R. W Scholz (Ed.), *Decision Making Under Uncertainty*. Elsevier North-Holland.

Tversky, A., & Kahneman, D. (1982). *Judegement under uncertainty: Heuristics and Biascis*. Cambridge University Press.

MULTIPLE PERSPECTIVES ON TEACHER CHANGE

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The study of teacher professional development frequently has been limited in its subjects, its tools, and the duration of the research. This paper describes a study to examine teacher change from the perspectives of the teachers, their students, and the researcher, through the use of a variety of research tools. Each of these instruments was used in a variety of different forms addressing different specific issues. The guiding aim of the study is the comprehensive portrayal of the change process in teaching practice, knowledge and beliefs as experienced by eight junior secondary mathematics teachers. The longitudinal nature of this study was a deliberate attempt to describe the process of teacher change in the long-term. A professional development program for secondary mathematics teachers (the ARTISM program) provided an appropriate context because its design was informed by recent research into teacher professional development and because the program took a long-term perspective on teacher change. This combination of design factors was conceived to give due recognition to the complexity of the teacher change process.

Paper not available for publication

SECONDARY MATHEMATICS TEACHER EDUCATION: REPORT ON A DEVELOPING PROJECT

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At the Auckland College of Education, secondary teachers are prepared for the classroom in a one-year post-degree course. This involves lectures and 13 weeks classroom practice. In 1991 it was realised that the model being used was at odds with the classroom experience of the students.

An action research programme involving lecturers, students and associate teachers in the schools is in process. Its aim is to link the classroom experience to the model of teaching demonstrated at College. The programme has gone through four cycles in the action-research model.

This paper describes the four cycles and reports on lessons learnt during the research. The exciting part of the emerging model is a collegial process which involves lecturer, students and associate teacher working together in the classroom with more experimentation and reflection on teaching as an art. Other benefits are increased cost effectiveness and closer links between College and schools.

At the Auckland College of Education a one year secondary teacher training course is offered to graduates. They are required to do two curriculum subjects, professional studies and an education course. Fourteen weeks, in three blocks, are spent in schools for the teaching practicum. This project arose out of an increasing dissatisfaction with the teaching practicum for the mathematics students. During 1990, the students were reporting difficulties matching their practicum experience with the College experience, and the lecturers were feeling that the models of mathematics education being presented to the students in schools were generally far from ideal. Improving teaching practicum became a major focus for development.

Glossary:		
	pupil	- secondary school students in the classes
	student	- student teachers at ACE
	lecturer	- lecturer at ACE
	TP.1, 2, 3	- teaching practice blocks when students are in schools

Statement of the Problem

The teaching experience is the time when students experiment, take risks, and challenge existing ideas about teaching. Thus they will find a teaching style and methodology which fits their developing philosophy. In schools some students are with innovative, experimental teachers. Such students flourish. In contrast, many students are encouraged to emulate the associate rather than find their own style. These students do not gain the confidence to experiment. A further problem is a lack of a collegial style of assistance.

This research is based on the belief that there are two major factors which will make better teachers: an increased focus on LEARNING and an attitude of continual self-development. It is further assumed that the teaching experience is the setting which is most likely to provide opportunities to develop these characteristics, and that the collegial relationship is the means which is most likely to help students develop in these areas.

Working within the constraints of a school's right to choose the associate teacher, and the student's right to choose their TP school, how do we create a teaching experience which will push students to their limits, consolidate a personal philosophy and style of teaching based on student learning, and develop an attitude of self-reflection and analysis of pedagogy?

Context

Since 1990 the secondary teacher mathematics education course has been based on a model proposed by Maher [1988]. Its premise is that teachers' beliefs, knowledge, judgments, thoughts and decisions have a profound effect on the way they teach as well as on student learning in their classrooms. The model has three interrelated parts (see fig.1.):

STUDENT AS RESEARCHER

a) Students experience their own learning in problem solving sessions. This is recorded in their journals, and discussed in class. It is analysed to illuminate the the roles of the learner and teacher.

b) Students give diagnostic tests to pupils, interview them, then work with them on overcoming misconceptions. This is an observe - analyse - guide - facilitate cycle. They write a report on the implications for teaching.

STUDENT AS PRACTITIONER

In the Hewet Project model [de Lange, 1988] students learn to design problem driven lessons, implement them using a variety of teaching approaches, and evaluate the results (see fig.2.).

STUDENT AS PHILOSOPHER AND MATHEMATICIAN

Students write down their thoughts on the teaching and learning of mathematics and contribute to discussion sessions on this issue. They are encouraged to create and solve mathematics problems and to view themselves as practising mathematicians.

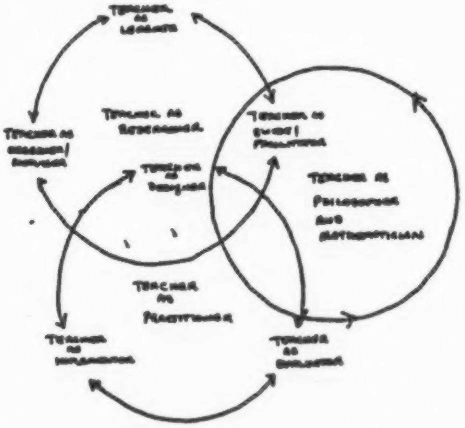

FIG.1 MODEL BASED ON [illegible] ([illegible], p499)

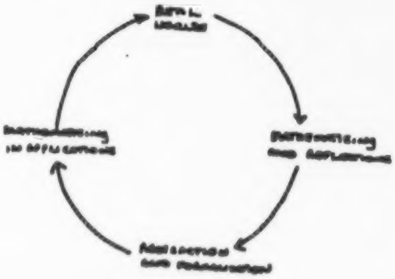
REAL WORLD
MATHEMATISING AND REFLECTION
ABSTRACTION AND FORMALISATION
MATHEMATISING IN APPLICATIONS

FIG.2 FROM DE LANGE (1987, p72)

The ongoing development of this course caused a mismatch between the College course and TP, a universal problem identified by Zeichner [1990]. The course focuses the student on the reflection and analysis of classroom learning. Simon and Schifter [1991] describe a teacher as the creator of problem solving situations grounded in real world experiences that are well known to the pupils, thus enabling them to build on already present cognitive structures. The teacher must balance the interests and questions generated by the pupils with the goals of the curriculum. The teacher is the facilitator, asking probing questions, encouraging the exploration of potential pitfalls and misconceptions with the aim of developing more resilient concepts. This helps the pupil to develop bridges between the mathematics in the real world and formalised mathematical ideas. Jaworski [1992] proposes a teaching triad comprising the management of learning, sensitivity to students, and mathematical challenge. "Only students themselves can construct their mathematical knowledge, relative to their own individual experience. In every moment of classroom action, some sort of construal occurs. A teacher needs to influence and interact with this construal."(ibid, p14)

On the other hand, New Zealand mathematics teachers are in danger of become professionally deskilled by increasing administrative and non-teaching pressures and by the provision of cookbook resources. A smoothly operating class is often considered paramount. Teaching proceeds with an assumption that the pupils are learning and teachers often have little opportunity (or encouragement) to diagnose what is blocking the learning of the pupils.

Associates' focus for the training of students on teaching practice is on classroom management and teacher-controlled learning. They pass on the 'tips of the trade', how to survive in the classroom, and how to become a good technoteacher. Zeichner [1986] describes this as a 'utilitarian teaching perspective'. This view of teacher training is grounded in teachers' own training experiences. Grossman [1992] sees this reproduction of the traditional role of the

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associate teacher as a formidable obstacle. Taking on a student teacher is seen as an 'on-site sabbatical', not an opportunity for personal learning. The other obstacle cited by Grossman is the 'expert - novice' relationship which deters the teacher from sharing the implementation of new teaching approaches.

In Zeichner's [1986] view, efforts to improve TP must be closely linked to efforts to reform the entire enterprise of teacher education, in which the teacher's role is linked to continued learning over the course of a career. He reports [Zeichner 1990] that a large number of current innovations in TP have been aimed at preparing students to be researchers of their practice. Thus teachers' practical theories are accorded a legitimacy. He has labelled this an inquiry-oriented practicum, where the school no longer serves as a model for practice but becomes a social laboratory for study. The obstacles experienced in Auckland during TP relate to Zeichner's list [1986 & 1990]. They are:

(1) the dominant view of TP as an exercise in apprenticeship;
(2) the lack of an explicit curriculum for the TP and the lack of connection between what the students are taught at College and in the schools;
(3) the uneven quality of TP supervision and the lack of formal preparation for both College lecturers and associates;
(4) the fact that schools are not set up for learning to teach; and
(5) the discrepancy between the role of the teacher as a reflective practitioner (as embedded in the goals of teacher education), and the role of the teacher as technician (which is dominant in practice).

In this action research programme the TP has been altered to increase the opportunities for research and reflection. Our belief is that this can happen through collegial models of interaction and by freeing the students from classroom management fears.

The Four Cycles of the Programme to Date

PLATFORM

Four selected students and four selected associates were paired with a lecturer for TP.3. 1991. Group meetings and team planning sessions were held and all three shared in the planning and teaching of the unit.

TEAM-TEACHING

Where possible on TP.2. 1992 lecturers visited students more than once and on the second or later times the lecturer became actively involved in the teaching of the lesson.

STAGE

On TP.3. 1992 students who were at the same school were paired together for one maths class. They shared planning and teaching

CROSS-CURRICULAR PAIRING

On TP.1. 1993 some students were paired to observe, team-teach and give mutual feedback.

In addition students were given the opportunity to pair with others in different subject areas.

INTERNSHIP???

Cycle.1. PLATFORM

Students reported teaching practice as unsatisfactory. It was observed that students did not focus on learning, nor were they seeing good mathematics teaching. It was decided to work intensively with a few students. The objectives were to work closely with associates whose teaching was consistent with the College programme, and for lecturers to become involved as teachers in the classroom. All participants were to be partners in a process of personal development.

Four selected students and four selected associates were paired with a lecturer for TP.3. 1991. Group meetings were held, ethnographies of the classrooms were made, and team planning sessions developed the units of work. All shared in the teaching and kept journals. A report was written describing the project and making a series of recommendations [Pfannkuch & Barton, 1991].

All participants were positive about the project, and the opportunity it provided for everyone to think about teacher development. All felt they had gained considerably. The students improved their teaching techniques in that the pupils were given work that was accessible, that challenged them to think, and that included strategies for generating mathematical enquiry. In the development and teaching of a unit the realistic learning model (see Context) was successfully used. The lessons that were formally evaluated were some of the most creative and interesting lessons that the lecturers observed that year.

The recommendations from the Report included:

◊Associates should be offered training, should be rewarded so that their work was valued.
◊Associates need to be involved with the student and lecturer, and should meet students before TP.
◊Lecturers should visit each student more than once each TP.

It was concluded that working with students rather than watching them was likely to lead to cooperative styles of teaching.

Cycle.2. TEAM TEACHING

After the successful experience of the PLATFORM cycle, it was decided to develop a more collegial working relationship with all students. The reason this was not happening with the associate was probably because of the mismatch between the associates and lecturers view of TP.

During TP.2. 1992 the lecturers attempted to visit every student twice with the same class. On the first visit the lecturer gave a formal observation, on the second visit the lecturer became involved in the planning and then taught with the student. Students particularly enjoyed planning together, and they also watched the lecturer demonstrate skills, e.g. drawing ideas from pupils' discussion. This cycle also included an attempt to establish a closer relationship with associates. Associates were encouraged to be present both during the lesson attended by the lecturer and at the debriefing afterwards. They became more interested in the process of teacher education and it was found that the points of view of the associate and the lecturer were closer than had previously been realised. More than previously, the discussions after the lesson centred on learning rather than management.

Students and associates responded positively to these attempts at a collegial TP. In particular the associates became actively involved in the practicum and more aware of the orientation of the lecturers. In turn, the lecturers renewed their experience of the realities of the classroom, and were perceived to be practitioners still in the process of learning. Thus the mismatch was reduced through movement from both sides. The benefits of a collegial model of TP were reinforced.

Cycle.3. STAGE

Despite the success of the team teaching, it was still impossible to plan and teach with all the students. In TP.2. two students placed in the same school had worked closely together and tried out each others ideas in their classrooms. And in July the students worked in pairs on a learning investigation assignment which went extremely well. These two events provided a foundation to try team-teaching in pairs. (Pairing students has been attempted in the primary section of ACE and overseas [Zeichner, 1990, p116, Copeland & Jamgochian, 1985]). Some of the anticipated benefits were: more communication about the topic being taught; students moving into collegial mode; opportunity to take more risks and try better teaching strategies; and more time to focus on learning.

Eight pairs of students worked together. They were asked to experiment, to try problem driven lessons and to use a variety of teaching strategies. Two visits were made by the lecturer to each pair, who became involved in the teaching of the lessons under the direction of the pairing. Both lecturers and students kept a journal, and specific feedback was sought from students and associates. At a meeting for associates, the experimental nature of the plan was explained and feedback sought. The associates welcomed the opportunity to be part of the experiment in team-teaching, and expressed surprise that the mathematics education focus was learning not classroom management, and collegial rather than expert/novice.

The pairing trial was effective mainly because students were together all the time: planning, teaching, and talking after the lesson. Compared with lecturer/student team teaching, there was much more continuity. This was useful for both non-innovative and innovative students, the former because they could 'see' what had been described at College when given the opportunity to try things out, the latter because this was a perfect opportunity to experiment with new-found skills. The main advantages were: that for one class they could concentrate on learning rather than management; that together they took risks with new approaches that they would not have countenanced on their own; that it was stimulating to bounce ideas and discuss how to teach a class with someone who had a similar view on the teaching of maths; that they supported one another in all aspects of the lesson; and that they learned strategies for teaching different things and handling management situations from one another.

Lesson planning unexpectedly took longer as more discussion ensued, so that the reasons for teaching in a particular way had to be clarified. This resulted in what the students thought were better quality and more interesting lessons. The lecturers noted that the paired students created more problem-solving situations in which pupils were active participants in the maths learning.

Associates reported that the students were more intensely on task, were thinking more, took more risks and the learning for pupils was improved. The discussions afterwards, often involving the associate, were interesting and focussed on improvements to the lesson for better learning. Some associates were worried about students not learning management skills.

The team-teaching worked best where the students were compatible and had worked out how to team-teach together. One disadvantage was that the level of uncertainty was increased in the class and each had to think quickly on their feet as they took cues from each other. The general impression of the pairs was the increased energy level and the enjoyment of teaching and supporting one another. A few students felt that the team teaching was not completely successful for them because of personal differences with the other student. They recommended more care with, and student control of, the pairing process. Despite this, they all felt that they benefitted from the experience and recommended that team teaching should continue next year.

From the lecturers' perspective the pairing worked well where the students were similar in ability and confidence about teaching. In cases where there was discrepancy in ability the weaker person definitely benefitted more than the other. It was noted that the associates were present more often, probably because they felt engaged in the process of teacher education. They were also drawn in by the more frequent and participatory lecturer visits and the idea of the classroom as a place for teacher development, equally for the associates as for the students and lecturers. Thus teacher change was a positive outcome of the trial for associates. However the question remains as to whether they will continue to experiment with their teaching as a result.

There is no doubt that students put much more into their learning, and that they made more progress towards effective teaching those from the previous year. This may have been because they were further forward than last year's group, or it could have been a result of the opportunity to operate in a collegial style. It was also apparent that students were more critical of their educational practice, and could justify their critiques in a way they were not able to do earlier in the year.

This trial began to address each of the five barriers to student learning mentioned above:

1). The dominant view of TP as apprenticeship was circumvented because students were not as reliant on their associates. Furthermore, with two working together, management was not a problem and therefore the feedback from the associate had to be on the teaching strategies used and the learning taking place. This also helped to break down the expert/novice relationship because it took the focus off teacher skills and the associate's style.

2). Students were given explicit instructions to use the pairing trial to experiment, and the associates were drawn into another orientation by becoming party to the trial and the philosophies which generated it. Thus the mismatch between school and College was reduced.

3). Those students on the trial were given special attention by both lecturers and associates. It is possible that some of the success of the project resulted from a Hawthorne effect, however a subjective assessment is that the quality of supervision was greater than for other students as the associates were drawn into the student - lecturer - associate partnership.

4). Although the schools were still not set up for learning to teach, special arrangements were made for the pairing trial.

5). The discrepancy between teacher as reflective practitioner and the teacher as technician was resolved by putting the latter concept on one side for the duration of the trial.

Students were developing a different model of mathematics teaching and developing an attitude of self-reflection and analysis of pedagogy. A major issue which remains is that pairing may be a high-order skill so that learning how to team-teach effectively takes the focus away from the other aspects of the TP. If so, can this be alleviated by team planning and a slow build-up towards team teaching?

Despite the success of the trial, and its feasibility as a permanent feature of TP.3., there are some implications to be considered:

1). If pairing enables experimentation and trying out new ideas, the practicalities of teaching must also be part of a student's experience. Students involved in this programme need to have another mathematics class on TP.

2). Students will not gain as much if they are in a relationship which they find difficult. Pairs should be chosen by the students, and, if possible, given a trial in some assignment work prior to TP.
3). Students need to be trained to team-teach and to observe. This can possibly take place before and during TP.2. with the lecturer, as well as in sessions prior to TP.3.
4). The increased links with associates have been shown to be most valuable. The pairs which worked best had associates who both participated in the classroom and agreed with the philosophy of team-teaching and self-reflection. All means to promote this relationship should be pursued.

Where To Now?

For TP.1. the exploration on improving the TP experience has become cross-curricular and pairs of students in two professional studies groups will observe each other and give feedback on teaching performance. Four mathematics students are trialling team teaching at their request. In preparation the students and lecturers learnt observation and feedback techniques which are designed to give the student autonomy over the feedback process and which encourage self-assessment.

On all TPs associates will be encouraged to adopt a collegial style with the students. Before TP.2. team-teaching will be built into the mathematics course by focussing on what happens when students pair for planning, experimentation, presentation; and what happens when the lecturers team-teach. On TP.2. lecturers will team-teach with students on the second visit. On TP.3. the paired teaching model will be used. The aim will be to promote a partnership between associate-students-lecturer for the TP in which all are expected to plan, teach, evaluate, observe, analyse and experiment.

Ways to link College with the students' first year of teaching are being explored. Is it possible to work more closely with the mathematics adviser or to provide in-service time at intervals throughout the first two years of teaching?

Mathematics teacher training in Auckland is moving towards an inquiry oriented and reflective model of learning to teach. Looking beyond this year, perhaps the next step is to have an internship model similar to that at the University of California, San Diego [Smith & Souviney, 1993] where two students share a class for the whole year with a lecturer working collegially with the students in the classroom. It has been noticed that students who have taken mathematics education papers at university bring a wider perspective to mathematics education and are more responsive to innovation. Therefore, in the year before internship, the students would be required to take such papers along with a practical component in schools.

References

Copeland,W. & Jamgochian, R (1985) Colleague Training and Peer Review, *Journal of Teacher Education*, 36(2) 18-21

Grossman, P. (1992) Changing Roles and Relationships of Teachers as Learners and as Leaders, *The Changing Contexts of Teaching*, Ann Lieberman (ed), University of Chicago Press

Jaworski, B. (1992) Mathematics Teaching: What Is It ?, *For the Learning of Mathematics* 12, 8-14

de Lange, J. (1987) *Mathematics Insights & Meaning*, OW & OC, Utrecht University

Maher, C. A. (1988) The Teacher as Designer, Implementer, and Evaluator of Children's Mathematical Learning Environments, *Jour. of Mathematical Behaviour* 6, 295-303

Pfannkuch, M & Barton, B (1991) *PLATFORM Report*, Auckland College of Education

Simon, M. A & Schifter, D. (1991) Towards a Constructivist Perspective: An Intervention Study of Mathematics Teacher Development, *Ed. Studies in Mathematics* 22(4), 307-332

Smith, J. & Souviney, R. (1993) Apprenticeship and Teacher Education Unpublished

Zeichner, K. (1986) The Practicum as an Occasion for Learning to Teach, *South Pacific Jour. of Teacher Education*, 14(2), 11-27

Zeichner, K. (1990) Changing Directions in the Practicum: Looking ahead to the 1990's, *Jour. of Education for Teaching*, 16(2), 105-119

AFFECTIVE DEVELOPMENT IN PRIMARY STUDENTS: AN INSTRUMENT FOR TEACHER AND STUDENT USE IN THE MEASUREMENT OF AFFECTIVE DEVELOPMENT IN PRIMARY SCHOOL STUDENTS

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This study resulted in the development of two instruments for teacher and student use in assessing the affective development of year 5, 6 and 7 Primary school students. Eighty-seven teachers rated the students in their class on a set of 22 affective characteristics which they deemed important to develop in their students as a result o+f their teaching. Factor analysis of these ratings resulted in two factors. Twenty-two teachers wrote classroom descriptors for the 11 characteristics making up the two factors which formed the basis for the instruments The descriptors were validated and the reliability of the two instruments was determined using a 25% sample of the children in these 22 classes. These instruments will help sensitise teachers to achieve a more appropriate balance between cognitive and affective objectives in their classrooms, and where there is dialogue between teacher and student following the student's self evaluation, will result in student empowerment.

> Affective issues play a central role in mathematics learning and instruction. ... Although affect is a central concern of students and teachers, research on affect in mathematics education continues to reside on the periphery of the field. If research on learning and instruction is to maximize its impact on students and teachers, affective issues need to occupy a more central
>
> (McLeod, 1992, p. 575)

In a major report prepared by the Schools Council for the National Board of Employment, Education and Training concerning Australia's teaching profession and the problems confronting schools, the authors asserted that:

> Yet no matter how frequently and cleverly schools and systems are restructured, little of import will change until the working relationships between teachers and students become more effective and productive.
>
> (1990, p.49)

It is our view that in part we, as educators, have brought the problem upon ourselves. We have been so convinced that schools exist to "push in" the essential content, process and skills of our disciplines, particularly mathematics and the sciences, that we have been deaf and blind to the human drama enacted before our very eyes. Day by day our students in many and various ways try to tell us that they are more than just "heads" to be filled with the school curriculum, but being "moulded in our own image" have feelings, attitudes, beliefs and needs. Perhaps above all they want to be valued and feel that, within school especially, their opinions count and their needs can be met. And day by day we ignore them.

It is our contention that how we feel is almost always more important to us than what we know, and that since behaviour is frequently determined more by how we feel about a situation than by what we know about it , the affective dimension of our lives will play a major role in our everyday living. Tyler (1973) comments in this regard that the most profound challenges to our society are not cognitive. They are challenges to our social unity, and to our individual righteousness, to our ethical standards and to our moral values, to our courage and to our compassions. If our schools dwell too much on cognitive outcomes, they will fail to contribute as they should to meeting those other important challenges. In this regard Tyler is supported by the research of Knight (1974), Wright and Headlam (1976), Popham (1978), Black and Dockrell (1980), Glasser (1986), Braithwaite (1988), and Kohl (1991).

What we are addressing here is the balance between cognitive and affective goals in everyday classrooms. Annesley and Clark (1990) have argued, and supported their argument from an extensive review of the literature, that what enhances the relationship between teachers and students and promotes a view of school that leads to student empowerment, is a situation where there is in each classroom a balance between the pursuit of cognitive and affective goals and where each goal is pursued as both a means and an end of education. They stress that neither should be seen as subservient to the other but many times will be blended naturally into each other during the teaching of any particular lesson.

Because we hold and support the view that little of import will change in schools until there is a change in teacher - student working relationships, and that some intervention is necessary to encourage a greater balance between cognitive and affective outcomes, we sought to design an assessment instrument for charting student growth in the affective domain which could be used by teachers and students. In this way students would learn to make judgements about their own development in this area of classroom learning. Indeed, since teacher judgements deriving from this instrument are essentially for formative purposes it seemed to us that a teacher and student could compare judgements, and this would allow each to share their perceptions on a wide range of affective outcomes. This interchange can only make the student feel that he or she is a partner rather than a recipient in the educational enterprise, thereby improving morale.

THE STUDY

The present study sought to investigate the portability of the Secondary school instrument developed by Annesley and Clark (1990) to the upper part of the Primary school.

The Sample

One hundred and ten teachers in grades five, six, and seven in 24 Primary schools in Townsville and Cairns were invited to participate in the study. Seventeen of these schools comprising two Roman Catholic and eight State Primary schools in Townsville and seven State Primary schools in Cairns formed the final sample. Eighty-seven teachers completed an initial survey questionnaire.

The Survey Questionnaire - Part A

The Survey Questionnaire was in two parts. Part A of the Survey Questionnaire was designed: (i) to determine the current situation in a sample of Primary schools in North Queensland regarding the development of affective characteristics at the upper grade levels; (ii) to gain some background details on the teachers participating in the study; and (iii) to ascertain what teacher educators can do at both preservice and inservice levels to assist teachers in developing the affective characteristics of the children they (will) teach. Results from Part A are not reported in this paper but can be found in Annesley and Putt (1992).

The Survey Questionnaire - Part B

Part B of the questionnaire sought information from the teachers concerning the affective characteristics they would like to develop in their students through their teaching. The list of 22 affective characteristics in Part B (see Table 1) together with a definition for each characteristic was compiled from those generated by Annesley and Clark (1989) together with those affective outcomes which were included in the current State Primary school syllabus documents for the various subjects. Respondents were asked to indicate each of the characteristics which they desired to see developed in their students as a consequence of their teaching. They were also given the opportunity to add other characteristics which were not on the list but which they regarded as important.

Insert Table 1 about here

The data indicated that there were no significant additions to the original list of 22 affective characteristics supplied to teachers. This was an interesting outcome since it suggested that the original list of 22 characteristics was comprehensive and had a measure of face validity.

The 87 teachers were invited to rate each student in their class on a scale from 5 (most satisfactory) to 1 (least satisfactory) on each of the 22 affective characteristics. Sixty-eight of the 87 teachers (a 78% response) returned ratings for each student in their class on the 22 characteristics. This represented a total of 1740 students being rated. Correlations between each of the variables are shown in Table 2. A principal component factor analysis was run on the data using both orthogonal and oblique rotations. A scree plot of Eigenvalues (as originally proposed by Cattell (1966)) and the Eigenvalue-one criterion (proposed by Kaiser (1960)) both suggested that a two factor solution best described the data. The first three Eigenvalues were 14.472, 0.963, and 0.285.

Insert Table 2 about here

The first factor was comprised of the 6 characteristics - Enjoyment of learning, Independence, Initiative, Appreciation of Language, Appreciation of Mathematics, and Curiosity. Inspection of these characteristics suggests aspects of general classroom attitudes that impact on academic performance. The second factor was comprised of the 5 characteristics - Caring for others, Obedience, Honesty, Courtesy, and Responsibleness. These characteristics are of a much more personal nature and could be components of a social behaviour factor. The criterion which was used for grouping characteristics was that the factor loadings obtained from the normalised varimax rotation should be at least 0.72. The oblique rotation yielded a similar result to the varimax. It would have added "Self Esteem" to factor 1 and left factor 2 unchanged.

A comparison of the two factors and the characteristics within each factor obtained from the Primary teachers in this study with that of Secondary teachers in a similar study reported by Annesley and Clark (1989) is illuminating. In their research with State, Catholic and Independent Secondary schools using some 122 teachers and 2044 students they also reported a two factor solution. Three of the six characteristics in the first factor were the same as for the Primary teachers namely, Enjoyment of learning, Independence and Initiative. The remaining characteristics for the Secondary teachers were, Positive attitude, Self motivation and Participation, while for the Primary teachers they were, Appreciation of Language, Appreciation of Mathematics and Curiosity. It is not hard to see reflected in the last three characteristics of the Primary school a "picture" of the essential curricula of that sector of education.

Of the five characteristics in the second factor, four were common to the Primary and Secondary teachers namely, Obedience, Honesty, Courtesy and Responsibleness. The fifth characteristic in factor two for the Primary teachers was Caring for others, while for the Secondary teachers it was Self discipline.

Having established a two factor model, the next task was to develop a draft of the instrument for rating students on the two factors. This involved a subset of the teachers writing a series of classroom/school behaviours (descriptors) for each characteristic within each factor. The behaviours were to be written in a way that would allow a teacher to assign a student to a position on a 5 to 1 scale for that characteristic where 5 is most satisfactory and 1 is least satisfactory. Twenty-two teachers were selected from the 68 who completed the matrix task (eight from Cairns and 14 from Townsville).

The researchers and the teachers met one afternoon after school in each centre to arrive at consensus on the descriptors for each of the levels of each characteristic associated with the two factors.

INSTRUMENT VALIDATION

While reliability forms one important dimension of an assessment instrument's characteristics, an even more important dimension is its validity, since it matters little if the instrument is consistent in its measurement characteristics if it does not measure what it purports to measure. A considerable amount of time was spent in determining the content and construct validity of the Measure of Affective Development in Primary School Students (MAPS). In essence we have followed a philosophy espoused by Salvia and Ysseldyke: "In a real sense, one does not validate a test, one conducts experiments to demonstrate that the test **is not** a valid measure of the trait or construct." (1988, p.139)

Each of the teachers was supplied with a copy of the descriptors for the 5 points on the scale for each factor, and were asked to indicate whether each statement was acceptable or whether it could be improved. These suggestions were considered by the researchers in compiling the final version of the instrument for use in establishing its reliability.

RELIABILITY TRIAL - TEACHERS' TASK

Having established a valid set of descriptors, test-retest reliability of the instrument was determined. For each of the 22 teachers involved in this phase a random sample of 25% of the students in each class was taken. The teachers rated the sample of students on both factors on two occasions separated in time by three weeks. Ratings were obtained on 122 students from 18 of these teachers.

Pearson correlation coefficients for the teachers' ratings on each factor were 0.85 for Factor 1 and 0.85 for Factor 2. These data provide evidence of the reliability of the assessment instrument which has been developed for the affective domain, especially when account is taken of the lack of experience of most teachers in working in a quantitative manner in this area of schooling. The reliability coefficient of 0.85 for Primary teachers is in keeping with that obtained by Annesley and Clark (1990) on their Measure of Affective Development in Secondary students (MOAD). They reported a reliability coefficient of 0.80.

When teachers used the instrument the second time they were given the opportunity to raise issues about the instrument and also to make any comments they wished. The issue which was raised most often by some of the teachers related to the constraints imposed by the format of the instrument.

RELIABILITY TRIAL - STUDENTS' TASK

In an earlier study Annesley and Clark (1990) invited a sample of Secondary school students to assess their own progress in the affective domain by rating themselves on the same instrument that their teachers had used to rate them. It was decided to use a similar procedure with the random sample of 122 upper Primary school students in the present study. The sets of descriptors were altered so that each statement was written in the first person. For example, Factor 1, level 5 was as follows:

5. * *I am almost always an excited and enthusiastic learner, and I take pleasure in pursuing most subjects.*
- *I am almost always confident to think, plan and work independently.*
- *I am almost always self-directed in making decisions and taking action, and I require minimal supervision and guidance.*
- *I almost always seek language experiences and find enjoyment and interest in their many forms.*
- *I almost always find enjoyment in maths activities and I often pursue maths topics further.*
- *I am almost always motivated to know more about a topic to enhance my own knowledge.*

This was done so that students could more easily relate to each statement. The students were asked to rate themselves on both occasions that their teachers had rated them.

Pearson correlation coefficients for the students' ratings on each factor were 0.66 for Factor 1 and 0.65 for Factor 2. The reliability coefficient of 0.66 for Primary students is comparable to 0.70 obtained with 316 secondary students reported by Annesley and Clark (1990) on their Measure of Affective Development in Secondary students (MOAD).

On the first reliability trial the students were also asked to write any comments they wished about the rating task after they had completed it. The number of students who responded was 102 or 84 %.

When the students rated themselves a second time three weeks later, they were asked also to underline those words in each of the statements for both factors which they had difficulty understanding. The final version of the student instrument incorporated simpler words for some of those identified.

On the second reliability trial the students were asked to write on things they liked and disliked about the task. Some of the comments showed a high level of maturity by some students in this age group. Clearly, they appreciated the opportunity to examine the affective side of their school lives.

CONCLUSION

Black and Dockrell (1980), in an earlier attempt to work with teachers in the affective domain, observed that when teachers make judgments in this area they appear to employ what Bruner and Taguiri (1954) have called "naive implicit personality theory'. Taguiri (1961), in an extensive review of "Person Perception" describes implicit personality theory as:

> a concept used primarily in connection with individual differences in person perception to refer to the assumptions we make about the nature of other persons ,:.. these assumptions presumably affect the way we perceive and understand others, much in the same way our conceptions about any phenomenon influence **what** we perceive, and **how** we perceive it and understand it. (p. 423)

Support for Bruner and Taguiri's (1954) theory was evidenced by a two factor solution which contained 11 of the 22 original affective characteristics judged by teachers as being important to develop in their students through their teaching.

We believe that use of the Secondary and Primary measures of Affective Development will help sensitise teachers to achieve a more appropriate balance between cognitive and affective objectives in their classrooms, provide a reliable and valid measure of student growth in the affective objectives, and where there is dialogue between teacher and student following the student's self evaluation, will result in student empowerment. Clearly, there is much that links the Primary and Secondary sectors of schooling in the area of assessment in the affective domain. The fact that there is now a reliable and valid assessment instrument available for both Primary and Secondary teachers and students is a further link between Primary and Secondary schools.

DIRECTIONS FOR FUTURE RESEARCH

There are some clear directions for future research arising out of this study. One is to explore a Likert type format. The outcome of this would provide a 'profile' on student development in the affective domain on the characteristics within each factor.

Given the significance of the student response when they were asked to say what they liked about the Measure of Affective Development in Primary Students (MAPS), it is important to explore further:

(i) the effect of teacher-student discussions on their mutual ratings;
(ii) the impact on student performance in the classroom, both academically and behaviourally;
(iii) the change in students' liking for school and feelings of satisfaction; and
(iv) the parents' perception of change in their children.

REFERENCES

Annesley, F.R., & Clark, J.F. (1989). *The measure of affective development in secondary school students.* Townsville: James Cook University of North Queensland.

Annesley, F.R., & Clark, J.F. (1990). *Affective assessment for secondary schools.* Townsville: James Cook University of North Queensland.

Annesley, F.R., & Putt, I.J. (1992). *The measure of affective development in primary school students.* Townsville: James Cook University of North Queensland.

Black,H.D., & Dockrell, W.B. (1980). Assessment in the affective domain: Do we? Can we? Should we? *British Educational Research Journal*, 6 (2), 147-208.

Braithwaite, J. (1988). Disadvantaged students and their parents' perception of education and schooling in the post compulsory years. *Curriculum Perspectives*, 8 (2), 20-29.

Bruner, J.S., & Tagiuri, R. (1954). The perception of people. In G. Lindzey (Ed.), *The Handbook of Social Psychology* (pp. 634-654). Cambridge, Mass. : Addison - Wesley.

Cattel, R.B. (1966). *Handbook of multivariate experimental psychology*. Chicago : Rand, McNally.

Glasser, W. (1986). *Control theory in the classroom*. New York : Harper and Row.

Kaiser, H.F. (1960). The application of electronic computers to factor analysis. *Educational and Psychological Measurement*, 20, 141-151.

Knight, I. (1974). Powerlessness and the student role: Structural determinants of school status. *The Australian and New Zealand Journal of Sociology*, 10 (2), 112-117.

Kohl H. (1991). Toward educational change and economic justice. *Phi Delta Kappan*, May, 72 (9), 678-681.

McLeod, D.B. (1992). Research on affect in mathematics education. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning (pp. 575-596)*. New York: Macmillan.

Popham, W.J. (1978). *Criterion referenced measurement*. Englewood Cliffs, New Jersey: Prentice-Hall.

Salvia, J., & Ysseldyke, J.E. (1988). *Assessment in special and remedial education*. Boston : Houghton Mifflin.

Schools Council (1990). *Australia's teachers: An agenda for the next decade*. A Paper prepared by the Schools Council for the National Board of Employment, Education and Training. Canberra: Australian Government Publishing Service.

Tagiuri, R. (1961). Person perception. In G.Lindzey and E. Aronson (Eds.), *The Handbook of Social Psychology*. (2nd ed.). Cambridge, Mass. : Addison-Wesley, (Vol. 3) , 395-449.

Tyler, R.W. (1973). Assessing educational achievement in the affective domain. *Measurement in Education*, 4 (3), 1-8.

Wright, A.F., & Headlam, F. (1976). *Youth needs and public policies*. Melbourne: Department of Youth, Sport and Recreation.

Table 1:
List of Affective Characteristics

	Characteristic	Definition
1.	Caring for others	• an accepting and positive attitude
2.	Self esteem	• has favourable opinion of him/herself
3.	Enjoyment of learning	• takes pleasure in most subjects
4.	Participation	• actively joins in
5.	Values excellence	• strives for high quality work
6.	Independence	• the state of thinking for oneself
7.	Obedience	• willingness to follow directions
8.	Self motivation	• self initiated action
9.	Honesty	• in the context of intellectual honesty
10.	Courtesy	• respectful in manner and action
11.	Responsibleness	• trustworthiness and accountability for one's actions towards others and the environment
12.	Perseverance	• sticking to the task
13.	Appreciation of cultures	• in the context that one's way is not the only way
14.	Initiative	• acts without direction from others
15.	Willingness to seek help	• in most subjects
16.	Positive attitude	• in response to most subjects
17.	Appreciation of language	• values the richness of the English language
18.	Appreciation of the use of Maths	• values the application of maths in real life situations
19.	Co-operative effort	• willingness and ability to work co-operatively with others and to value the contributions of others
20.	Self Evaluation	• knows one's strengths and weaknesses, and not just in the academic sense
21.	Curiosity	• an eager desire to know
22.	Open-mindedness	• willingness to consider new ideas

Table 2
Correlation Matrix for Affective Characteristics (No. of cases =1740)

Chara- cteristic	car foth	self -est	enjl earn	part icip	val exc	inde pend	obed ienc	self mot	hon esty	cour tesy	resp ness	pers ance	app cult	init atve	wils khlp	posa ttde	app lang	app math	coop eff	self eval	curi osty	opmi nnes
carfoth	1.00	.38	.60	.58	.59	.51	.73	.58	.63	.69	.67	.58	.60	.54	.52	.58	.52	.45	.67	.55	.49	.56
selfest		1.00	.61	.60	.54	.60	.39	.58	.39	.35	.48	.54	.47	.58	.47	.61	.53	.54	.49	.54	.58	.54
enjlearn			1.00	.79	.77	.74	.67	.81	.58	.58	.70	.77	.64	.77	.66	.77	.73	.72	.69	.69	.75	.68
particip				1.00	.72	.71	.62	.75	.60	.56	.66	.72	.58	.73	.66	.73	.65	.66	.69	.66	.70	.66
valexc					1.00	.73	.68	.78	.60	.58	.71	.77	.65	.72	.62	.71	.73	.67	.67	.69	.68	.64
independ						1.00	.61	.80	.58	.52	.70	.76	.59	.79	.59	.69	.69	.69	.62	.68	.70	.65
obedienc							1.00	.69	.73	.77	.75	.70	.63	.62	.56	.67	.61	.55	.70	.60	.56	.63
selfmot								1.00	.65	.61	.75	.82	.65	.82	.66	.76	.75	.72	.69	.72	.73	.69
honesty									1.00	.70	.68	.61	.59	.59	.53	.60	.56	.52	.62	.55	.55	.59
courtesy										1.00	.75	.62	.62	.55	.54	.63	.56	.46	.66	.55	.51	.58
respness											1.00	.76	.69	.70	.60	.71	.68	.61	.73	.68	.62	.65
persance												1.00	.66	.78	.66	.76	.73	.69	.70	.71	.70	.67
appcult													1.00	.63	.55	.65	.66	.57	.67	.65	.60	.69
initatve														1.00	.66	.73	.73	.70	.65	.71	.75	.68
wilskhlp															1.00	.70	.60	.58	.64	.62	.63	.61
posattde																1.00	.72	.67	.71	.69	.71	.70
applang																	1.00	.75	.65	.68	.72	.67
appmath																		1.00	.62	.67	.72	.64
coopeff																			1.00	.71	.63	.69
selfeval																				1.00	.70	.70
curiosty																					1.00	.74
opminnes																						1.00

THE PSYCHOLOGY OF PROBLEM SOLVING AS A VEHICLE FOR THE ANALYSIS OF PROFESSIONAL DEVELOPMENT OF INSERVICE TEACHERS.

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*The main features of a professional development program for inservice teachers (**ITAM**) are described and a subset of the data from two case studies is presented to highlight some of the strategies used by teachers while undertaking the program. The case studies demonstrate an isomorphism between the stages of growth experienced by the teachers in the program and the four problem-solving phases popularised by Polya. These phases provide a useful framework to analyse the success or failure of inservice experiences and provide some guidelines for those designing professional development programs. This research also highlights the usefulness of using our knowledge of student growth in acquiring mathematics concepts to assist in our understanding of teacher growth.*

Pre-service teacher preparation courses can only provide, at the very best, a platform upon which future professional growth can occur. Learning to be an effective teacher requires an ongoing commitment to knowing more about teaching, subject matter and how students learn. Knowing ways to assist teachers to face the challenges of growth is an important focus for research. However, it often appears to be a poor relation to many of the major research thrusts in mathematics education.

There has been a noticeable trend in recent years to use understanding of how children learn mathematics to provide insights into the area of teacher professional development (Pegg 1989; Begg 1992; Rice 1992a, 1992b; Mousley 1992a, 1992b). Pegg (1989) used this philosphy in analysing mathematics lessons. He stated (p.19) that

> teachers of mathematics...... grow in acquiring competence in planning and presenting mathematics lessons in a manner similar to the ways in which students (in the classroom) grow in acquiring mathematical skills. Consequently, much of what we currently know about how to help students grow in their knowledge of mathematics is relevant and of use in helping teachers grow in becoming better mathematics educators.

Rice (1992, p.250) placed professional development in a consructivist context by arguing that

> ...if constructivist principals are to underpin mathematical learning environments for children, they should also provide the basis for professional development programs.

Mousley (1992) invoked the work of Voight in adapting the perception of teaching as 'negotiation' in describing a teacher development course. Similar frameworks were used to underpin a professional development program for secondary teachers at the University of New England, Armidale. The purpose of the program was twofold. First, to help teachers identify and confront issues or problems that they felt needed to be addressed, and second, to record and analyse how the teachers involved in the program acquired new skills and understandings. To assist in monitoring the process of development within the program a framework was borrowed from the problem-solving literature.

Following a brief overview of the program, and a description of the problem-solving framework used, it is the second of these issues, namely, the monitoring of the behaviours and thinking processes used by the teachers, on which this paper reports.

BACKGROUND

The professional development program is referred to as **ITAM** (Improving **T**eaching **A**pproaches to **M**athematics)(Pegg and Redden, 1991). It was developed by (i) identifying and trying to minimise many of the disadvantages of current inservice packages offered to teachers in country regions; and, (ii) building into the program additional features and perspectives that have been identified as valuable in the research literature. The program was designed primarily for teachers in country areas who suffer from problems of isolation. This occurs not only in the practical sense of large distances between centres but also in the professional sense. Conducting inservice programs is difficult in these areas: the costs are proportionately much greater than in larger centres; the organisation is more difficult; fewer teachers from the schools can be involved; the issues covered are usually decided by persons outside the school; and, there is seldom any backup or long-term follow-up, i.e., most courses are simply 'one-day wonders' (Owen *et. al., 1987*).

ITAM attempted to minimise these problems by providing all teachers in a mathematics staff at a school with access to support for a full year on a problem or issue that the staff themselves had identified and wished to address. That is, the program incorporated the idea that teachers had to have not only ownership of the issue to be investigated, but also, the solution and the type of support required.

The physical aspects of the program are shown in Figure 1. Two teachers were selected from each school to act as the driving force or coordinators of the program for their school. They were referred to as the key group. The key groups from all the schools involved in the program met on two occasions for a two-day workshop. The whole mathematics staff also attended two meetings, referred to as collegial meetings, but these were out of school time. It was at these meetings that the staff from adjacent schools met to hear what others were doing and to report on their own decisions and progress. The University personnel organised and attended all meetings but their major role was as catalysts for change. This involved encouraging participants to reflect on the nature of change and providing methods of inducing and managing change. The methods discussed included pupil-centred, department-centred and curriculum-centred approaches. These approaches place emphasis on ownership of the issues identified and the processes used by the teachers. However, there was a second major issue which underpinned the thinking behind the program's development. It concerned the problem-solving nature of the whole exercise.

There has been a very large body of research undertaken into problem solving in schools. However, most of this has been directed at students. Further, a great deal of this work has revolved, in some form, about the ideas of Polya (1957) and, in particular, his four phases. Despite the generally favourable reception of these ideas, some writers have identified problems (see e.g., Schoenfeld, 1985). Nevertheless, it would appear that as long as the phases of Polya are seen as descriptive and not prescriptive, i.e., the phases are present in any problem-solving activity but a problem solver does not necessarily proceed in a step by step fashion through them, they represent an important vehicle to help describe and analyse behaviour.

The following discussion looks at the way two schools, which have taken part in the **ITAM** program, have proceeded. To assist with the analysis, the experiences of the staff will be considered within the framework offered by Polya's four phases, namely,

understanding the issue
developing a plan of attack
implementing the plan
reflecting back on the issue and the process.

School A

A department-centred approach was chosen initially to identify and *understand the issue* that was to be central to their **ITAM** project. Each member of the faculty was asked to submit five concerns to be placed on a common list. Using a modified Delphi technique, the items on this list were reduced to a central issue involving the "interest and motivation of students in the intermediate mathematics course", which became the focus of the project. [The intermediate course is a programme of study in Years 9 and 10 in NSW secondary schools. The target audience for this course is those students whose mathematics performance lies in the middle 50% of the cohort.]

The whole mathematics staff attended the first collegial meeting at which the key group presented a report on the problem identification process. However, when they were questioned by the rest of the participants at the meeting for clarification of the general issue, it became apparent that they had identified their problem only at the most global level. They were unable to be specific about the nature of the "interest and motivation of students in intermediate mathematics". They were unable, also, to identify any causes or elements of the problem. Because of this lack of clarity they were unable to plan any specific strategies for solving the problem. Rather their attempts at a solution plan were global and vague. That is, *developing a plan of attack* proved impossible. It became

ITAM

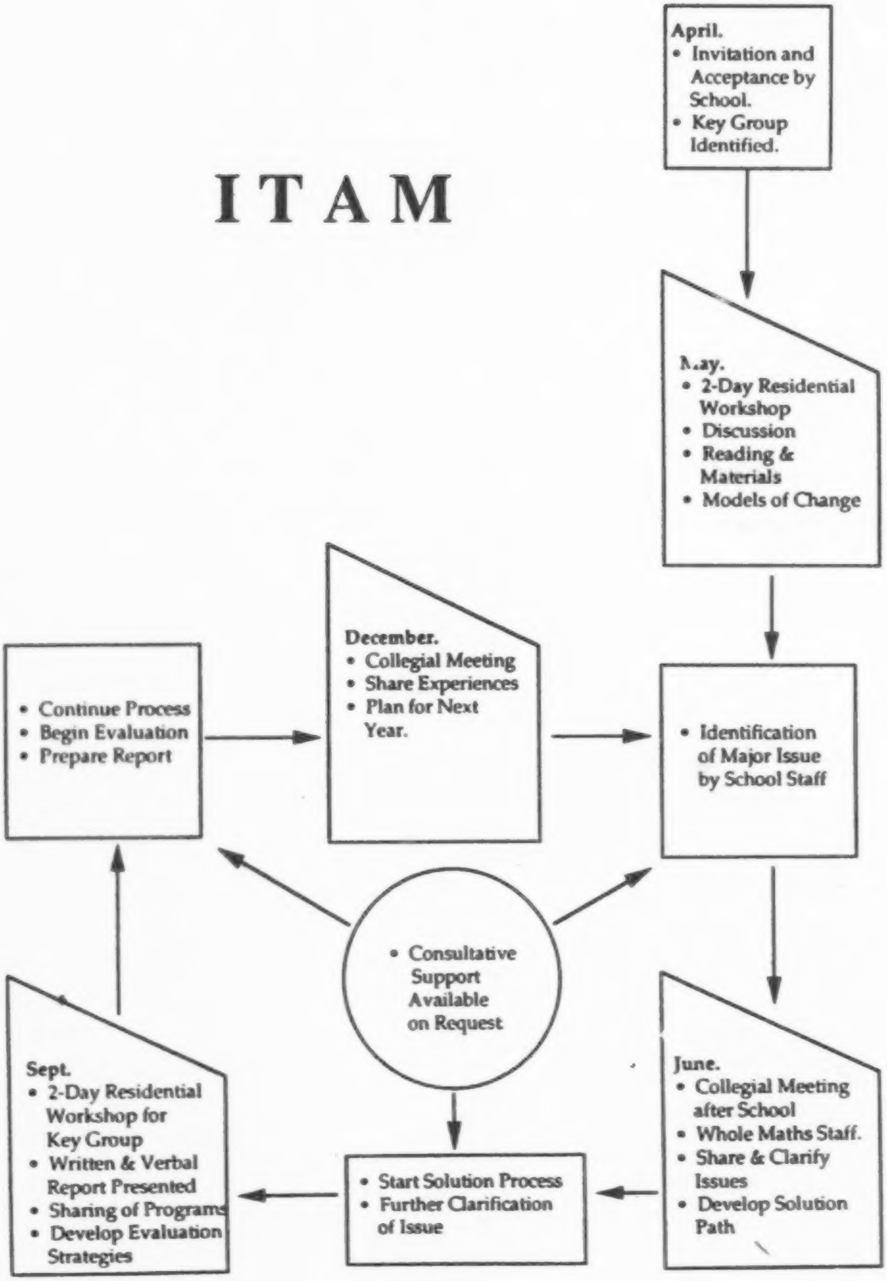
April.
• Invitation and Acceptance by School.
• Key Group Identified.
May.
• 2-Day Residential Workshop
• Discussion
• Reading & Materials
• Models of Change
• Identification of Major Issue by School Staff
June.
• Collegial Meeting after School
• Whole Maths Staff.
• Share & Clarify Issues
• Develop Solution Path
• Start Solution Process
• Further Clarification of Issue
Sept.
• 2-Day Residential Workshop for Key Group
• Written & Verbal Report Presented
• Sharing of Programs
• Develop Evaluation Strategies
• Continue Process
• Begin Evaluation
• Prepare Report
December.
• Collegial Meeting
• Share Experiences
• Plan for Next Year.
• Consultative Support Available on Request

Figure 1

evident that insufficient time and energy had been spent on phase one, *understanding the issue.* The staff were reflecting the behaviour of many students when confronted with non-routine problems. They were tending to rush towards Polya's third phase, *implementing the plan,* without adequate preparatory work.

To address this problem it was decided to return to phase 1 to find out what the issue really involved. A three-pronged attack was decided upon which consisted of surveying and interviewing past and present students and conducting a staff survey. These surveys identified four specific issues of importance to students and ten specific possible changes that the staff believed would improve the "interest and motivation" of students in the targeted mathematics course. The ten changes were broadly based and focused on using community resources, changing teaching organisation, content changes, class restructuring, excursions and the incorporation of technology. The time spent collecting data to better *understand the issue* had a number of benefits which were appreciated by the staff. A much clearer focus was identified and a greater staff commitment to the project was noticed.

Before the project was completed School A was confronted with a major structural change that was imposed on it from an external source. The mathematics staff reacted to this change in an interesting way. First, they felt such commitment to their original project that they continued *implementing the plan* despite the external forces. Second, they were able to apply the skills gained during the **ITAM** program to react to the new situation positively and in a constructive way.

School B

The key group of School B reacted in a much more 'open' way to the invitation to identify a problem to be addressed within the context of **ITAM**. They adopted a pupil-centred approach initially, in an attempt to ascertain childrens' perceptions of mathematics at their school. The methodology adopted was for every mathematics student to keep a diary of his/her mathematics lessons for a period of one week. Analysis of these diaries led to the formulation of four hypotheses about students' perceptions of mathematics and mathematics lessons. While these were not particularly original ideas, the fact that they were generated within the school gave them added importance in the eyes of the teachers and, hence, a greater commitment to instituting a program of change to address the issues identified.

Then, students were surveyed in an attempt to confirm or reject these hypotheses and to investigate if these views were consistently held within the student body. Additional surveys of parents and teachers were conducted to gain further data. Of interest is the great amount of effort that the mathematics staff put into *understanding the issue* prior to embarking on a solution strategy. It is clear that the value of this thoroughness is evidenced in a number of aspects. They include a strong commitment to the project by the whole staff and clear support shown by the school executive. Further, a clear understanding of the problem facilitates the *development of a plan of attack* for solving the identified issue.

The *plan of attack* involved the formulation of three aims which were addressed by adopting a changed approach to both lesson structure and assessment. The lesson structures were changed to incorporate problem solving as a mechanism for introducing and developing concepts. That is, teaching *through* problem solving rather than the more traditional teaching *about* or *for* problem solving was encouraged. The espoused philosophy of the staff was to "encourage students, through planned activities, to discover mathematical concepts for themselves and hence be encouraging the development of our students' feeling of ownership of their mathematics and coming to think of themselves as mathematicians rather than regurgitating machines." The adoption of problem solving as a philosophy underpinning the teaching process had implications for the school's traditional assessment procedures which had appeared to encourage rote learning.

As a consequence, assessment techniques that encouraged class participation and consideration of methods and procedures of problem solving by the students were introduced. The surveys had confirmed the importance that students and parents placed on formal assessment procedures that were recorded and reported to parents. Hence the staff ensured that attempts were made to quantify as many of the new criteria for assessment as possible.

The *implementing the plan* phase was an ongoing process requiring continual monitoring and modification. Polya's *reflecting back* phase became an ongoing evaluation process that identified new problems requiring new solutions. While School B's project had begun with a student-centred approach, the *reflecting back* phase had pointed to a set of structural problems within the school's administration and resource allocation policies. However, because of the long process that had preceded the identification of these new problems, there was substantial evidence for their existence and also a commitment for their resolution. The earlier part of the **ITAM** project had indeed become the *understand the issue* and *develop a plan of attack* phases for these latter issues.

IMPLICATIONS AND CONCLUSION

These two case studies provide an opportunity to consider some general issues in relation to the professional development of teachers.

- Time spent carefully identifying the issue, which may include collecting data, is critical in developing a greater understanding of the issue to be addressed. School A lacked a clear goal in their initial attempt at *developing a plan of attack* while School B's more thorough and considered analysis provided a clear view of how to proceed.
- It takes considerable time to move through the phases. Both schools spent seven months in the first two phases of *understanding the issue* and *developing a plan of attack.*
- Movement through the phases does not need to be linear as in the case of School B. School A's lack of initial success in developing a plan of attack indicated the need to return to the first phase.
- Within the context of **ITAM,** Polya's phases are not discrete but appear to provide the opportunity for ongoing development. This can happen in two ways. The *reflecting back* process provided the new issue for School B while School A was able to use the skills developed in the process to react to an issue generated from outside. Thus, while not having ownership of the issue initially, they had developed ownership of a process for handling such issues.

Polya's problem-solving phases provided a useful framework for analysing problems associated with teacher inservice and professional development. As with research observations with school children, the phases were descriptive rather than prescriptive. A too rigid adherence to step by step hierarchical phases in a short time frame will lead to a superficial treatment of an issue and not develop in teachers the skills necessary to support ongoing teacher growth.

This research has provided further evidence that using theoretical perspectives from children's learning is both valuable and appropriate for planning and reviewing teacher development programs.

REFERENCES

Begg, A. (1992). Gaining Insights by Comparing Processes. In Southwell, B., Perry, B. and Owens, K. (eds), *Proceedings of the Fifteenth Annual Conference of Mathematics Education Research Group of Australasia.* University of Western Sydney, Richmond.

Mousley, J.A. (1992a). Constructivism: Epistemology to Practice. In Southwell, B., Perry, B. and Owens, K. (eds), *Proceedings of the Fifteenth Annual Conference of Mathematics Education Research Group of Australasia.* University of Western Sydney, Richmond. p. 398-409.

Mousley, J.A. (1992b). Teachers as Researchers: Dialectics of Action and Reflection. In Gleeson, W. and Graham, K. (eds) *Proceedings of the Sixteenth PME Conference* . University of New Hampshire. Durham. p. 136-143.

Owen, J., Johnson, H., Clarke, D., Lovitt, C., and Morony, W. (1987). *The Mathematics Curriculum and Teaching Program - Guidelines for Consultants and Curriculum Teachers.* Canberra, Curriculum Development Unit.

Pegg, J.E. (1989). Analyzing a Mathematics Lesson to Provide a Vehicle for Improving Teaching Practice. *Mathematics Education Research Journal.* 1 (2), 17-33.

Pegg, J.E. and Redden, E. (1991). *Analysis of a framework for professional inservice development of secondary mathematics teachers in a rural environment.* Paper presented to the 14th annual MERGA conference, University of Western Australia, Perth.

Polya, G. (1957). *How To Solve It.* New Jersey, Princeton University Press.

Rice, M. (1992a). Towards a Professional Development Ethos. In Southwell, B., Perry, B. and Owens, K. (eds), *Proceedings of the Fifteenth Annual Conference of Mathematics Education Research Group of Australasia.* University of Western Sydney, Richmond. p. 470-477.

Rice, M. (1992b). Teacher Change: A Constructivist Approach to Professional Development. In Gleeson, W. and Graham, K. (eds) *Proceedings of the Sixteenth PME Conference* . University of New Hampshire, Durham. p.250 - 257.

Schoenfeld, A.H. (1985). *Mathematical Problem Solving.* Orlando, Academic Press.

RECENT EFFORTS AT POPULARISING MATHEMATICS IN BRUNEI DARUSSALAM (A CASE STUDY)

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The year 1990 was officially declared "The Year of Enchantment of Mathematics" in Brunei Darussalam. The Fifth South East Asian Conference on Mathematics Education (SEACME 5) was successfully hosted in June. School based promotional activities were carried out effectively. Chief among them was a National Exhibition: "Mathematics in School". A National Mathematics Quiz was also held. It has since been established as an annual event. A National Mathematical Society has also been formed. Overall, an attempt has been made to ensure that the benefits of hosting SEACME 5 would be lasting. That private enterprise too, can stimulate national educational growth has been demonstrated.

The year 1990 has been an important milestone in the annals of Mathematics Education in Negara Brunei Darussalam, indeed in the entire South East Asian region.

The Hon. Minister of Education of Brunei Darussalam in October 1989, formally declared the year 1990 to be the "Year of Enchantment of Mathematics" in Brunei Darussalam (Year of Enchantment of Mathematics (Y.E.M) Newsletter No.7, Dec. 1990). The impetus for this declaration stemmed chiefly from a recognition of the importance of an international event in the arena of Mathematics Education which was due to take place in Brunei Darussalam towards the middle of that year. This was the Fifth South East Asian Conference on Mathematics Education (SEACME 5).

A committee called "The Year of Enchantment of Mathematics Committee" was instituted at the request of the Hon. Minister "to coordinate and oversee the completion of the various projects commenced during the year to promote Mathematics in Brunei Darussalam" (Y.E.M Newsletter No.6, Nov. 1990).

The theme of the international conference SEACME 5 itself was "The Enchantment of Mathematics". The Hon. Minister's declaration reinforced it. It also provided a focal point for most of the activities planned by the SEACME 5 Programme Committee. The main aim of these activities was to make the best use of the opportunity provided by the Conference to promote Mathematics Education in Brunei Darussalam. The benefits should be lasting.

To most people, especially to those not well versed in Mathematcs, the title "Enchantment of Mathematics" would seem a contradiction in terms! Many a parent would say "Oh! we never found mathematics interesting", when confronted with their children's poor grades in the subject.

The successful learning of mathematics involves a systematic building up of mathematical concepts, skills and facts (Skemp, 1971; Ruberu, 1987). Also, it should occur in an order commensurate with the intellectual development of the learner (Piaget, 1950; Collis, 1972).

It is not unusual for people who have not had a proper mathematics training during their school days, to develop a sort of fear of the subject, or even worse, a strong dislike for it. This has been cited by Professor McKay (1968) as one of the reasons why much eloquence is not possible on mathematics as a discipline. McKay cites two further reasons. First, he says mathematics training is not always conducive to oratorical skill. He quotes from Professor J. J. Sylvester who had once said:

> *"... An eloquent mathematician must, from the nature of things, ever remain as rare a phenomenon as a talking fish and it is certain that the more anyone gives himself up to the study of oratorical effect the less he will find himself in a fit state to mathematize. It is the constant aim of the mathematician to reduce every superfluous word and phrase, and to condense the Maximum of meaning into a Minimum of language ..."*

Second, McKay says that the non-descriptive, abstract nature of mathematics makes it a very difficult subject to present verbally. "In fact mathematicians are well known for their copious use of pencil, paper and waste-paper baskets or alternatively, of chalk, blackboards and dusters". To some people mathematics is only an art of reckoning and computation, and mathematicians are a weird tribe sitting all day making enormous calculations!

These erroneous and negative perceptions of mathematics and mathematicians have persisted through the ages, especially in the less developed countries of the world. Even today there are many people who look back on their school experiences with mathematics as so distasteful that they were glad to be finished with the subject.

On the other hand, there are also those who find mathematics interesting. To them there is something intrinsically beautiful in the subject. They perceive order, clarity and economy in mathematics and accept it as a powerful medium for a precise and coherent expression of their thoughts. To them the aesthetic value of mathematics is self-evident. These are the lucky ones! They have been properly initiated into the subject either through their own effort, or through the efforts of enthusiastic teachers; probably both. However, such people are only a minority in most countries.

Against such a background to present mathematics as an enchanting activity is indeed a tremendous uphill task. Yet a group of academic and education leaders of Brunei Darussalam decided to do just that! A courageous and innovative group of people indeed. How far were they successful in their efforts? Let's find out.

The Fifth South East Asian Conference on Mathematics Education (SEACME 5) was planned to be held in Brunei Darussalam in June 1990. To organise this Conference a Steering Committee was formed under the chairmanship of the Permanent Academic Advisor of the University of Brunei Darussalam (UBD). The Committee included members from the Department of Mathematics and the Department of Science and Mathematics Education of the UBD, Heads of schools, and officials from the Ministry of Education. This Committee formally adopted the title : "Enchantment of Mathematics" as the theme of the Conference (Minutes of Meeting 1 held on 10/04/1989).

The major target group of the Conference was the primary and lower secondary teachers. It was decided to conduct two workshops for these groups prior to SEACME 5, with assistance from foreign experts. The upper secondary, tertiary and vocational levels were also targeted.

The Steering Committee of SEACME 5 realised at the outset that their task was an innovative and ambitious one. Their Conference was going to be different. "It would ambitiously aim to communicate the pleasure and the fascination of Mathematics, to share enjoyment and enthusiasm, and to give numerous constructive ideas for passing on these positive attitudes in the classroom" (Y.E.M. Newsletter No.7, Dec. 1990).

Three Subcommittees were formed. The organisation of the academic Programme of the Conference was the responsibility of the Programme Subcommittee headed by the Professor of Mathematics at UBD (Minutes of Meeting 1 of SEACME Steering Committee, 10/04/1989).

For a start, this Subcommittee decided to clarify the term "enchantment". They used the definition: the feeling of great delight which would stem from something beautiful or fascinating or mysterious", and reference was made to G. H. Hardy's words : "A mathematician, like a painter or poet, is a maker of patterns ... The mathèmatician's patterns, like the painter's or poet's, must be beautiful; the ideas, like the colours or words, must fit together in a harmonious way" (Y.E.M. Newsletter No.7, Dec. 1990).

The overall aim of the Conference was well formulated in the Programme Subcommittee's statement:

> *"The intention of the Conference is to show not only that mathematics can be enjoyed, but to demonstrate how this enjoyment can be brought into the classroom at all ages and levels of ability" (Minutes of Meeting 3 of Programme Subcommittee, 01/06/1989)."*

A series of school-based promotional activities to boost the Conference was planned, and financial assistance for their execution was sought from Brunei Shell Petroleum Company (Minutes of Meeting 8 of Programme Subcommittee, 16/08/1989). A "Task Force for School-Based Promotional Activities" was also formed.

Meanwhile, the SEACME Steering Committee at its meeting held on 19/08/1989 requested the Professor of Mathematics, UBD, "to prepare an outline of possible activities which would justify 1990 being proclaimed the Year of Enchantment of Mathematics". The Hon. Minister of Education had agreed with the proposal that 1990 be proclaimed the "Year of Enchantment of Mathematics" in education in Brunei Darussalam (Minutes of Meeting 9 of Programme Subcommittee, 30/08/1989).

The workshop for secondary school mathematics teachers was held on 5-9 September 1989, with the assistance of a Curriculum Development Officer from Singapore. The attendance was good, and there was general agreement that on the whole the workshop was successful. Apparently, a majority of participants had been looking for more of "teaching method" than activities for enchantment of mathematics! (Minutes of Meeting 10 of Programme Subcommittee, 13/09/1989).

Local and regional needs in the sphere of mathematics education were also probed as a prelude to the Conference and documented (SEACME 5 - PS 6) (ibid.).

The newspaper "Borneo Bulletin" of October 28, 1989 reported on the Announcement made by the Hon. Minister of Education declaring 1990 as the "Year of Enchantment of Mathematics". As further activities during 1990, the formation of a mathematics association in Brunei Darussalam and the production of a SEACME 5 Newsletter were considered. It was decided that the Newsletter be called "Year of Enchantment of Mathematics Newsletter". The Mathematics Association would be primarily for the mutual benefit of teachers of mathematics at all levels of the education system. A document titled "Education in Brunei Darussalam" (SEACME 5 - PS 11) would also be prepared, for advance circulation to prospective speakers and participants in SEACME 5 (Minutes of Meeting 14 of Programme Subcommittee, 04/11/1989).

A large number of projects, to be carried out by school children, were planned as a part of the proposed school-based promotional activities. A special issue of the Newsletter was proposed as a vehicle to convey to schools the idea behind those projects. Selected projects from the schools would form a major component of a National Exhibition, entitled "Mathematics in School" to be held during 1990 (Minutes of Meeting 20 of Programme Subcommittee, 15/02/1990).

As yet another means of promoting interest in mathematics, representatives of Brunei Shell Petroleum Company proposed the production of a Year of Enchantment of Mathematics badge. They also came up with the idea of an interschool mathematics quiz to be organised after SEACME 5 (Minutes of Meeting 22 of Programme Subcommittee, 15/03/1990).

Efforts were made by Brunei Shell Petroleum Company to procure materials from abroad for building up of a "Mathematics Resource Package". The Company had also produced two mathematical board games which, after trialling, would be distributed to the local primary schools. The idea of the proposed badge was subsequently modified to two types of stickers containing the Year of Enchantment of Mathematics logo with the phrases "Enchantment of Mathematics" and "Pemikatan Matematik" round it

(Minutes of Meeting 24 of Programme Subcommittee, 12/04/1990).

A series of school-based exhibitions was planned. Their organisation would be left to the schools themselves. The schools were to be encouraged to subsequently display a selection of many of their best exhibits at the proposed National Exhibition. To maximise the impact of the national exhibition, attempts would be made to publicise it by radio, television and the local press. TV coverage of the opening ceremony would be sought (ibid.).

Another activity proposed for the year 1990 was for Bruneian children to participate in the Australian Mathematics Competition held in July that year (ibid.).

The Ministry of Education would invite a large number of teachers from state primary and secondary schools and also from private schools to participate in SEACME 5. They would be exempted from payment of registration fees (Minutes of Meeting 27 of Programme Subcommittee, 10/05/1990).

Discussions with Radio Television Brunei (RTB) to televise a mathematics quiz series involving students from local schools were successfully concluded. Seven programmes involving eight secondary schools would be screened on RTB in November and December 1990 (Minutes of Meeting 28 of Programme Subcommittee, 19/05/1990).

As an additional activity of the Year of Enchantment of Mathematics, an exhibition of Fractals was planned to be held at the University of Brunei Darussalam in July 1990. This would consist of 96 framed pictures of fractals comprising the touring "Beauty of Fractals" exhibition of the West German Goethe Institut (ibid.).

By the middle of 1990 membership of the newly formed Mathematical Society was expanding and a second chapter of the society was formed for the Kuala Belait / Seria region (ibid.).

A public lecture was delivered at the University of Brunei Darussalam on 30/05/1990 by Professor Celia Hoyles of UK on the theme "Popularising Mathematics on TV : The Story of Fun and Games". A video recording of two series of her TV programmes was also available (Minutes of Meeting 30 of the Programme Subcommittee, 06/06/1990).

All preparations for the National Mathematics Exhibition were completed, and the exhibition was opened on the date decided : Sunday 10 June 1990 (ibid.).

The National Exhibition "Mathematics in School" was well received by the public and was given good coverage by Radio Television Brunei. Arrangements were made to distribute the prizes to the schools as soon as possible.

The Mathematical Society of Brunei Darussalam would be represented at the South East Asia Mathematical Society meeting to be held in Hong Kong in August 1990. Professor Roger Eggleton, the then Professor of Mathematics at the University if Brunei Darussalam, who had been the prime mover in most of the activities already described, was nominated for the task (Minutes of Meeting 31 of Programme Subcommittee, 27/06/1990).

SEACME 5 was held during 14 - 16 June 1990. Most participants were highly complimentary on its success. Papers presented were to be collated and published as a "Conference Proceedings" book. It was suggested that the next Conference, SEACME 6 to be held in Surabaya, Indonesia (1993), should have its scope extended from Tertiary Education to include also school education. This would give it wider appeal in the region (ibid.). The "Conference Proceedings" of SEACME 5 have since been published (Programme Committee, 1991).

An important spin-off from the Year of Enchantment of Mathematics was the Concept of a National Mathematics Quiz for secondary schools as an on-going activity. As mentioned earlier, the quiz conducted under this title in 1990 was the brain child of representatives of Brunei Shell Petroleum Company. Sponsorship from Brunei Shell Petroleum Company and Radio Television Brunei received at a very early stage of its development helped to make it a reality. This mathematics quiz generated considerable interest and enthusiasm among members of the public towards mathematics. The Mathematical Society of Brunei Darussalam Newsletter No.10 of January 1992 commented :

> *"It showed that Mathematics can be entertaining, can be successfully presented as a competitive skill by pupils in Brunei, and can be presented as a spectator sport! The serious immediate results of this Quiz included the stimulation of public interest in this mathematical activity, the raising of the public profile of mathematics, and the cultivation of a more positive public image of mathematics. The longer term effect of the Quiz is to assist in improving the acquisition of mathematical knowledge and skills in Brunei Darussalam".*

This positive feedback from the schools and the public created an additional impetus and motivation for the organisers of that Quiz to conduct such competitions on a more systematic basis as a continuing annual activity.

With this end in view a Committee was formed in 1991, at the request and with the support of the Permanent Secretary to the Ministry of Education. It comprises University staff, mathematics teachers, School administrators and representatives from Brunei Shell, Radio Television Brunei, the Ministry of Education and the Mathematical Society (The mathematical Society of Brunei Darussalam Newsletter No.10, January 1992). Called the "National Mathematics Quiz Steering Committee", this body has already successfully organised and concluded the National Mathematics Quiz 1992 in collaboration with Maktab Sains Paduka Seri Begawan Sultan, a leading secondary school in Brunei Darussalam (Minutes of Meeting 11 of the National Mathematics Quiz Steering Committee, 20/08/1992). The 1993 Quiz is now in progress, in collaboration with another leading secondary school in the country - Sekolah Menengah Berakas .

In conclusion, it must be stated that

(1) 1990 - the "Year of Enchantment of Mathematics" has been a tremendous success in boosting the image of Mathematics in Brunei Darussalam, with impact extending to all the South East Asian Region, and

(2) University Departments and private enterprise (evidenced by the role played by Brunei Shell Petroleum Company) can effectively join forces with the government sector to make substantial advances in the educational development of a country.

Acknowledgments :

The author is indebted to Professor Roger Eggleton, the former Professor of Mathematics at the University of Brunei Darussalam for giving him access to relevant documents, and for comments on an earlier version of this paper.

Acknowledgments are also due to Professor Patrick Duignan, the Dean of the Faculty of Education, UBD, for providing the initial impetus to prepare this paper depicting a recent educational advancement in the arena of Mathematics Education in Brunei Darussalam.

REFERENCES :

Borneo Bulletin. (1989, Oct.). Newspaper of 28/10/1989.

Collis, K. F. (1972). A Study of Concrete and Formal Operations in School Mathematics. Ph.D. Thesis, University of Newcastle, NSW.

Eggleton, R. B. (1992, Jan.). National Mathematical Quiz 1992-The Mathematical Society of Brunei Darussalam Newsletter No.10.

McKay, M. H. (1968). What is New in Mathematics? - Inaugural Lecture by Foundation Professor of Mathematics,University of Papua New Guinea.

National Mathematics Quiz Steering Committee. (1992). Minutes of Meeting No.11.

Piaget, J. (1950). The Psychology of Intelligence. London: Routledge and Kegan Paul, Ltd.

Ruberu, J. (1987). An Investigation of Concept Attainment in Secondary School Mathematics. Ph.D. Thesis, La Trobe University, Melbourne.

SEACME 5 Programme Subcommittee. (1989, 90). Minutes of Meetings 3, 8, 9, 10, 14, 20, 22, 24, 27, 28, 30, 31.

SEACME 5. (1989a). PS 3, Local and Regional Mathematical Needs.

SEACME 5. (1989b). PS 6, Resource Materials.

SEACME 5. (1989c). PS 11, Education in Brunei Darussalam.

SEACME 5 Programme Committee. (1991). Proceedings of the Fifth South East Asian Conference on Mathematical Education. University of Brunei Darussalam.

SEACME 5 Steering Committee. (1989). Minutes of Meeting No.1.

Skemp, R. R. (1971). The Psychology of Learning Mathematics.Harmondsworth: Penguin Books.

Sylvester, J. J. (1968). Baltimore Address : Mathematical Papers, Vol.3, pp. 72-73. As quoted by M. H. McKay, What is New in Mathematics? - Inaugural Lecture by Foundation Professor of Mathematics, University of Papua New Guinea.

Year of Enchantment of Mathematics. (1990a). Newsletter No.6.

Year of Enchantment of Mathematics. (1990b). Newsletter No.7.

CHANGING CHILDREN'S APPROACHES TO MATHEMATICAL PROBLEM SOLVING

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A year-long teaching experiment explored the possibility of changing fourth grader's approaches to mathematical problem solving. A metacognitive question-and-answer technique was used to negotiate meaning, explore problem representation, discuss possible solution strategies, and reflect on the problem-solving enterprise. Analysis of the transcript data, classroom observations and childrens' work samples revealed that while those with most to gain and nothing to lose demonstrated the greatest shift in approach, each child's approach was successfully challenged to some extent.

It would appear that there are two relatively distinct views of mathematical problem solving. One view, which might be characterised as the information processing view, describes problem solving as the process of moving from a given state (where you are) to a goal state (where you want to be) without the benefit of a pre-conceived solution path (Mayer, 1985). The other, which might be characterised as a constructivist view, regards problem solving as a particular form of learning.

It should be clear that, for the constructivist, substantive mathematical learning is a problem solving process ... In this context, substantive learning refers to cognitive restructuring as opposed to accretion or tuning (Cobb, Wood & Yackel, 1991, p.158).

These two views have variously impacted the research on mathematical problem solving as has the notion of metacognition, the term introduced by Flavell to describe one's awareness of one's own cognitive processes and products (Flavell, 1976, p.232). If an information processing perspective is adopted, metacognition tends to be seen in terms of the control and regulation of cognition (for example, Garofalo & Lester, 1985; Schoenfeld, 1985). However, if a constructivist perspective is adopted, metacognition can be viewed as cognition informed by problem-solving-relevant and/or learning-relevant knowledge and experience. That is, as an interaction between cognitive goals, cognitive actions, metacognitive experiences and metacognitive knowledge (Flavell, 1981). Viewed in this way, problem solving can be seen as a form of learning influenced by one's goals, knowledge, beliefs, actions and experiences in much the same way as any other form of learning. This implies that what an individual has learnt to value, attend to, monitor and accept in a particular setting will shape the individual's approach to learning and/or problem solving in that or a related setting.

Although children's approaches to mathematical problem solving in out-of-school settings has been the subject of considerable research in recent years (for example, Carraher, Carraher and Schliemann, 1985), what children might be observed and/or inferred to be doing while they are engaged in problem-solving episodes in school mathematics classrooms over a long period of time has only just begun to emerge as a focus of research (for example, Cobb, Wood & Yackel, 1991). Long-term teaching experiments which recognise the complexities of the classroom environment and acknowledge the critical role of the teacher in generating, modifying and endorsing the type of social interactions and norms which shape childrens learning appear to offer the best means of describing and explaining childrens approaches to mathematical problem solving in school settings.

Four distinct approaches to school mathematical problem solving were confirmed by a year-long teaching experiment involving a grade four class and their teacher (Siemon, 1992). The approaches varied according to the extent to which the children appeared to be valuing, attending to, and monitoring their procedural and/or conceptual knowledge. For example a Low Conceptual-High Procedural approach (referred to as the Player's approach) was characterised by relatively few instances of cognitive monitoring with respect to the identification and evaluation of appropriate cognitive goals (conceptual knowledge), but by relatively high levels of cognitive monitoring in relation to the implementation of a range of cognitive actions (procedural knowledge). Where there were instances of cognitive monitoring, these were more likely to be directed at cognitive actions than at cognitive goals.

Beliefs and values about oneself as a problem solver, one's role as a student of school mathematics, and about mathematics were recognised as important factors in childrens' approaches to problem solving. Additional characteristics of each approach were derived from the literature and confirmed by the year-long teaching experiment (for example, differential approaches to learning (Marton & Saljo, 1976) and the tendency to premature closure

CONCEPTUAL KNOWLEDGE:
metacognitive knowledge and cognitive goals

Diver:
High Conceptual-Low Procedural

cognitive goals attended to more than cognitive actions
comprehension strategies: checks, monitors, plans, predicts, links and reflects more on knowledge than on actions
access to a variety of strategies, not always well used
identifies goals
tends to synthesise and analyse data
some tendency to conceptually-driven premature closure
questions directed more at goals than actions
some undirected actions
uses labels, understands structure
deep approach to learning
extended locus of control for actions

Solver:
High Conceptual-High Procedural

attends to cognitive goals and actions
comprehension and regulation strategies: checks, predicts, monitors, plans, links, introspects and reflects on knowledge and actions
uses a variety of strategies knowingly
identifies goals and appropriate actions
synthesises and analyses data
strong tendency to persist until reasonable solution obtained
questions directed at goals and actions
directed manipulation
uses numbers, labels, structure
deep-achieving approach to learning
internal locus of control

Survivor
Low Conceptual-Low Procedural

unlikely to attend to either cognitive goals or cognitive actions
tendency to remember and replicate, but experiences difficulty
strong tendency to premature closure
tends not to check, monitor, reflect or predict on actions or goals
experiences difficulty identifying goals
experiences difficulty identifying appropriate actions
more likely to synthesise than analyse
little or no questioning
surface approach to learning
external locus of control for knowledge and actions

Player:
Low Conceptual-High Procedural

cognitive actions attended to more than cognitive goals
remembers and replicates, often quite effectively
tendency to procedurally-driven premature closure
regulation strategies: checks, monitors, predicts and reflects, more on actions than on goals
tends to assume goals
tends to try a range of actions
more likely to synthesise than analyse
questions directed more at actions than at goals
surface-achieving approach to learning
external locus of control for knowledge

PROCEDURAL KNOWLEDGE:
metacognitive experiences and cognitive actions

Figure 1. Children's approaches to mathematical problem solving (Siemon, 1992).

(Biggs and Collis, 1982)). The Player's approach and the remaining approaches, that is, the Solver's approach (High Conceptual-High Procedural), the Diver's approach (High Conceptual-Low Procedural), and the Survivor's approach (Low Conceptual-Low Procedural) are described in Figure 1. The suggestion of orthogonality is deliberate as the approaches are not meant to imply discrete, mutually exclusive entities, but tendencies towards some particular behaviours rather than others in relation to a specific task at a given point in time. Organised in this way, the model provides a framework which allows children's approaches to a range of problem-solving tasks to be identified and monitored over time.

CHALLENGING CHILDREN'S APPROACHES

Given that different children appear to be valuing, attending to and monitoring different aspects of the mathematics classroom environment, it is reasonable to ask what can be done to challenge and change those approaches which do not value the negotiation of shared meanings and/or the use of more powerful generalisable strategies, that is, the Player's, Diver's and Survivor's approaches. In the context of the year-long teaching experiment, this question took the form, to what extent could children's approaches to mathematical problem solving be changed by a program designed to enhance metacognition, and it is this question which will be addressed here.

As indicated above, Flavell's (1981) model of cognitive monitoring was used as a basis for understanding what metacognition might mean in this context. The model consists of four interactive components, cognitive goals, cognitive actions, metacognitive knowledge and metacognitive experiences. The interaction provides for a powerful and dynamic link between cognition and metacognition and accommodates a distinction between conceptual and procedural knowledge. The model also acknowledges the critical role of prior knowledge, beliefs and values (conscious or unconscious, cognitive or metacognitive) in the generation of cognitive goals and in the implementation of cognitive actions. While metacognitive experiences allow for the conscious recognition of problem-solving-relevant or learning-relevant properties of a particular experience, this view of metacognition does not presume the degree of consciousness implied by more recognised views of metacognition in the mathematics education literature, for example, "knowledge of cognition" and the "regulation of cognition" (Garofalo & Lester, 1985, p. 164).

The problem solving program used in the year-long teaching experiment explored a variety of problem types and strategies and provided an accessible model of cognitive monitoring in terms of the ASK-THINK-DO problem-solving cycle (see Barry, Booker, Parry and Siemon, 1985). Key components of the program were,

(i) the mathematics content of the particular problems considered was selected by the teacher to support her curriculum objectives,
(ii) the problems were varied according to the amount of information provided, the degree of ambiguity about what procedures or strategies might be required, the transparency of the relationships between data, and the number of steps involved,
(iii) the problem-solving process was specifically talked about in terms of the ASK-THINK-DO cycle and modelled by the teacher whenever a strategy was reviewed or introduced,
(iv) pupils were encouraged to reflect on their problem solving both individually and collectively, and
(v) key questions, strategies and observations about problem structure and process were discussed and recorded on a large ASK-THINK-DO problem solving chart which was on constant display in the classroom.

Problems were worked on individually, in small groups or as a class. Twelve children were selected on the basis of their mathematical performance (high/low attainment) and interviewed every three to four weeks. Six of the twelve children, three girls and three boys, were interviewed on a more frequent basis. Interviews generally consisted of a reflective review of the problem considered in class followed by an attempt at a similar or related problem. A response mapping technique which reflected Flavell's (1981) model of cognitive monitoring was developed and used to analyse the individual interview data (see Siemon, 1992). Patterns in the response map data were used to identify childrens approaches to mathematical problem solving.

Kieron - The Solver

At the beginning of the teaching experiment Kieron's response maps indicated that while he appeared to be monitoring his cognitive goals and actions quite effectively, he seemed to be unaware of what he was doing. For example, in response to the problem, *If Mr Applebee arrived at the bus-stop at 7:15, how long will he have to wait to catch the next bus at 8:33?*, Kieron replied, "From 7:15 to 8:15 is one hour, from 8:15 to 8:33 is .. eighteen minutes, one hour and eighteen minutes". Asked how long from 8:33 to 9:05, Kieron said, "8:30 to 9:02, thirty-two minutes". Although Kieron's approach remained fairly consistent over the course of the teaching experiment, the program appears to have had an impact nonetheless in that it seems to be associated with him becoming more aware of what he was doing and why. For example, in the fourth month of the teaching experiment, the class considered the problem, *Greg was given $175 by his uncle to buy a bike for his birthday. How much money did Greg have left over?* (an illustration provided the information that a 5-speed racer was $158 and a BMX was $137), Kieron recognised this problem as similar to a problem considered two months earlier, "It's like the ice-cream one", and suggested that an "If ..., then ..." sentence was required. Reporting what he had done in the class lesson, Kieron indicated that he subtracted to find the change for both options and wrote, "If Greg bought the BMX, he'd have $38 change. If he bought the racer, he'd get $17". In this instance, Kieron systematically partitions the problem to consider all options in turn (formulates appropriate cognitive goals), uses a computation strategy (employs a relevant cognitive action) and reports his answer in a conditional manner (applies a problem-solving-relevant item of metacognitive task knowledge). Asked to reflect on his grade four maths experience to date, Kieron acknowledges that "when I first came, I wasn't sure, now I'm really enjoying it, ... I like the way Mrs M argues with us, ... I've learnt to think for myself".

Possibly as a result of the fact that Kieron generally met with success he seemed to become complacent about attending to those aspects of the program which focussed on the analysis of problems, particularly the questioning designed to connect cognitive goals to cognitive actions. By the end of the teaching experiment, it was apparent that Kieron was much more inclined to monitor his goals and actions in situations where the relationships between data were fairly transparent (familiar problems) than he was in situations where the relationships between data were fairly ambiguous or for which a representation was required. The effect of the program in this instance seems not to have changed Kieron's approach but to have clarified and qualified certain aspects of it.

Kristie - The Diver

At the beginning of the teaching experiment, Kristie's response maps indicated that she was much more inclined to value, attend to, and monitor her understanding of the problem conditions and what was needed (cognitive goals) than she was to value, attend to, and monitor her cognitive actions in relation to achieving those goals. For example, in response to the problem, *Maria was given $50 to buy a doll for her birthday. If a baby cabbage-patch doll costs $35 and an ordinary cabbage-patch doll cost $49, how much change would she have left?*, Kristie said, "I'd do two sums (indicating subtraction but not proceeding) ... then, if Maria bought the baby doll she would have so and so much change, and if she bought the big doll she would have so and so". More or less as an afterthought, she adds, "If I think it over, I'll be able to work it out". Her apparent lack of interest in actually working it out and her strategy for avoiding it provide a good example of conceptually-driven closure, that is, closure prompted by the recognition that while she knows what to do she is not so confident about her ability to actually do it.

By the end of the teaching experiment, Kristie's response maps indicated that she was more inclined to value, attend to, and monitor both her cognitive goals and her cognitive actions For example, it was quite apparent that Kristie was becoming increasingly impatient with her rather inefficient and time-consuming computation strategies. In the final interview, not only did she recognise that subtraction was required, she recorded the problem using place value columns and talked her way through a decomposition recording process. Asked if there was another way she could have done it, Kristie replied, "Yes, I could have drawn, sort of like numbers for dollars (presumably referring to a tally), but that would take for ever".

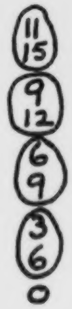
Carlo - The Player

At the beginning of the teaching experiment, Carlos' response maps indicated that he was much more inclined to value, attend to, and monitor his cognitive actions than he was his understanding of the problem conditions and what was needed (cognitive goals). For example, asked what he does when he gets stuck, Carlo replies, "I try times tables .. I try lots of things (indicating algorithms)". Asked if it takes seventeen minutes to walk to school, how long would six trips take, Carlo immediately records six seventeens (vertically) and says, "six sevens are forty-two ... fifty-two minutes". Asked to explain how he arrived at his answer, Carlo replies, "Well, I said six seventeens and six sevens are forty-two, you get two down here and four up there and from the seventeen there was a ten and so four and one is five and I just put five down in their's column and I got my answer". While Carlo appears to be monitoring his actions in relation to his view of formal multiplication, it is quite apparent that he is not valuing, attending to or monitoring the meaning of what he is doing beyond the purely procedural. Carlos' knows his number facts and demonstrates a sound knowledge of place-value when asked a direct question, however, Carlo does not appear to be able to draw on this knowledge independently to validate or check his actions. Rather, he seems content to assume that his answer is correct on the basis that he has done what he believed was appropriate.in the circumstances.

By the middle of the year, Carlos' approach tended to be more typical of a Survivor's approach than a Player's approach. He appeared to be confused and threatened by program's focus on and valuing of cognitive goals (something Carlo clearly did not value or see the sense of initially). For example, in response to the Spiders and Beetles problem, *If ten bodies and sixty-eight legs, how many spiders and how many beetles?*, Carlo says, "I hate things like that ... first of all it's very confusing, and second of all I haven't done it before and I can't ... I can't remember straight off. I gotta do it many times and get it right and wrong to remember it".

However, by the end of the teaching experiment, Carlos' response maps indicated that he was more inclined to value, attend to, and monitor both his cognitive goals and cognitive actions. For example, faced with the problem, *How many days will it take Clara Caterpillar to climb a tree trunk seventeen metres high if she climbs six metres up the tree trunk during the day but slips back three metres at night ?*, Carlo provides the following response.

> **Carlo**: Id do sixes in seventeen, I think, which is two sixes, that's twelve, that's five left over, oh, no! I'm stuck, seventeen metres to the tender young leaves, so how many days did it take her ... days?, Oh, is that including days, that is, whole days and nights?
> **Interviewer**: What do you think?
> **Carlo**: I think day and night maybe ... so that's six metres up the tree and that's seventeen, so minus six from seventeen, that's six from seventeen, that's eleven, then you add three which is fifteen (records and circles 11 and 15), so that's already fifteen metres, ... then you minus six from fifteen which is nine, then you add three again which is, which is twelve (records and circles 9 and 12), then you minus three and then, no, then you minus six, get six, plus three is nine (records and circles 6 and 9) plus, minus six which is three, plus three which is six (records and circles 3 and 6), minus six that's zero, perhaps he (possibly drawing on a similar problem involving Freddie Frog) would have taken one day (counting), that's two days, three days, four days.



Carlos' unique solution to this problem was not previously modelled or discussed in class although a similar problem had been considered in class five months earlier. An analysis of the transcript reveals many instances of conceptual and procedural monitoring. It is evident that Carlo is not only keeping track of his strategy but testing it out against the conditions of the problem as he goes, that is, he is monitoring his cognitive actions in relation to his cognitive goals. That he now appears to recognise and value such strategies as drawing a diagram, recording data, and working backwards represents a significant shift in Carlos' beliefs about the nature and purpose of school mathematics and his role as a student of mathematics.

David - The Survivor

At the beginning of the teaching experiment, David's response maps indicated that he was not particularly inclined to value, attend to, and monitor his understanding of the problem conditions and question (cognitive goals) or his cognitive actions in relation to achieving those goals. For example, in response to the problem, *If Mr Applebee arrived at the bus-stop at 7:15, how long will he have to wait to catch the next bus at 8:33?*, David replied, "from seven to eight is one hour and thirty minutes more ... ninety, and three there, ninety-three minutes". Asked if was necessary to take the fifteen minutes (7:15) into account, David said, "no, not really". In responding to the problem, *How much could you earn on a paper-round if you delivered eighty-two papers per day from Monday to Saturday at five cents per paper?*, David says, "Five dollars two cents". Asked to explain his answer, David says, "I said six eights are forty-eight and I said six twos are twelve, and that gave me fifty-two, so I said five dollars and two cents". By the end of the teaching experiment, David's response maps indicated that he was more inclined to value, attend to, and monitor both his cognitive goals and his cognitive actions. He was particularly inclined to do this in relation to problems which prompted from some form of diagramatic representation. For example, David experienced little or no difficulty with the Spiders and Beetles problem or the Fence problem, *A farmer wants to make a rectangular fence using twelve lengths of timber. Each length is nine metres long and costs $7. If fence posts cost $5 each, how much will it cost the farmer to build the fence?* David used the diagrams to clarify his cognitive goals and evaluate his cognitive actions. Asked, "What made you think of doing it like that?", David responded by saying, "I thought sketches would help me a lot more than trying to do it in my head and writing sums. It's a lot easier doing that I think" (that is, using the drawings).David had nothing to lose by attending to the program and modelling his behaviour accordingly. His overall shift in approach is best summarised in his own words at the end of the last interview, "If you think you can always aork out the answer to a sum".

CONCLUSION

While the nature and extent of the shifts in approach were different for different children, it would appear that the program did have an impact on childrens' approaches to problem solving. Given that aspects of the program were directly targeted at challenging children's knowledge and beliefs about school mathematics, such an outcome is consistent with what might be expected from a constructivist perspective of teaching and learning, that is, that learners value, attend to, monitor, and store different aspects of their shared experience depending on their prior knowledge, goals, beliefs and values. If the program had no effect on childrens' approaches to problem solving, it would have to be argued that something else occurred which did prompt the change in approach as one would hardly expect such well-established responses to school mathematics to be changed as a result of maturation or history alone. The value of the approaches for classroom teachers is that they help identify the different, goals, beliefs and values which prompt different children to make different sense of the same experience and explain why, in response to the sense they make of that experience, different children draw on different knowledge, skills and strategies in order to deal with that experience. Having some insights into the different goals, beliefs and values operating in the classroom provides a basis for challenging and changing those which actively operate against the negotiation of shared mathematical meanings and the use of more powerful generalisable strategies.

REFERENCES

Barry, B., Booker, G., Perry, B. & Siemon, D. (1983-) *HBJ Mathematics Levels 4-7.* Sydney: Harcourt Brace Jovanovich.

Biggs, J.B. & Collis, K.F. (1982) *Evaluating the quality of learning: The SOLO taxonomy.* New York: Academic Press.

Carraher, T.N., Carraher, D.W. & Schlieman, A.D. (1985) Mathematics in the streets and schools. *British Journal of Developmental Psychology, 3,* 21-29.

Cobb, P., Wood, T. & Yackel, E. (1991) A constructivist approach to second-grade mathematics. In E. von Glasersfeld, (Ed), *Constructivism in mathematics education,* Dordrecht: Kluwer, 157-176.

Flavell, J. (1976) Metacognitive aspects of problem solving. In L. Resnick (Ed.) *The nature of intelligence.* Hillsdale, NJ: Lawrence Erlbaum.

Flavell, J. (1981) Cognitive monitoring. In W. Dickson (Ed.) *Children's oral communication skills.* New York: Academic Press.

Garofalo, J. & Lester, F. (1985) Metacognition, cognitive monitoring and mathematical performance. *Journal for Research in Mathematics Education, 16,* 163-176.

Marton, F. & Saljo, R. (1976) On qualitative differences in learning - I: Outcome and process. *British Journal of Educational Psychology, 46,* 4-11.

Mayer, R. E. (1985) Implications of cognitive psychology for instruction in mathematical problem solving. In E. Silver (Ed.) *Teaching and learning mathematical problem solving: Multiple research perspectives.* Hillsdale, NJ: Lawrence Erlbaum.

Schoenfeld, A.H. (1985) *Mathematical problem solving.* Orlando, Florida: Academic Press.

Siemon, D. E. (1992) Childrens' approaches to mathematical problem solving. In W. Geeslin & K. Graham (Eds.) *Proceedings of the Sixteenth Conference of the International Group for the Psychology of Mathematics Education,* University of New Hampshire, New England.

PROBLEM SOLVING DIFFICULTIES OF ABORIGINAL STUDENTS

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Underachievement in mathematics by Aboriginal students is widely acknowledged and appears to more extreme in the crucial area of mathematical problem solving. This paper will report on research conducted on Aboriginal students in the Northern Territory and present evidence that such difficulties do exist and are more noticable among rural rather than urban Aboriginal students. In particular the evidence indicates that problems involving the concept of difference provide even greater difficulty.

Can the results be explained by current knowledge and what are the implications for curriculum writers and educators of Aboriginal students?

The thematic approach of the mathematics curriculum introduced at Alice Springs High School from 1987, required all year 9 students to undertake a 5 week problem solving course. As the teacher of this course for several years it became apparent that many of the schools' 40% of Aboriginal students had difficulties with the course inconsistent with their arithmetic skills.

Are the factors which affect achievement in mathematics in general likely to have an even greater affect on an Aboriginal students' ability to solve mathematical problems?

BACKGROUND

The under achievement in mathematics by Aboriginal students has been recognised for some time, certainly since Bourke and Parkin (1977) reported that overall performance *"....was very low indeed (p.152)."* The reasons for such low achievement are generally attributed to the broad areas of language and culture.

The recent review of mathematics and language by Ellerton and Clements (1990a) attested that *"the burgeoning literature on everyday cognition and mathematical learning points to the conclusion that language, culture and mathematical learning are intimately related (p. 233)* " This paper along with the earlier review by Austin and Howson (1979) covers the broad range of research into the relationship between language and mathematics and includes works on linguistics, psycholinguistics: difficulties with polarised comparatives and logical connectors, bilingualism, sociolinguistics: language deprivation in English and/or the vernacular, a lack of equivalence between English and the vernacular and the cultural aspects of Aboriginal language. All of these factors are likely to cause greater difficulties due to the greater language demands in many problem situations.

For Aboriginal people their language and culture are intimately related: described by Pam Harris (1980) as *"language problems are just the tip of the iceberg - spoken language is part of the surface structure of culture which is obvious to everyone (p.30)".* Other important aspects of Aboriginal culture likely to influence mathematical achievement include earlier socialisation into meaningful rather than purposeful behaviour (Christie, 1985) and differences in traditional learning styles which Stephen Harris listed as:

- Observation and imitation rather than through verbal instruction
- Personal trial and error rather than verbal instruction
- Real life performance rather than practice in contrived settings
- Mastery of context - specific skills, rather than abstract, context-free principles.
- Orientation to persons rather than to information.

(S Harris, 1980, p.130-132).

Such cultural reasons for poor achievement have been compounded by the long held belief by western educators that mathematics was culture-free. Whilst this belief may have been repudiated by such eminent educators as Ellerton and Clements (1990b) and Alan Bishop (1988), such repudiation has yet to reach many classroom teachers, textbook writers and examinators. The effect of this belief is that Aboriginal students are usually assessed in an alien language and culture and then compared with 'European' students. This was acknowledged by Bourke and Parkin who commented in their report that

> ...if the items used to test performance were appropriate, the evidence indicated that most students at Aboriginal schools in the N.T. would not be able to lead a 'normal' life in Australia without serious disadvantage.....One may, however, ask whether these students should be required or would ever desire to lead the pattern of life followed by most Australians and described by them as 'normal'! (p. 153)

However the aspirations of many members of Aboriginal communities are such that they want their children to participate and succeed in the 'white' schools in addition to retaining their Aboriginal culture, referred to by the National Aboriginal Education Committee (1982) as the desire of Aboriginal parents for their children to be able to function in both their own culture and the wider Australian community.

THE RESEARCH

The research undertaken focused on the student's ability to apply their arithmetic knowledge and skills to simple problems. The skills assumed were ordering numbers, addition, subtraction, multiplication and using simple fractions.

The research method used was to individually interview twenty seven students and present them with a series of problems. The twenty seven students were all Aborigines. Twenty four were residents of Yirara College, a facility near Alice Springs that provides accommodation and schooling to Aboriginal students from rural communities. Of the twenty four Yirara students, four were year 8 students at Alice Springs High School, whilst the remaining eighteen were undertaking various levels of courses at Yirara College: six in Access, five in Foundation, five in General and four in Initial Secondary Studies. The three non-Yirara students consisted of two who lived a semi-traditional lifestyle in a town camp and one urban Aboriginal student. The interviewer spent time working with the students in their normal classes prior to the interviews and all students interviewed volunteered to participate.

The problems attempted to reflect themes that might occur in both Aboriginal communities and western society: distance, money, interpreting bills, program guides and reports. The interviews were video taped for later analysis.

The problems from the interviews were replicated as closely as possible in written form and put to a group of seventy one students in year 8 at Alice Springs High School. The group consisted of forty six non Aboriginal students, nineteen urban Aboriginal students and six rural Aboriginal students. Both groups also undertook an arithmetic skills test, with two students appearing in both groups.

The initial prime aim of the research was to look for a correlation between a student's ability to use a skill in an unfamiliar situation and their degree of acculturation, however the purpose of this paper is to draw attention to a more fundamental point which I hope will become apparent from the video extracts.

THE RESULTS

The video extracts shows a sample of the student responses to two problems in the interviews. In the first problem the students were shown a card which named 5 places and their distances from Alice Springs (figure 1).

The students were asked to identify:

1. The place closest to Alice? - usually answered correctly.
2. The place furthest from Alice? - usually answered correctly.
3. Which two places were about the same distance from Alice? - again usually answered correctly.

The final question required the students to identify the difference between the two distances which were about the same distance from Alice Springs, usually put in the form: "If you travelled to Yuendumu and I travelled to Barrow Creek, how much further would you travel than me?" Most students interviewed were unable to correctly answer this question with the common responses being 296 km (the larger distance) or 586 Km (the sum).

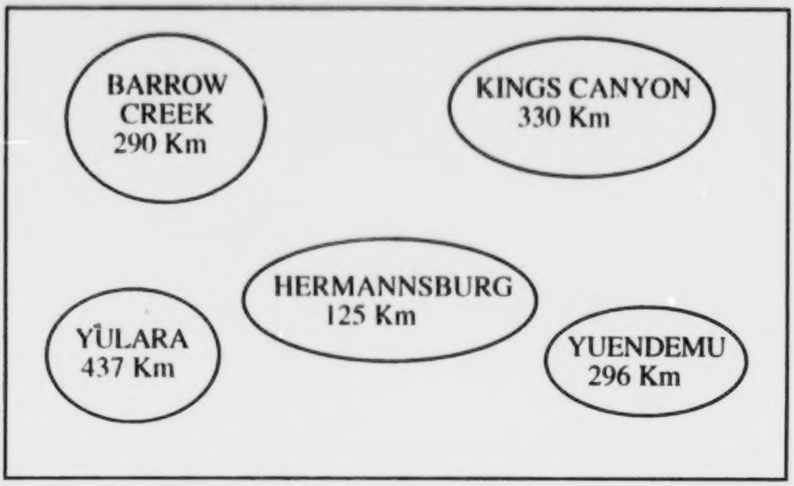

Fig. 1

In the second problem shown, the students were shown a map which showed the position of two 'Gaps' on the Ross River Highway. All students were familiar with the gaps which are popular swimming holes around Alice Springs. Emily Gap is 17 Km from Alice Springs and Jesse Gap is 22 km from Alice. The question posed was "what is the distance between the two gaps?" Once again the responses were either 22 Km (larger) or 39 Km (the sum).

The concept of difference appeared in four other questions relating to:

- difference between sums of money spent on clothes (although usually able to correctly sum the money spent)
- difference between grants to Aboriginal Housing Associations
- increases in airfares
- program length from TV. guide.

In almost all cases the students were unable to correctly answer any of the questions relating to difference.

This difficulty also manifested itself in the written testing. Figure 2 shows the percentage of correct responses to questions.

	Arithmetic skills	Problems	Difference problems
Non Aboriginal	96%	86%	88%
Urban Aboriginal	92%	80%	81%
Rural Aboriginal	86%	53%	25%

Fig. 2

As the table shows, the arithmetic skills of the rural Aboriginal students are not too dissimilar.from that of the non Aboriginal and urban Aboriginal students. The problem solving ability is however way down and **why is the ability to solve difference problems so low?**

All of the students interviewed were undertaking courses where problem solving is meant to be an integral component. Indeed the Yirara students were undertaking courses specifically prepared for them by the NT Open College which all espoused

> "Students will be involved in solving a variety of real life problems... Language usage should be varied so that students can become familiar with a wide range of mathematical terms and the associated processes. New vocabulary needs to be identified and explained." (NTOC-1989ab&c).

The average age of the students interviewed was 15+. All had spent many years in the western education system learning to perform such rituals as subtraction. And yet most still had no ability to apply their skills to a problem situation. Language and culture may play their part but why the extreme difficulty with difference problems in particular?

Can the difficulties be explained by previous research? Are the courses and methodologies wrong? Should the amount of mathematics instruction be reduced in favour or more language development? What are the implications for curriculum writers?

The Northern Territory considers itself in the forefront with regard to the development of appropriate material and curriculum for Aboriginal students. Current iniitiative include the Aboriginal Schools Curriculum Materials Project of the Aboriginal Education Program and the Mathematics Teaching Learning & Assessment Project. A report on the progress of these projects in addressing the issues raised by the research will be made during the presentation of this paper.

REFERENCES

AUSTIN, J.L. and HOWSON A.G.: 1979, Language and Mathematical Education, Educational Studies in Mathematics 10: 161-197.

BISHOP, A.J.: 1988, Mathematical Enculturation: A Cultural Perspective on Mathematics Education, Dordrecht, Kluwer Academic.

BOURKE, S.F., and PARKIN, B.: 1977, The Performance of Aboriginal Students. In S,F, Bourke and J.P. Keeves (Eds.), Australian Studies in School Performance: Volume 3, The Mastery of Literacy and Numeracy: 131-155. Canberra: Aust. Govt. Printing Service. (ACER).

CHRISTIE, M.J.: 1985, Aboriginal Modes of Behaviour, The Aboriginal Child at School 13(5): 3-10.

ELLERTON, N.F. and CLEMENTS, M.A.: 1990a, Language Factors in Mathematics Learning: A Review. Papers presented to the 13th Biennial Conference of A.A.M.T.: 230-259, Hobart.

ELLERTON, N.F. and CLEMENTS, M.A.: 1990b, Mathematics, Culture and Curriculum , Paper presented at the 13th Biennial Conference of A.A.M.T, Hobart.

HARRIS, S.: 1980, Culture and Learning: Tradition and Education In Northeast Arnhem Land, N.T. Department of Education, Darwin N.T.

N.T. OPEN COLLEGE OF TAFE: 1989a, Certificate in Initial Secondary Studies , N.T. Department of Education, Darwin,

N.T. OPEN COLLEGE OF TAFE: 1989b, Certificate in Foundation Studies , N.T Department of Education, Darwin.

N.T. OPEN COLLEGE OF TAFE: 1989c, Certificate in General Studies , N.T. Department of Education, Darwin.

NATIONAL ABORIGINAL EDUCATION COMMITTEE: 1982, Rationale, Aims and Objectives in Aboriginal Education , In Sherwood, J. (Ed.) Aboriginal Education: Issues and Innovations, Creative Research, Perth.

METACOGNITIVE AWARENESS AND MATHEMATICAL PROBLEM SOLVING IN THE SENIOR SCHOOL

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A major aim of the study reported here was an investigation of the relationships between mathematical and cognitive processing and metacognitive activities during problem solving by Year 11 Mathematics I students. During the first year, one problem was used for video-taping sessions whilst another was used in an examination situation followed by free response interviews where students reviewed their examination scripts retrospectively. In the second year, the role of the problems was reversed and a structured interview was used. The interviews probed the students' metacognitive knowledge, strategies, decision making, beliefs and affects. Results indicated that the students possessed quite a store of metacognitive knowledge which had the potential to influence their problem solving activities. Orientation activities were crucial with many students failing to inhibit impulsive responses to initial reading of the problem. Students were more concerned with the mechanics of solution execution and the tyranny of time than with planning, monitoring and verification strategies. The study supported the notion that students acquire and develop their store of metacognitive knowledge through metacognitive experiences and social interaction. There were indications that classroom practice and assessment techniques emphasized the use of automatic routinized application of formulae and procedures at the expense of experiences where students needed to reflect on, monitor and evaluate their progress.

The importance of conscious self regulation or executive control in the problem solving process has been noted by Ashman and Conway (1989), Campione and Brown (1978), Flavell (1979), Lester, Garofalo and Kroll (1989) and Schoenfeld (1987) amongst others. The term "metacognition" has been used to refer to an individual's awareness and conscious control of these monitoring and regulating activites.

Although many authors stress the importance of metacognitive activities in problem solving, students appear to be deficient in monitoring and self regulatory skills. Lester and Garofalo (1982) found that elementary school students do not routinely analyze problem information, monitor progress or evaluate results. Schoenfeld (1983) reported the same result with American college students. Crawford (1986) attributed a similar finding in her study with upper primary school students to socialization processes in the classroom. Schoenfeld (1988) argues that, largely as a result of what goes on in classrooms, students develop a view of mathematics that is detrimental to their problem solving activities. According to Flavell's (1979) classification, metacognitive knowledge consists of both knowledge and beliefs such as this which affect the course and outcome of cognitive activities.

In order to study the role played by metacognition in mathematical problem solving, it is necessary to identify a framework or model which incorporates both the cognitive and metacognitive aspects of the problem solving process. The prototypical model on which most recent problem solving research has been based is Polya's (1957) four phase description of the problem solving process. The model does not, however, deal explicitly with the metacognitive aspects of problem solving which remain implicit. Garofalo and Lester (1985) have proposed what they call a "cognitive-metacognitive framework" which incorporates Polya's ideas but also "specifies key points where metacognitive decisions are likely to influence cognitive actions" (p. 171). The framework consists of four categories of activities that are involved in a mathematical task, namely, orientation, organization, execution, and verification. Each category is explicitly defined and illustrative subsets of the possible cognitive and metacognitive activities associated with each category are identified.

PURPOSE OF THE STUDY

The study reported in this paper was designed to investigate the relationships between mathematical and cognitive processing and metacognitive activites during problem solving by senior secondary students. The specific research questions addressed were as follows:

1. What metacognitive knowledge and activities do senior secondary school students exhibit when attempting to solve mathematical word problems?
2. How does this metacognitive knowledge and activities interact with the students' mathematical and cognitive processing?

METHODS

As the goal of the study was to examine how students think, it was decided that case study would be the most suitable method for such an investigation. A variety of data collection techniques was used as this allowed the researcher to triangulate results and interpretations and resulted in a more detailed portrayal of the problem solving process.

The study was carried out over a two year period at a Catholic Girls' School. In the first year 121 Year 11 Mathematics I students attempted a problem under formal examination conditions. Seventeen of these students were randomly selected to participate in audio-recorded open response interviews where they reviewed their scripts. A further 26 students participated in self-selected pairs in video-taped cooperative problem solving sessions using a second problem.

In the second year the problems were reversed with 120 students sitting for the second problem under examination conditions whilst 22 students were involved in the video-taped sessions using the first problem. Twenty-three randomly selected students participated in audio-recorded structured interviews which replaced the open interviews.

During the first year of the study, an unstructured approach was used in the interviews as the researcher wanted to remain as unobtrusive as possible, allowing her subjects to respond as spontaneously as possible. As preliminary analysis of the previous years' data was completed by the beginning of the data collection stage in the second year, it was decided to use the information already accumulated to develop a structured framework to assist the interviewer maintain consistency between subjects during the second round of interviews. An interview protocol was devised using Garofalo and Lester's Framework (1985). Questions were used to elicit responses by students reviewing their solution scripts within the four categories of orientation, organization, execution and verification. Students were also asked probing questions to determine state and trait variables affecting their performance. In addition, a final series of questions were used to determine students' beliefs about mathematical problem solving in general.

RESULTS AND DISCUSSION

All students in the study who participated in the interviews or the video-taped sessions possessed a store of metacognitive knowledge. This knowledge base had the potential to affect all four categories of problem solving activities identified by Garofalo and Lester (1985) in their framework. Flavell's (1979) person, task and strategy categorization of metacognitive knowledge provided a suitable basis for examining the metacognitive knowledge possessed by the students in the study.

Person knowledge exhibited by the students included assessment of their ability as problem solvers and assessment of their ability to cope in differing problem solving situations. This personal assessment was affected by the influence of such variables as motivation, anxiety and persistence on the particular individual involved.

Task knowledge included beliefs about the nature of the expected task they had to perform and assessment of task difficulty. Their assessment of task difficulty was affected by such features as content, context, structure, surface characteristics of the problem and the problem solver's preference for working on particular types of tasks.

Knowledge of strategic behaviour to assess and understand a problem included strategies to assess the level of difficulty of the problem, to aid understanding of the problem, to organise information and to aid recall of information from Long Term Memory. Strategies to aid understanding included rereading of the problem statement, jotting down key words and numbers, mulling over the question, starting from what the student knew, relating the problem to other problems encountered in the classroom and the textbook, and looking for further information. Students were aware that organizing information into a table or some form of systematic format before attempting analysis greatly facilitated the analytic process and the development of a suitable model. Although some students were aware of the utility of graphs and diagrams for organizing information or clarifying relationships between pieces of information, these were used less frequently. Mnemonic devices were the main aids mentioned or used by students in memory recall.

In general, students lacked an awareness of strategies for generating and executing plans. In fact, almost half of those interviewed believed such planning was unnecessary. Although students often used an overall strategic plan, many were unaware they did so. The highly ``event-driven" nature of mathematical problem solving precludes the implementation of a rigid plan and this could have contributed to students' beliefs that they did not use a plan or that plans were unnecessary. Similarly, if students' past experience of problem solving in mathematics had merely been the instantiation of schema-driven solutions, this reinforced the notion that planning was unnecessary.

An important area of planning where students lacked metacognitive knowledge was strategies for executing plans. For plans to become operational in a controlled and efficient manner, executive decisions had to be made about allocating resources, deciding which aspects of the plan needed to be focussed on and refined at any partciular time and the scheduling of actions. Students appeared to be totally unaware of such strategies as mental simulation which would have allowed them to review their plans for specific segments of their problem solving performance to ensure these decisions were appropriate and the best that could be made under the circumstances. Knowledge of, and experience with, strategies in this particular area of problem solving would have allayed many students' fear that they would not have sufficient time to complete their solution satisfactorily.

On the whole, students' store of metacognitive knowledge about specific strategies for monitoring progress was not well developed with many students saying they did little more than check they were still on track by checking the reasonableness of local results against a mental benchmark or "gut feeling", and/or they guarded against mechanistic errors by redoing calculations or reworking algebraic procedures with the same algorithm. As well as monitoring execution activities, students need to be able to successfully overcome feelings of stress, frustration or panic when things go wrong. To be effective problem solvers, students must be aware of the signs that show they are experiencing their typical stress-response pattern and learn how to control those responses themselves and, in so doing, better prepare themselves for improved performance in the future. Our thoughts affect our feelings and ultimately our actions. Scrambled thoughts are hardly going to lead to controlled and efficient problem solving performance. One quite useful strategy in this regard of which some students were aware, is self-talk. Controlled self-talk can be used by problem solvers to keep their attention on the task at hand, to maintain their focus by using affirmations and so control their confidence and guard against any tendency to just give up and say it is all too hard.

Despite the fact that most students appeared to believe in the importance of verification of their final solution, very few did this in any organised or systematic fashion even when they had sufficient time to do so. The reasons for this appeared to be that students' knowledge of suitable strategies was fairly limited, past experience had not reinforced the need for such verification, and verification received little attention in class and was therefore perceived by students to be of lesser importance.

How can we address these perceived deficiences in students' store of metacognitive knowledge? If the constructive learning theory is to be adopted, then we know that this knowledge can not simply be given to students. The students need to construct it for themselves; therefore, rather than attempt to teach a set of isolated strategies to students, teachers should provide their students with problem solving opportunities that allow them to reflect on and reorganize their current ways of thinking for themselves. To be able to provide students with suitable situations for this to happen we must know how students acquire and develop their store of metacognitive knowledge and how they come to use it efficiently and effectively. These questions, which arose from the investigation of students' store of metacognitive knowledge, were examined in the light of observations from the video-tapes and the interview data.

It was found that the study supported the notion that students acquire and develop their store of metacognitive knowledge through metacognitive experiences and social interaction. These findings were consistent with the views of Luria (1973), Vygotsky (1978) and Reeve and Brown (1985) who all argue that awareness of self-regulatory activity has its roots in our social interactions with others. According to Vygotsky, instruction should lead development. Learning situations should be devised so that they promote movement within a student's "zone of proximal development". The learning situation should awaken within the student a variety of internal developmental processes which are immediately possible for the student through classroom interactions with the teacher or able peers. With time, the degree of aid from the teacher decreases until students take over full responsibility for articulating their own metacognitive processes.

The degree to which students come to use their knowledge efficiently and effectively appears to bear some relationship to the quality of social interaction in the mathematics classroom. There were some indications that classroom practice and assessment techniques emphasized the use of automatic routinized application of formulae and procedures at the expense of experience in true problem solving where students needed to reflect on, monitor and evaluate their progress.

This is not advocating that automaticity of basic skills be ignored. When problem solvers perform basic skills automatically, little attention is required, freeing up memory capacity for performance of other problem solving activities. In fact, problem solvers who do not curtail ``wild goose chases" (Schoenfeld 1987) may fail to do so, not because of poorly developed managerial skills, but simply because in the particular circumstances the performance of basic algorithmic skills requires so much attention that there is no free attention to allocate to the monitoring aspects of problem solving. Thus, automaticity of basic skills affords the more able problem solver the opportunity to give attention to higher order skills. There needs to be a balance, then, between practising and testing basic skills, and giving opportunities to apply these meaningfully in a problem solving context.

The video-tapes revealed how these metacognitive behaviours interacted with the students' mathematical and cognitive processing. In addition, inferences made by the researcher in viewing the video-tapes were confirmed by comments students made in the interviews. The problem solving activities of the pairs of students in the video-tapes were consistent with a cyclic model of information processing as shown in Figure 1. Information gathering, representation, processing and validation rarely occurred in a linear progression. Instead, students would cycle back and forth through these processes as they attempted to build up their model of the problem and progress towards their goal in the light of the results they obtained and the new information that arose as their solution progressed or met a dead-end.

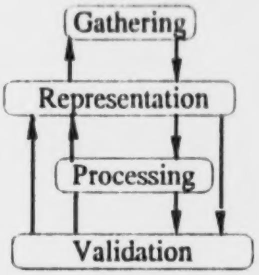

Figure 1. Information Processing Cycles during Problem Solving

Metacognitive activities were involved in all phases of the solution process. Garofalo and Lester's Framework (1985) proved valuable in identifying key points in students' solutions where metacognitive decisions were likely to influence cognitive actions by their presence or absence. Metacognitive activities during orientation appeared to be of crucial importance to the success of the solution. Many students in the study did not attempt to inhibit impulsive responses to their initial reading of the problem, jumping headlong into solution attempts without pausing for reflection. Students appeared to be more concerned with the mechanics of executing their solution and the tyranny of time than with planning, monitoring and verification strategies.

IMPLICATIONS FOR CURRICULUM DEVELOPMENT AND CLASSROOM PRACTICE

There is an obvious need to incorporate into our teaching, methods that facilitate students' problem solving performance by allowing them to make full use of their available metacognitive and cognitive strategies. If we are to adopt current theories of learning such as the constructivist perspective, we should be more concerned with how students will interpret their classroom or educational experiences than with how much of them they will absorb by maximum immersion. Students should be provided with learning situations that are going to give them experience in knowing how to use the metacognitive information they have available to them to predict future events and to plan appropriate responses in order to enhance their problem solving performance. This can be achieved by an astute selection of classroom problems and allowing follow-up time for reflection and discussion of both appropriate and inappropriate solutions including metacognitive activities that were or could have been used. It is crucial that students focus on the selection of relevant information, the construction of a feasible model and testing of that model as well as monitoring and verification strategies.

One method of raising the consciousness of students to the importance of these strategies and making them aware of the implications of their use, or lack of use, would be to view video-tapes of students in problem solving situations such as those produced in the study. Schoenfeld (1987) has found that when students view video-tapes of other students solving problems, it is easier for them to analyze and be objective about that behaviour when it is not their own but then they begin to empathize with the students and see that such an analysis can also be applied to their own problem solving behaviour. For instance, this study has shown that students should be made aware that persistence is not necessarily a virtue in problem solving. Viewing video-tapes from the study would certainly convince students of this.

Metacognition concerns decision making. Decision making involves using both strategic and temporal resources to determine the appropriate response. Both the speed and accuracy of this process of response selection is influenced and linked to the number of decisions to be made, the number of options to be selected from, the total time perceived by the student to be available for decision making and the perceived time-cost associated with incorrect decisions. As has been observed in this investigation, students' decision making about temporal resources is crucial to their success. Students need sufficient experience in problem solving during an extended time interval in order to develop appropriate mechanisms for dealing with efficient use of time

REFERENCES

Ashman, A.F. & Conway, N.F. (1989). Cognitive Strategies for Special Education London: Routledge

Campione, J.C. & Brown, A.L. (1978). Towards a theory of development: contributions from research with retarded children **Intelligence**, 2, 279-304.

Crawford, K. (1986). Simultaneous and successive processing, executive control and social experience: individual differences in educational achievement and problem solving in mathematics. Unpublished doctoral dissertation, University of New England, Armidale.

Garofalo, J. & Lester, F.K. (1985). Metacognition, cognitive monitoring, and mathematical performance. **Journal for Research in Mathematics Education, 16** (3), 163-176

Lester, F.K. & Garofalo, J. (1982). Metacognitive aspects of elementary school students' performance on arithmetic tasks. Paper read at the March meeting of the American Educational Research Association, New York.

Lester, F.K., Garofalo, J. & Kroll, D.L. (1989). The Role of Metacognition in Mathematical Problem Solving. Final Report. Bloomington, IN: Indiana University, Mathematics Education Development Centre. (Eric Document Reproduction Service No. ED 314 255).

Luria, A.R. (1973). **The Working Brain: An Introduction to Neuropsychology**. Translated by Haigh B. London: Penguin Books.

Polya, G. (1957). **How to Solve It**. Princeton, NJ:Princeton University Press.

Reeve, R.A. & Brown, A.L. (1985). Metacognition reconsidered: Implications for intervention research. **Journal of Abnormal Child Psychology, 13** (3), 343-356.

Schoenfeld, A.H. (1983). Episodes and executive decisions in mathematical problem solving. In R. Lesh & M. Landau (Eds.), **Acquisition of Mathematics Concepts and Processes** (pp. 345-396). New York: Academic Press.

Schoenfeld, A.H. (1987). **Cognitive Science and Mathematics Education.** Hillsdale, NJ: Lawrence Erlbaum Associates.

Schoenfeld, A.H. (1988). When good teaching leads to bad results: The disasters of "well-taught" mathematics courses. **Educational Psychologist, 23** (2), 145-166.

Vygotsky, L.S. (1978). **Mind in Society: The Development of Higher Cognitive Processes**. In M. Cole, V. John-Steiner, S. Scribner & E. Souberman (Eds. and Trans.). Cambridge, MA: Harvard University Press.

DESCRIBING TEACHING: CATEGORIES FROM TEACHER EDUCATORS' DESCRIPTIONS OF QUALITY PRACTICE

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Survey responses of 125 teacher educators and experienced teachers on an open-response item on aspects of mathematics teaching are presented. A qualitative analysis of the responses using the NUDIST program resulted in six major categories of responses. These were Communication, Problem solving, Building understanding, Engagement, Task orientation and Teacher concern. Within each of these categories, the frequency of use of particular phrases and descriptors indicates general beliefs about the important characteristics of quality mathematics teaching. These categories and qualifications are presented as a starting point for further discussions about the development of a common language for describing teaching.

This paper is an attempt to initiate discussion on ways to describe teaching. It presents a list of categories which arose from a survey of teacher educators on ways of describing quality teaching, and then discusses how these can inform our understanding of the task of teaching. The survey, which had mainly fixed format items and some open response items, was first trialled by twelve teacher educators who were asked to complete the questionnaire with an observer present. The educators were requested to think aloud while responding to the items to determine the appropriateness of questionnaire items and wording. All comments and queries were recorded, then after appropriate revision the trial process was repeated with a further four teacher educators completing the survey and being interviewed.

The revised survey was mailed to three groups. There were 40 survey responses from graduate students in mathematics education (100% return), 56 from Victorian teacher educators (80% return), and 29 from American teacher educators (40% return). These groups were selected because they represented an informed view of current issues in teaching and learning.

The focus here is one open-response item in which respondents could give their own impressions of the important features of teaching. There was a huge diversity in the actions which were rated as important as well as the language used to describe these. This article is an attempt to extract some common themes from the diversity.

ANALYSING FREE FORMAT RESPONSES

The data presented here are the responses to one open-response item, presented as follows:

> Throughout this survey we want you to imagine a mathematics lesson, at any year level, where the students are learning, for example, to estimate the mass of various objects, or to add fractions, or to record given information as a graph. Before turning the page, please write down the most important characteristics which a **quality** mathematics lesson on any of these concepts/skills would usually have.

Respondents were asked to complete this item first, before they read any categories or actions presented by the researchers. In this way, it was hoped that responses would represent natural views.

Initially, all 125 responses were typed and categorised by hand under general sub-headings. In summary, this stage of the research involved the following actions:

Responses were typed and stored on computer disks.
Key ideas were identified; these are called nodes (e.g. *communication*).
Subsidiary ideas were noted; these are called sub-nodes (e.g. *recording*).
Each phrase in the responses was coded according to the node and sub-node.
The qualitative analysis program NUDIST was used to group the phrases together.

The collections of phrases were examined to determine whether they formed a coherent set. The phrases which were not congruent with the others in the set were moved to a different sub-node.
The sub-nodes were regrouped.
These last two steps were repeated until we were satisfied.
The data are presented.

RESULTS

As with most research, our goal is to organise, summarise, and present the data in a coherent way in order to allow readers to interpret the findings in the light of their own practices. The data are organised here into six nodes. These nodes are presented in the order of the frequency with which phrases were coded under the node. While there is no suggestion that these nodes should be compared quantitatively, it gives some insights into the priorities of the respondents.

There is a summary table for each node to indicate the frequency of phrases which were assigned to each sub-node. For the commonly cited sub-nodes, a list of some phrases is presented. It is not appropriate to produce lists of frequency tables for each sub node, but to give an indication of the frequency of each type of statement, frequently occurring statements (say greater than 10 times) are printed in larger font, statements given often (3 - 9 times) are normal size, and statements given infrequently (1 or 2 times) are presented in small font.

Building understanding.

The frequency of use of phrases categorised under each of the sub-nodes of building understanding are presented in Table 1.

Table 1: Number of entries under each sub-node of **building understanding**

Sub-node	No. of entries
Materials	36
Prior knowledge	21
Mathematical thinking	15
Connections	14
Conceptual understanding	10
Reflection	6
Sequence	4
Review	6

Building understanding relates to comments which seemed to refer to a role which the teacher assumes in order to convey some pre-determined meaning to students. It is a recognition of a content to be covered, and of strategies to achieve this end by building on existing knowledge, using material to explain and clarify concepts, choosing appropriate sequences, helping students to make connections and form relationships, and to know the meaning of terms. There is a strong inference of decision-making, direction, explanations and control by teachers.

We coded as "materials" statements like the following:

Concrete representations
Aids that can be manipulated

This was the most commonly listed sub-node overall.

A key component of learning is making links with the students' existing knowledge. We coded as *prior knowledge* phrases like the following :

Prior knowledge of children, existing understanding of concepts
Appropriate to learner

The making of connections is an activity which the student does, but which can also be the focus of the attention of the teacher. These can also have a mathematical emphasis. We coded as *connections* statements like:

Mathematical connections
Constructing relationships

We coded as *mathematical thinking*, including the use of appropriate language, phrases like the following:

Significant mathematical issues arise in a natural way out
Present a strong "sense" of number

We coded as *conceptual understanding*:

Build understanding of concepts or procedures including meaning
Coherence and development of ideas and concepts in both teaching and learning

This, as with most sub-nodes in this node, is about concepts, mathematical ideas, developing links and perhaps most refers to the Skemp (1976) notion of relational understanding. The node overall refers to an orientation on the part of the teacher to plan and teach mathematics in an orderly, coherent, sequenced, and connected way.

Communication.

Communication is widely acknowledged as an essential component of learning mathematics, but for which there is often inadequate definition or explanation which allows an appreciation of what is actually meant by the term and associated issues. In grouping the responses, this node was used to include statements related to opportunities for talking, explaining, describing, listening, asking, clarifying, sharing, writing, reporting, and recording. Table 2 presents the sub-nodes which form this node and indicates the frequency of occurrence of phrases which were placed within each sub-node.

Table 2: Number of entries under each sub-node of **communication**

Sub-node	No. of entries
Discussion between pupils	34
Sharing strategies	20
Co-operative situations	15
Recording	6

Under the sub-node *discussion between pupils* phrases like the following were used:

Discussion can occur between children
Lots of discourse

This is an example of the dilemmas we meet when describing teaching. Clearly discussion is an avenue to other goals. It has limited value in itself. This suggests that the language we use to describe this aspect of teaching needs to be more explicit and less open to interpretation by the listener.

The second sub-node was termed *sharing strategies*. Within this the various terms used were:

...alternate solution strategies
Sharing children's own strategies

This is a development of the first sub-node, with an assumption that the teacher is facilitating some structured sharing of ideas. Again it is an avenue to a broader goal.

Under the sub-node termed *co-operative situations* phases like the following were used

Co-operative group work
Interacting with peers

The same comment as above is relevant. The goal is not the co-operative situation, nor even the interaction, but how such situations lead to opportunities for learning. While the development of co-operative skills is useful and is seen as a necessary competency for industry (e.g. The Mayer Committee, 1992), it is also a strategy designed to achieve some other goal.

Engagement.

This node is about the students' involvement in their own learning. This has always been recognised as important by good teachers but has had renewed impetus with recent conceptions of knowledge as constructed by the learner. Table 3 shows the nodes used to compose this factor.

Table 3: Number of entries under each sub-node of **engagement**

Sub-node	No. of entries
Active involvement	34
Personally relevant	16
Enjoyment	5
Real world	7
Motivation	6
Variety	6

We coded as application to *real world* the following:

Based on real world situations

Opportunities to apply skills and understandings

A somewhat significant dilemma arises here. Is the use of real world examples merely an aid to some other goal, or is it in fact a goal in itself, or some combination? This perhaps is an issue which could provoke some debate among mathematics educators, but it is clear that the issue is not resolved by examination of curriculum documents.

We coded as *personally relevant*:

Experiences/activities have relevance to children

Personally meaningful and relevant to students

A link to a situation children can relate to

The notion of active involvement was the second most frequently cited phrase. However, by itself it has no real meaning. Does it presume some physical activity? Is physical activity enough? It is another example of a term which is used loosely and which serve to reinforce the readers' or listeners' predisposition but which does not necessarily convey the meaning intended by the communicator. We coded as *active involvement* terms such as the following:

Students will be actively involved in the learning process

Engagement in solving interesting problems

Physical involvement

Students are "immersed" in the mathematics they will be learning

We coded as *motivation:*

Interested in what they are doing

Engagement in solving interesting problems

Excitement at exploring

Students enthusiastically embracing the task

The refers to a rationale for encouraging the students to become engaged in their learning or, in other words, to want to learn.

Problem-solving.

Problem-solving has been a major strand in considerations about mathematics teaching and learning for over 30 years (e.g. Polya, 1957). Nevertheless the term problem solving has been used to mean many things in many situations. It is not surprising that some variation of the term problem-solving was used frequently by respondents. As with other terms discussed here, use of the term problem-solving does not convey to an intended audience any specific meaning.

In the analysis, the phrases grouped in this node were those which relate to students working out for themselves how to perform mathematical tasks in such a way that it is the students' own work and they know that it is. It refers to activities such as risk-taking, challenging, exploring, investigating, thinking, asking, and posing. Table 4 presents the sub-nodes which form this node and indicates the frequency of occurrence of phrases which were placed within each sub-node.

Table 4: Number of entries under each sub-node of **problem-solving**

Sub-node	No. of entries
Investigation/problem-solving	27
Open-ended activities	14
Challenging	6
Problem posing	6

The *investigation/problem solving* node was made up of the following comments.

> Student active involvement in investigations
> Engagement in solving interesting problems
> Promotes thinking

Under the heading *open-ended*, the following descriptions were coded

> Open-ended questions
> Opportunities to explore

The use of a term such as problem solving is not adequate in itself, and it may be necessary to use qualifying phrases in order to make the meaning clear.

Teacher concern (for students).

The reviews of teacher effectiveness use the term "family like atmosphere" as being a component of classrooms where the children have comparatively high achievement. It is hard to know the significance of that phrase but a recurring thread in the listed features of a quality lesson were characteristics which suggested that the teacher is sensitive to the needs of the students as individuals.

Table 5: Number of entries under each sub-node of **teacher concern**

Sub-node	No. of entries
Catering for levels of ability	11
Non-threatening	10
Rapport	9
Relationships	4
Goal setting	2
Enthusiasm	8

The phrases related to *catering for abilities* were as follows

Allowing for levels of ability
Challenging but caters for individual differences

The phrases related to *non-threatening* were like:

Some in-built success
Students should feel comfortable within taking risks

The sub-nodes are explicit about a relationship developed between the teacher and pupils. The task is not just to teach mathematics but to teach students as well. While the best ways to teach mathematics is to cater for the range of abilities and develop rapport, nevertheless we suspect that the node **teacher concern** is a more a recognition that teaching and learning is a two-way process and that there is something natural in the expert/novice relationship which includes a nurturing component.

Task orientation.

Task orientation, a factor often cited in reviews of teaching effectiveness (e.g. Good, Grouws & Ebmeier, 1983), refers to actions designed to keep the students working towards achieving the lesson's goals. We took it to mean decisions made by the teacher about a specific focus for what would happen in the lesson and a commitment to pursuing that focus and to communicating the focus to the students.

Table 6: Number of entries under each sub-node of **task orientation**

Sub-node	No. of entries
Clear purpose	14
Clear instruction	7
Organisation	4
Questions	3
Assessment	4

The phrases coded as *clear purpose* were like the following:

The teacher making clear what is the purpose of the lesson
A set of clear objectives which connect to what pupils already know

The obvious assumption is that the teacher will have selected a direction or goal for the lesson, that this is communicated to the students in some way, and even that it could be identified by an observer. A similar comment can be made about the next two sub-nodes, *clear instructions* and *organisation*.

SUMMARY

The statements listed above are responses from 125 experienced mathematics educators to an open survey item which sought views on features of quality lessons. The first observation was of the diversity of replies. Given that teaching is highly complex it was anticipated that a broad range of factors would contribute to an impression of a quality lesson. Nevertheless we were surprised by the lack of commonality in the responses. It seemed that there was little professional language taken as shared by the community represented by the replies. We see this as an important issue. The purpose of our professional interactions, in papers and reports, at conferences and elsewhere, is to develop sets of shared meanings. We suggest that there needs to be attention to the practical and theoretical meanings of the constructs which comprise our understanding of the tasks of teaching and research about teaching. We need to subject our professional language to detailed examination and to test our individual interpretations in practical situations.

In an attempt to seek some commonality in this diversity we dissected responses into phrases and then grouped them together to seek trends and features. Six factors emerged as components of the replies. The six nodes, along with summary descriptions are as follows:

Building understanding: This is about a recognition of a content to be covered, and of strategies to achieve this end by building on existing knowledge, using materials to explain and clarify concepts, choosing appropriate sequences, helping students to make connections and to form relationships, and knowing the meaning of terms. In this node, strong inferences of teacher decision, teacher direction, teacher explanations and teacher control are evident.

Communication: Under this node were included statements related to opportunities for talking, explaining, describing, listening, asking, clarifying, sharing, writing, reporting, and recording. The emphasis within this node is on expressing and communicating mathematics.

Engagement: This is about facilitating student involvement in their own learning. It includes actively involving the students in their learning, and motivating students to learn. This can be done by using personally relevant material or real world situations, and by seeking to make learning enjoyable.

Problem solving: This refers to activities such as risk taking, challenging, exploring, investigating, thinking, asking, and posing. It is about students using their own conceptions to interpret unfamiliar situations and becoming comfortable with their own ability to do this.

Task orientation: This includes a focus by the teacher on specific goals which are made explicit, clear instructions, good organisation and some assessment of the achievement of the lessons' aims.

Teacher concern: This is about treating the students as individuals, the creation of non-threatening environments which support opportunities for success by all students, the development of mutually positive relationships between teacher and students, and about shared goal setting.

While these six factors present a summary of the responses, the factors are different in both the focus and the locus of responsibility. For example, the node **building understanding** was outstanding in terms of the number of times respondents referred to its features (112 times). Upon reflection, it seemed that each of the other nodes could be considered a vehicle for building mathematical understanding. For instance, the phrases listed under **task orientation** seemed directed at teacher actions which in turn would lead to building understanding. There were even sub-nodes of **building understanding** which were merely mechanisms for achieving the goal; *materials* is such a sub-node. While other nodes, such as **communication**, may be significant in themselves, the main purpose of the comments within the node seem to be aimed at building understanding.

Overall, it seems that the challenges for teacher educators are to find ways to define and communicate goals of teaching and to differentiate between goals and tools.

REFERENCES

Good, T. L., Grouws, D., & Ebmeier, H. (1982). *Active mathematics teaching.* New York: Longmans.

Mousley, J. & Clements, M. A. (1990). The culture of mathematics classrooms. In M.A. Clements (Ed.) *Whither Mathematics?* (pp.397-406). Melbourne: Mathematical Association of Victoria.

Mousley, J. & Sullivan, P. (1992). *Quality mathematics teaching: Consensus and contradiction.* Proceedings of the Australian Association for Research in Education Conference (disc), Deakin University.

Mousley, .J , Sullivan, P. & Clements, M. A. (1991). *The perceptions which student teachers have of teaching practices in classrooms observed during field experience.* Paper presented at the Annual Conference of the Mathematics Education Lecturers' Association, Perth.

Polya, G. (1957). *How to solve it.* New York: Doubleday.

Skemp, R. (1976). *Relational understanding and instrumental understanding.* Harmondsworth: Pelican.

The Mayer Committee (1992). *Employment-related key competencies: A proposal for consultation.* Melbourne.

NEGATIVE NUMBERS; CONCRETE AND FORMAL IN CONFLICT?

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Provisional results are reported here from developmental research on negative numbers in the upper grades of primary school. An attempt has been made to deal with learning strands both on concrete magnitudes or quantities and on formal constructs. the observation of individual learning process is the core of the research. The results show that - in the learners- a gradual shift takes place either from the realistic context to the mathematical one or from the known mathematical context to the new one.

Negative numbers occurred differently in the history of mathematics. On the one hand they were connected with concrete magnitudes and quantities (e.g. possesses and debt)[1] and on the other hand with by-products of, or constructs to support the execution of algebraic operations (Fischbein, 1987).

This means that negative numbers appeared both as concrete numbers and as formal mathematical constructs. Both manifestations provoked cognitive conflicts in mathematicians for centuries (Glaeser, 1981)[2].

For the formal one no intuitions, to use Fischbein's terms (1987) can be developed (or mental objects as Freudenthal (1983) would have called them).

Moreover the existence of concrete manifestations of negative numbers blocked the acceptance of negative numbers as meaningful mathematical through-things (or nooemena according to freudenthal (1983)). And 'less than nothing','$-3 <$ but $(-3)^2 > (2)^2$' are controversial expressions indeed from the perspective of concrete magnitudes or quantities.

It was Hankel who solved the problem of controversion in negative numbers definitely in 1867. In his 'Theory of complex numbers' he formulated his principle of permanence. This means that negative numbers ass an operational structure had to fulfil the demand of algebraic consistency, that is obeying the law of distributivity of multiplication over addition.

E.g. Distributive law: $(a + b)\,(c+d) = ac + ad + bc + bd$, thus

$$(4 + {}^{-}3)\,(5 + {}^{-}4) = 4x5 + 4x{}^{-}4 + {}^{-}3x5 + {}^{-}3x{}^{-}4 = ?$$

$4x{}^{-}4$ permits a concrete interpretation (that means intuitive in Fischbein's terms or at the level of mental objects in Freudenthal's terms), namely '4 times a debt of 4 (...)'. ${}^{-}3x5$ can only be interpreted in the same manner it is replaced by $5x{}^{-}3$ due to the law of commutativity and that is a formal, mathematical operation! Finally ${}^{-}3x{}^{-}4$ needs to be interpreted as ${}^{+}12$ in order to achieve the same outcome as $(4 + {}^{-}3)\,(5 + {}^{-}4)$ has.

So, according to Hankel's view, negative numbers are after all formal mathematical constructs.

THE TEACHING OF NEGATIVE NUMBERS

The historical dilemma as described briefly, has its educational counterpart. One the one hand one cannot avoid 'possess and debt', 'above and below sealevel' and so on, but at the same time it will be impossible to neglect the formal arguments to make the negative numbers 'full-grown' mathematically. This means that a course on negative numbers also needs to contain a learning strand aiming at formalisation, which means algebraising the structure of negative numbers (by-product approach).

Because of the cognitive controversion between the to manifestations of negative numbers and in order to avoid misconceptions in the learner because of false concretisations one also might choose for a by-product course only (cf. Liebeck, 1990, p. 238).

An important question is now: Will such a course with learning strands on both concrete magnitudes or quantities and by-products be possible? And derived from that: What additional features should the by-products strand necessarily have?

THE PRESENT RESEARCH

Kind

The research to be reported here is exploratory, developmental research (cf. Freudenthal, 1991, and Streefland, 1991). Such research aims both at the development of a prototype of a course (on negative numbers in the present case) and at bringing forth theory regarding the teaching and learning of the subject (this also includes extending the existing theory within the same general framework for mathematics education).

The Course

What will be reported here is based on a course consisting of two parts. Part I: Events at the busstop. The changes in occupation of the bus on ride, can be described by means of events, either expressed in number pairs of passengers getting on and off the bus, for instance (5,$^{-}$10) or in single numbers expressing the change, ($^{-}$5), in the number of passengers. Part II: Column subtraction from left to right, for instance:

$$\begin{array}{r} 324 \\ \underline{258}\text{-} \\ 134 \end{array}$$

This way of calculation can evoke the need for the construction of the learners' personal notation of the shortages. Will it be possible to subtract also numbers in shortage-notation? What happens in the hundreds', tens' and units' position, how can it be related to busstop-events? And what is the role of numbers and their opposites?

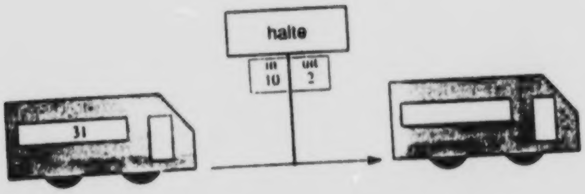
N.B. Different contexts will be explored later too, like 'temperature' and 'dry and wet numbers in locks'.

Additional Features of the Course

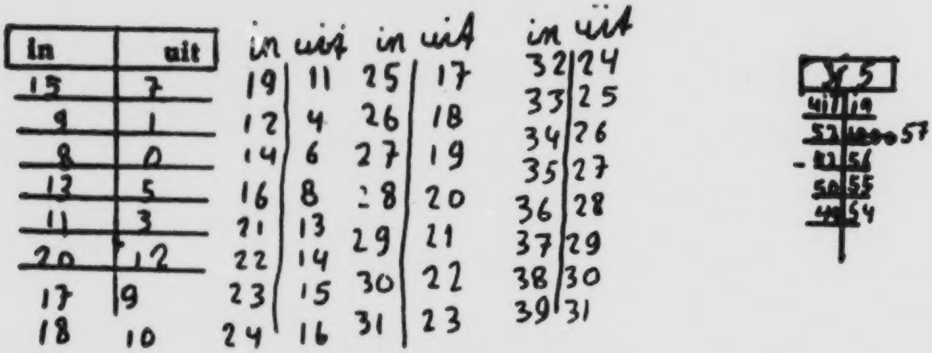
As the first example shows, the number pair appearance of negative numbers can be met easily; one single negative number 'hides' and (infinite) set of number pairs.

The second example implies that the learners can apply their known way of subtraction to find out how they should operate with shortages. Their normal procedures wither hands-on or with calculators act both as a framework for *making plausible* and for *verification*.

Problem:



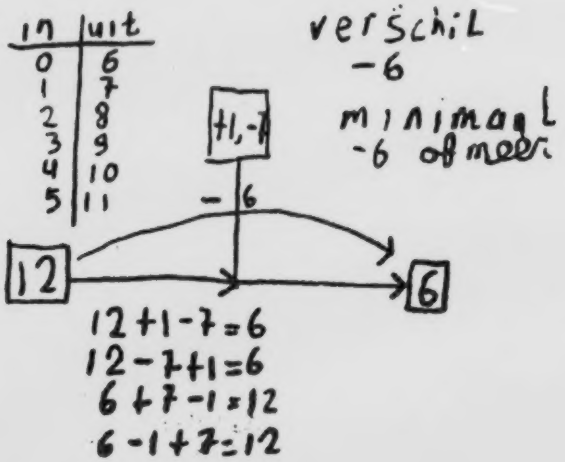
31 people on the bus; 10 on and 2 off at the stop; how many.........?



in	uit
15	7
9	1
8	0
13	5
11	3
20	12
17	9
18	10

in	uit	in	uit	in	uit
19	11	25	17	32	24
12	4	26	18	33	25
14	6	27	19	34	26
16	8	28	20	35	27
21	13	29	21	36	28
22	14	30	22	37	29
23	15	31	23	38	30
24	16			39	31

Paradigmatic examples of pupil's work



in	uit
0	6
1	7
2	8
3	9
4	10
5	11

12+1-7=6
12-7+1=6
6+7-1=12
6-1+7=12

The following characteristics draw attention:

- treating busstop-events from a change-in-occupation point of view;
- filling in busstop-tables either arbitrarily or systematically;
- shifting from filling in arbitrarily to systematically;
- meeting restrictions set by the context, for instance either starting or ending with the maximal numbers of passengers getting on and off the bus that are possible;
- (de-) contextualising and algebraising (cf. the number sentences).

The Group

The group (17) consists of the two combined uppergrades of a primary school (11-12 years) and is of average ability.

Methodology and Methods

In general the observed learning processes are considered to be the starting point for the determination of 'How to proceed?' As a consequence the question is not how to achieve a set of predetermined goals for negative numbers, operations included, but rather to find out how the learners' notions on negative numbers develop.

Therefore the observation of the teaching and the (individual) learning processes are at the core of the researcher's activities.

The ongoing activities consist of a teaching experiment intervened by individual interviews.

PROVISIONAL RESULTS

The results will be considered in relation to the aims of developmental research (prototype and theory). Since the results are still provisional (the research has not been finished yet), only some essential features can be touched upon here.

Individual Learning Processes

Part I: Events at the busstop.

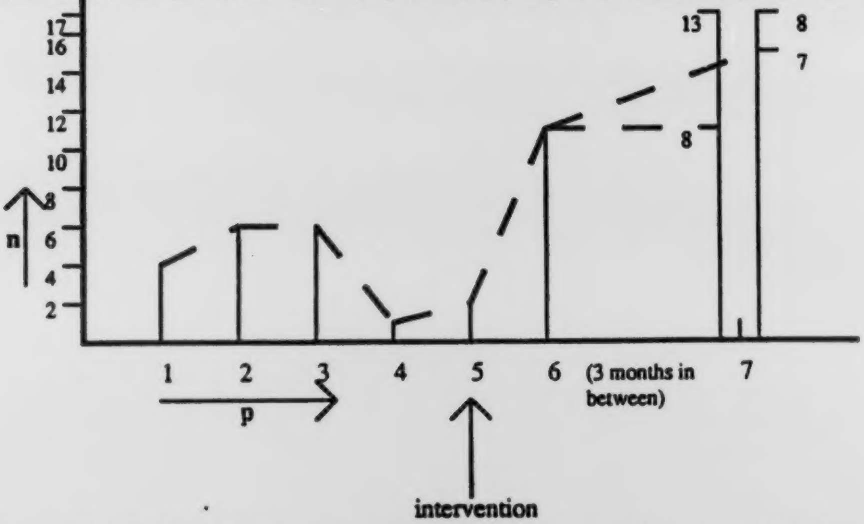
Different aspects of the educational context for this part of the course were:

- collaboration in practical work (worksheets) in small groups;
- interactive sessions on the constructions and productions by the learners which means discussing and negotiating them;
- producing problems of busstop events freely by way of assessment.

Interventions were made by means of (clerical) interviews. The buses 'drove during four lessons'. The assessment took place three months afterwards. The different aspects will be shown in an example of students work first. The learning history of the complete group will be considered from a double perspective, namely:

- the presence of (non)-systematic tables (1), and
- (de-)contextualisation and algebraisation (2).

(1): For six problems the pupils filled in tables. The fifth problem was discussed in an interactive session. It was then that filling in systematically was discussed and reflected upon for the very first time with the group as a whole. The following graph shows that working systematically improved considerably after the interaction.



Intervention by interactive session; assessment after three months; n = number of pupils, p = problem number; problem 7 was a free production assessment task concerning the invention of bus problems with tables and number sentences.

It came out from the interviews after the assessment that all the pupils who did their tables systematically party in advance of the interactive session said that they would have used an eventual second opportunity to change their non-systematical tables into sysematical ones. The aim was to develop the skills of connecting number

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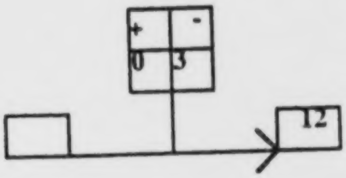
pairs and their (negative) outcomes.

(2): 'Having the bus driving backwards' (db) was invented spontaneously by 8 out of 17 pupils for the following problem:



Two of them were able to make fitting number sentences like 12 + 3 - 0 = 15.

The challenge of the assignment of the assessment provoked the production of varying number sentences according to features like (de)-contextualisation and algebraisation. The table on the next page contains the results. Three pupils even formulated rules like:

> 'If you calculate the other way round, that is from left to right, the pluses become minuses and the minuses pluses.'

task	db	alg
1	10	7
2	5	2

Number sentences (de)-contextualised and algrebraised

N.B. This concerns a free production task again. The fall back in registered results can be explained by the kind of assignment, the time factor and ciromstances like that.

Part II: Column subtraction from Left to Right.

Here it turned out that the problems were counter-intuitive as soon as an overall result was negative. For instance:

1823 "This one is wrong because there is nothing left from which you

$\underline{-9745}$ can subtract your debts. So I have a debt of $\overline{7}\overline{9}\overline{2}\overline{2}$

$\overline{8}1\overline{2}\overline{2}$ no debt $\underline{-100}$

So I have a debt of $\overline{7}\overline{8}\overline{2}\overline{2}$

As long as the shortages were not in the place with the largest value the operations performed by the pupils went well in general. 13 out of 17 pupils applied the notation with shortages in the assessment task, while the remainder fell back to their usual method. The rule was formulated that two shortages of the same 'size' have a difference of zero.

DISCUSSION

The research showed thusfar that the historical conflict of the two manifestations of negative numbers is also met by the pupils of the group that participates in the experiment. Moreover it came out that the context of the busstop provides for the possibility to notate number sentences in the opposite manner. This can be considered as a prestage to notating them formally, or to put it differently: It may be hypothesised that between the positive and the negative numbers there - from the learners point of view - seems to be an interjacent area where negative numbers occur implicitly. It will be our further aim to chart this area.

Finally it came out that the learning processes can be characterised by shifting gradually from a concrete to a partly formalised level. This shifting process consists of different, subsequent shifts, that are provolked by reflections in the learner on what is going on.

Notes

1. The names for negative numbers used by Brahmagupta (about 600 a.C) referred to debt and property.
2. Lazare Carnot (1753-1823), writes, referring to operations with negative number: "A multitude of paradoxes and relative absurdities result from the same notion; for instance, that -3 would be smaller than 2; nevertheless $(-3)^2$ would be greater than $(2)^2$, that it to say that, considering two unequal quantities, the square of the greater would be smaller than the square of the smaller: This shocks every clear idea one may get, referring to the notion of quantity" (quoted from Fischbein, 1987, l.c. p. 99).

REFERENCES

Chulvers, P. A consistent model for operations on directed numbers, Mathematics in School 14, 1, 1985.

Fischbein, E. Intuition in Science and Mathematics, Kluwer, Dordrecht, 1987 (ch. 8 The Practicality of Intuitive Meaning, Analysis of an example: The Negative Numbers, 97 - 103)

Freudenthal, H. Didactcal phenomenology of mathematical structures, Reidel, Dordrecht, 1983.

Freudenthal, H. Revisiting Mathematics Education China Lectures, Kluwer Academis Publishers, Dordrecht, 1991.

Glaeser, G. Epistemologie des nombres relatifs, Recherches en Didactique des mathemmatiques 2 (3), 1981, 303 - 346.

Liebeck, p. Scores and forfeits. An intuitive model for integer arithmetic, ESM 21, 1990, 221 - 239.

Streefland, L. Fractions in Realistic Mathematics Education A Paradigm of Developmental Research, Kluwer Academic Publishers, Dordrecht, 1991.

INVESTIGATING THE METHODS USED BY QUEENSLAND MATHEMATICS TEACHERS TO OBTAIN CURRICULUM INFORMATION.

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The researched discussed in this paper reports on the findings of a survey of Queensland mathematics teachers. The survey sought to determine the methods used by these teachers to obtain curriculum information. The results suggest that as a group Queensland mathematics teachers do not undertake professional development activities.

In the past ten years major curriculum changes have taken place in Queensland schools and change is still occurring. A new 1-10 syllabus has been introduced having as its lynchpin a problem solving approach to the teaching of mathematics. As a direct consequence of the implementation of this curriculum a new and different syllabus is at present being introduced into the senior high school. Each of these documents advocates the use of problem solving as the vehicle for teaching mathematics. For many teachers this represents a radical change in the way they will have to teach. Added to the change in emphasis there are also new content areas, such as discrete mathematics, being included in the curriculum. These new topics are often unfamiliar to teachers, who have been in the workforce for a number of years.

The changes to the Queensland mathematics curriculum necessitate that teachers undertake a personal program to update their knowledge in the curriculum area. It must be remembered " one cannot communicate what he does not know; one cannot communicate with maximum effectiveness what he does not understand" (Bell, 1982). In the sense it is being used here, curriculum is to be taken in its broadest sense and would include pedagogical knowledge, content knowledge, curricula knowledge, and pedagogical content knowledge. The purpose of the investigation reported here was to determine what methods were being used by Queensland mathematics teachers to gain the necessary new information and skills.

LITERATURE SURVEY

Research (Cogan & Anderson, 1977; Owen, Hall & Malcolm, 1981; Hamilton, 1983; Pearce, 1984) has identified methods used by teachers to keep up-to-date in their professional area. Included are professional journal reading, inservice programs, seminars, conferences, formal study, conversations with colleagues, and being active in professional associations or serving on curriculum committees.

In a survey of the professional journal reading habits of Australian mathematics teachers Swinson (1992) found that they are not regular readers of these journals. This finding is supported by earlier research carried out in Australia (Pryke, 1987; Owen, Malcolm & Hall, 1982; Finger, 1977) and overseas (Davis & Butler, 1986; Hurd, 1981; Cogan & Douglas, 1977). In each of these studies the findings were the same, namely, mathematics teachers do not regularly read professional journals.

Dynan and Treaqust (1985) researched conference attendance of Australian science teachers and found that 62% of those surveyed had attended three or more conferences in the previous year and a half. Also 80% of the respondents claimed the conference attendance was very valuable in contributing to their professional development. Some caution must be exercised when considering these claims as those returning the questionnaires were all participants at the Conasta conference for science teachers and perhaps could represent a biased sample. Owen, Hall, & Malcolm, (1981) conducted a survey of Victorian teachers and found that more than half of the teachers responding had attended a conference that year, and about two thirds had been involved at a seminar in their school during the period. When reporting on an investigation relating to mathematics teachers in Canada, Schroeder & Frame (1986) noted that in British Columbia only 30% of junior high mathematics teachers had attended a conference in the previous three years.As well Schroeder & Frame found that 36% of Alberta's mathematics teachers had not participated in any inservice activity in the previous three years. No Australian data was found which gave conference attendance information for mathematics teachers.

Conversations with colleagues is often stated as a method used by teachers to gather new information related to their professional activities (Jeffery, 1984). However from an extensive survey of the literature, Pearce (1984) concluded that teachers " don't talk much to each other about the science and the art of teaching." This conclusion of Pearce is supported by the work of Bell (1982) who found that teachers did not get professional information from conversations with friends.

The review of the literature suggests that, as a group, mathematics teachers may not be keeping up with changes in their professional area. Perhaps this should not be surprising for Romberg (1989) stated that " many teachers consider current practice as basically good and see no need to change the status quo." To test the assumption that Queensland mathematics teachers undertake professional activities which would update their knowledge and upgrade their expertise, a survey of these teachers was undertaken in the second half of 1992 to determine which methods were used.

SAMPLE

A random sample of 31 Queensland high schools was selected from both city and country locations. All mathematics teachers in these schools were invited to participate in the survey. A sample of this size would be large enough to give an indication of the situation. A letter describing the project and suggesting possible benefits to teachers was sent to the mathematics coordinator at each of the selected schools. Included was a set of questionnaires and an invitation to all mathematics teachers to participate. One week later the mathematics coordinator was contacted by phone and requested to encourage staff members to complete a questionnaire.

Two hundred and fifty-seven teachers from the 31 schools returned completed questionnaires. This sample contained in excess of 80% of the mathematics teachers in those schools.

The survey sample contained both single sex and coeducational schools from both the city and country and included state and private schools. Of the mathematics teachers in the sample (i) 64% were male, (ii) 43% had less than ten years teaching experience, (iii) 63% were under 40 years of age, (iv) 54% held a bachelors degree, 14% had a diploma and 18% were undertaking further study, (v) 26% belonged to a mathematics teachers' professional association and (vi) 20% stated that they were members of a professional committee such as a state curriculum committee.

INSTRUMENT

To determine how mathematics teachers were updating and upgrading their expertise a questionnaire was developed. The questionnaire sought information on professional journal reading, conference attendance, inservice attendance, seminar participation, and membership of professional committees. The teachers were also asked how much they thought each of these activities impacted on their classroom practice. Furthermore the teachers' views were sought on what they thought was the main method by which they normally gained information about new teaching ideas and new curriculum developments.

RESULTS

The results suggest that Queensland mathematics teachers are not regular participants in professional activities which would upgrade and update their expertise. In some cases the teachers were not even aware of all of the normal avenues available for doing so. The results are similar to those found in research previously reviewed.

Like their colleagues in other states and overseas few of the teachers surveyed read professional journals. Sixty-seven percent of the sample do not regularly read professional journals. Only 14% stated that they read the most of the articles in the Australian Mathematics Teacher, the national journal. More importantly 37% indicated that they were not familiar with this journal. For the Queensland state journal only 14% regularly read most articles and 54% were not familiar with this journal. These figures are very similar those obtained by Swinson (1993) for Australian mathematics teachers as a group.

In table 1 is displayed the statistics related to attendance at conferences, seminars and inservice programs and the impact that each is perceived to have on classroom practice.

TABLE 1

Attendance at professional meetings 1990-1991

Type	Attending	Impact		
		none	little	significant
	%	%	%	%
Conference	23	32	41	6
Seminar	49	17	58	12
Inservice	62	9	60	27

It can be seen that conferences do not play a major role in the professional life of mathematics teachers. Only 23% had attended a conference in the period 1990-1991.

Fifty-one percent stated that they never go to a conference and another 34% go to less than one a year with only 13% indicating that they attend one or more each year. Moreover in excess of 70% of respondents stated that conferences had little or no effect on their classroom practice.

Though seminar participation is better than that for conferences their effect on classroom practice is not significant. Here 75% indicated that seminars had little or no impact.

While 62% had attended an inservice program during 1990-1991 46% stated that they would not normally attend such a program every year. However 52% attended at least one inservice activity each year. Though attendance at inservice activities is much higher than for conferences and seminars, only 27% suggested that the impact of these programs in the mathematics classroom is significant and 69% claimed that these have little or no impact.

When indicating their major source of new information, 69% said other teachers, 13% gave inservice programs, 5% stated journals, 3% each for conferences and textbooks and 1% stated the Education Department. These figures confirm the finding of research quoted earlier (Jeffery, 1984) namely that mathematics teachers **believe** that their colleagues are the major source for new professional information. The only other source of new information that these teachers thought significant was from inservice programs.

DISCUSSION

If the initial hypothesis had been that Queensland mathematics teachers do undertake professional activities in order to keep up-to-date with current developments in high school mathematics teaching, then this hypothesis should, in all probability, be rejected. However the stated purpose of this investigation was to determine the methods used by mathematics teachers to gain new curriculum knowledge. It appears that many teachers see their main source of new information as being other teachers. Perhaps this is really a "Catch 22" situation. Few of the teachers surveyed are participating in professional development activities or formal study to keep up-to-date in their field. However, presumably many of these teachers who are not undertaking such activities are the source of curriculum information for other teachers!

Many professions use their journals as a means of communicating new and interesting information and as a medium for generating discussion. If this is true for other professions why doesn't it happen for mathematics teachers? Research carried out by Swinson (1993) and others suggests that if mathematics teachers are to read these journals then school time must be allocated for their reading. Furthermore there is a better chance that teachers will read journals if the school supplies them.

Mathematics teachers in Queensland do not appear to be in the habit of undertaking professional development activities. However if the changes required by the new syllabuses are to take place, teachers need to be encouraged to undertake such activities. Conference and seminar attendance are accepted forms of professional development, yet most Queensland mathematics teachers do not undertake these activities. The employing authorities need to look for ways to encourage teachers to attend, and the obvious method is to give the teachers leave, hopefully with pay, so that they can participate. In Victoria where teachers are given paid leave they do attend mathematics conferences.

The fact that inservice programs had a higher participation rate than the other activities is not surprising. Many of these programs are developed and conducted by the employing bodies. However the higher participation rate should not cloud the fact that many of the teachers claim that these activities had little or no impact on their classroom practice.

Finally both the Education Department and the Queensland Association of Mathematics Teachers need to think seriously about their role in getting information to teachers. It would appear that teachers see these two organisations as of little use as a source of curriculum information.

How can mathematics teachers in Queensland be encouraged to undertake professional development? This is an area that offers a rich field for future research.

REFERENCES

Bell, P. E. (1982) Teachers' expertise in subject matter: A catch 22. paper presented at the Annual Conference of the National Staff Development Council, Detroit, MI

Davis, B. & Butler, C. E. (1986) Special education journal reading patterns of senior level undergraduate education students at the University of Oklahoma, 1973-1982, Journal of Research in Science Teaching, 22, pp. 485-490

Dynan, M. B. & Treagust, D. F. (1985) Are conferences worthwile activities? Views of Conasta '84 participants. Australian Science Teachers Journal ,31, 2, pp. 41-44.

Finger, J. (1977) Who reads educational periodicals? Education News, 16, pp. 12-17.

Hamilton, J. B. & McElroy, J. (1983). Approaches to Technological update of vocational/technical teachers. Paper presented at the Annual American Vocational Association Convention. Anahiem, CA.

Hurd, P. D. (1981) The Status of Middle School and Junior High School Science. Louisville, Center for Educational Research and Evaluation.

Jeffrey, P. (1984) Research confirms school librarian's important role in dissemination to teachers. Journal of The School Library Association of Queensland, 16, 2, pp. 5-6.

Owen, J. M. , Hall, K. C. & Malcolm, C. K. (1981) How do teachers gather new educational knowledge. An Interim Report. Tertiary Education Research Unit Melbourne State College.

Owen, J. M. , Malcolm, C. K. & Hall, K. C. (1982) Alternative forms of communicated knowledge and their effects on Australian teachers. Melbourne State College, Tertiary Education Research Unit.

Pearce, J. (1984) Sharing the word: Reading for those who haven't the time. Paper presented at the Annual Conference of the California Reading Association. Oakland, CA.

Pryke, R. R. (1987) A preliminary study of the level of professional reading of a sample of New South Wales high school teachers. Unpublished Masters Project. Macquarie University.

Romberg, T. A. (1989) Can teachers be professionals? In D. A. Grouws, T. J. Cooney & D. Jones (Eds.) , Perspectives on Research on Effective Teaching. Reston Va: The National Council of Teachers of Mathematics.Schroeder, T. & Frame, M. L. (1986) The preparation and continuing education of mathematics teachers in Alberta: A status survey and needs assessment. (Final Report) Prepared for The Mathematics Council of the Alberta Teachers' Association.

Swinson, K. V. (In press) Professional journals: Do mathematics teachers read them? South Pacific Journal of Teacher Education.

DEVELOPMENT OF A MODEL TO ENHANCE MANAGERIAL STRATEGIES IN PROBLEM SOLVING

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The aim of this study was to investigate children's perseverance when solving difficult or unfamiliar number problems. It was concerned with those students who are referred to as 'perseverers' because they reached a stage in their problem solution where they recognised that they had not reached a satisfactory answer and decided to take some action - start again, modify their strategies or change to different strategies rather than giving up immediately.

The sample consisted of ten boys and ten girls in grade 6 and ten boys and ten girls in grade 10. The tasks consisted of number problems of varying difficulty. Data gathering took the form of clinical unstructured interviews with individual students in which they were asked to verbalise concurrently with solving a set of number problems.

Task analysis maps were used to provide overviews of the interview protocols. From observation of the maps of students who were ultimately successful, it became apparent that these children were more inclined than others to be flexible in their use of strategies. A model was developed which described the sequence of strategies used most consistently by successful students. This model formed the basis for a small-scale training programme to investigate whether the strategies could be taught. A descriptive analysis suggested that most of these children were able to be trained to use the model independently.

This paper addresses the difficulty experienced by many students with problem solving and the ease with which many give up an attempted solution when the answer is not immediately apparent. More specifically, it explores the question of managing problem solving strategies, an aspect to which comparatively little research has been directed. It identifies managerial strategies used by successful problem solvers and gives classroom teachers some direction for helping their students to enhance their own strategies. The problems used in the research are structured number problems which require some productive thinking, as opposed to either structured 'word problems' requiring just the recognition and use of appropriate rules, or open-ended problems. Before describing the research (Taplin, 1992), consideration will be given to other literature which has addressed this issue and explored why it continues to be a 'persistent and recurring problem in the practice of mathematics teaching'.

SYNTHESIS OF RESEARCH EVIDENCE

Although there has been considerable emphasis on the importance of problem solving in the mathematics curriculum (Stanic and Kilpatrick, 1989; NCTM 1980, 1989) we are in danger of not developing it to its maximum potential because not enough is known about how people best acquire problem solving skills or how they can best be taught them. There is still much to learn about how problem solvers can best develop and use appropriate processes and strategies. In particular there is a growing need to understand more about the nature of the problem solving process and specifically factors which interfere with its development. This has been approached from two main directions. First there are writers such as Polya (1957) who offer practical procedures and strategies to increase success in the problem solving process. Also, there are writers such as Newell and Simon (1972) who have engaged in scientific research investigations of the effectiveness of these procedures and strategies, trying to tease out underlying mechanics and psychological variables from which they can make generalisations. There are also some writers whose work appears in publications of classroom practice as well as in research journals. Their writing often intersects with each of the two groups described formerly, yet is separate in some respects from both. For example, the work of Schoenfeld (1985a, 1985b) which will be described in this section examines some of the research findings and suggests reasons why they cannot always be applied successfully in learning contexts. The points of view of some writers representing each of these groups will be described here.

Difficulties with Problem Solving

Although a great deal has been written and debated about problem solving, it is still clear that there are difficulties associated with teaching people how to succeed at it (Bransford, Hasselbring, Barron, Kulewicz, Littlefield and Goin, 1989; Kowplowitz, 1982; Cockcroft, 1982). For example, Bransford et. al. (1989) document evidence of students employing sophisticated task-avoidance strategies in order to prevent facing difficult or unfamiliar problems. This section will address some of the difficulties which have been described by both those who are teachers of problem solving procedures and strategies and those who are researchers. It will also consider responses to the research which have been made by writers who can be described as fitting into both categories.

Difficulties Associated with Practical Procedures and Strategies

Polya is probably the best known of the former of these groups, the practical teachers of problem solving. In his well-known discussion of problem solving strategies, How to Solve It, Polya (1957) stresses the importance of having a plan when tackling a problem. He suggests that one of the difficulties with problem solving is that even if a plan is instituted, students often fail to check its implementation during the problem solving process. He also believes that failure to check the 'reasonableness' of the answer is another common problem. 'The student is glad to get an answer, throws down his pencil and is not shocked by the most unlikely results' (p.95). Polya also offers some insight into why so many people find it hard to persevere in problem solving: 'it is easy to keep on going when we think the solution is just around the corner, but it is hard to persevere when we do not see any way out of the problem' (p.93). Not only is this often the case, but as Kowplowitz (1982) says, many students fail in the problem solving process because they do not have a sense of knowing when they have reached a correct solution and often stop when they think they have, although their solution may be inappropriate.

A further hindrance to successful problem solving is an unwillingness or inability to understand the problem and translate it into the appropriate mathematical terms (Polya, 1957; Cockcroft, 1982). In fact Polya describes this as the most common problem. It was pointed out earlier that he believes that many students either rush into an attempted solution without any overall plan, or 'wait clumsily for an idea to come and cannot do anything that would accelerate its coming' (p.95). Bransford et. al. (1989) and Sowder (1989) also suggest that many students rush into trying to solve a problem before they are really aware of what the problem requires them to do or what operations are involved. Lester (1985) takes this point further, saying that children very often rush in and do what they think the cues in a problem are telling them to do, rather than taking time to reflect on what is actually required. He uses the example of the following problem.

> 'Tom and Sue visited a farm and noticed there were chickens and pigs. Tom said, "There are 18 animals." Sue said, "Yes, and they have 52 legs in all." How many of each kind of animal were there?' (Lester, 1985, p.41).

Many of the children who attempt this problem interpret the phrase 'in all' as a cue to add, which they do without giving any further consideration to other cues.

Another possible block to successful problem solving which is often commented upon by this group of writers, and which may well be attributed to the traditional preoccupation with the outcome rather than the process, is an unwillingness by many students to regard themselves as the 'owners' of the problems they are attempting. Rather than accepting responsibility for the problems themselves, there is a tendency for students to think it is common to accept procedures at face value and not try to understand why they work.

Research and Discussion by Other Writers

Amongst the researchers, one of the lines which has been followed in the attempt to contribute to an understanding of the problem solving process is the comparison of the problem solving strategies of experts and novices in the hope that the procedures, or heuristics, used by the experts can be taught to the novices. Newell and Simon (1972), Greeno (1976, 1980), Simon and Simon (1978), Larkin, McDermott, Simon and Simon (1980a, 1980b) Chi, Feltovich and Glaser (1981), Chi, Glaser and Rees (1982), and Schoenfeld (1985a) are some who have researched in this area. However, there is little evidence to suggest that teaching the use of heuristics has effectively enhanced problem solving performance in the classroom (Schoenfeld, 1985a).

This 'novice/expert' research on heuristics has been explored extensively by many writers in an attempt to clarify understanding of why the application of heuristics does not immediately make problem solving performance more effective. One possible explanation that has been advanced is that because novice problem solvers have their own innate systems of 'raw' heuristics (Silver, 1985), the use of these might actually inhibit the development of expert heuristics (Owen and Sweller, 1985). It may also be that the 'expert-novice' studies focus on the end-point, the level of heuristics at which the experts have arrived, rather than accounting for the developmental stages which must be traversed in the transition from 'novice' to 'expert' (Silver, 1985). If this is the case, then teaching the experts' heuristics may be akin to the traditional model of teaching mathematical rules - the problem solver is expected to use the strategies derived by the experts with no insight into how they were derived. A further likelihood is that too much emphasis may have been placed on the actual heuristics and not enough on how to manage them. Schoenfeld (1985a) emphasises this point when he suggests that the knowledge of heuristic strategies alone does not lead to problem solving success, without an understanding of which ones should be used for particular situations. He indicates that the problem solver must be able to develop a structure for knowing when it is appropriate to use a particular heuristic and how to recover from making a wrong choice.

Marshall (1989) is another who believes that problem solvers not only need to be equipped with the right 'tools', but more importantly they also need to 'call upon the knowledge and skills in a nonpredetermined order to make sense of a new experience' (p.161). This will not usually happen by itself. Pupils need to be provided with the right experiences. Resnick (1989) believes that the way we structure our choice of problems can inhibit the development of these managerial skills if they are designed in such a way as to enable students to practise particular rules rather than in such a way as to encourage them to find their own.

It appears, then, that difficulties may arise, at least in part, because educators are not acknowledging that successful problem solving is in fact influenced by a combination of a number of things. Schoenfeld (1985a) suggests four factors which can interact to affect problem solving performance. These are the problem solver's mathematical knowledge, knowledge of heuristics, affective factors which influence the way the individual views problem solving and the managerial skills associated with selecting and implementing appropriate strategies. An examination of some of the research which has been done in the latter two categories will indicate the growing recognition of their importance.

Affective Factors

A set of factors mentioned above by Schoenfeld which has been discussed fairly recently are affective variables. These encompass all of those individual difference variables that have to do with affect, or feelings. They include constructs such as self-esteem, achievement motivation, anxiety, and depression, the effects of which mediate achievement in a number of ways.

To date, there has been comparatively little research exploring the role of affective factors in the problem solving process. Much of what has been done has been limited to exploring correlations between problem solving performance and attitudes, such as motivation, interest, confidence, perseverance, and risk-taking (Lester and Garafolo, 1987). Other work has been of a more speculative nature, based on the writers' observations and experiences and intended as a means of pointing out the directions in which this research should go. An example of this is a series of scenarios presented by Lester and Garafolo (1987) which suggest the 'sometimes dominant influence non-cognitive factors can have on problem solving performance' (p.10). Silver and Kilpatrick (1989) believe that it is time to develop and evaluate schemes of instruction for improving affective factors. It seems likely that 'attempts to influence a range of affective variables should have a reasonably powerful effect on problem-solving ability' (McLeod, 1985, p.276).

It has already been suggested that affective factors can assist or interfere with problem solving. For example, inability to handle emotional reactions to problem solving can hinder the search for a solution: 'students have difficulty persisting in problem solving if their reaction is intense and negative, so they tend to quit and reduce the magnitude of the emotion' (McLeod, 1988, p.134). The implications of this can be linked to Polya's (1957) statement that an environment which denies the student the opportunity to experience the range of emotions associated with problem solving is failing to contribute to a vital aspect of mathematics education. One of these emotions is related to a lack of courage to tackle a potentially difficult problem rather than taking the easiest path of stopping or asking for help (Wertime, 1982; McLeod, 1988). In fact, Wertime suggests that many children become reluctant problem solvers, to the extent that 'they would rather enjoy the consolation purchased by despair than endure the fruitful stress of confronting the [problem solving] process' (p.192). This reluctance is also mentioned by Lester (1985) and in Schoenfeld's (1985b) description of the common belief that 'mathematics problems are always solved in less than 10 minutes, if they are solved at all' (p.372). It seems true that many problem solvers give up rather than face negative emotions because they expect solutions to come to them quickly and easily (Scott, 1988). Schoenfeld (1985a) also states that many 'potentially valuable approaches are abandoned before they can bear fruit' (p.98) because students are not aware of when it is worthwhile to keep on exploring an idea and when it is appropriate to abandon it because it is leading in a wrong direction.

Also linked to affective variables are the belief patterns described by Schoenfeld (1985a) discussed earlier in this paper. One of these is the belief that not only is it important to get the right answer, but that 'getting an answer in the right way is what counts' (Thompson, 1989, p.235). For example, Silver (1985) suggests that many children approach a problem locked into the belief that they must do it in a particular way, that there is always a 'rule' to be followed and that any other approach is 'wrong' (Silver, 1985). This kind of belief pattern can be very inhibiting to a child who is finding it difficult to begin a solution or continue after encountering difficulties.

It seems that at least part of the challenge of problem solving instruction is to help problem solvers to break away from these belief patterns by developing an awareness of when it is appropriate to use a particular strategy, when it is worthwhile to persevere and when it is not. It is therefore necessary to consider the managerial factors which link together knowledge, heuristics and affective variables.

Management Factors

As with affective factors, the importance of understanding managerial factors has only been acknowledged recently. To date little research has been reported in this area, although some writers advocate the need for it. Lester (1985), for example, claims that there has been much emphasis in research on the 'discrete skills and procedures'(p.43) of problem solving, but little on the 'managerial aspects ... which serve as "guiding forces". He advocates the need for this to be done through qualitative as well as quantitative methods in order to obtain clear insights into what these variables are and how they can be influenced. One of these managerial aspects is the previously mentioned sense of when it is appropriate to use a particular strategy. Linked to this, and also closely linked to the emotional factors described above, is the knowledge of how to persevere when the problem solving process becomes difficult. Unless problem solvers have the desire to persevere, knowledge of heuristics and planning can become redundant because the will to use them does not exist. However, it does not appear to be enough to just help students to understand that frustration, for example, is a normal part of problem solving (McLeod, 1988), and to encourage them to spend time working on the task. They also need to know how to make '[managerial] decisions about whether to persevere along a possible solution path' (McLeod, 1988, p.138). It is desirable to encourage perseverance, but on the other hand it is possible to 'overpersevere', particularly if one becomes locked into one approach. 'Overpersevering' is an unproductive strategy to employ when it may be more appropriate when stuck to use other strategies, such as help-seeking (Nelson-Le Gall and Scott Jones, 1983). This poses not only the question of how to enhance perseverance, but as well the question of how to avoid time being wasted on 'overperseverance'. Since this appears to be an important question within the context of managerial strategies for problem solving, it is this which will become the main focus of the research described here.

THE FOCUS OF THIS STUDY

This study was directed towards exploring how the particular managerial factors associated with perseverance could contribute to problem solving success, since comparatively little is known about perseverance. It was decided to address an aspect of perseverance in solving problems by examining the ways in which persevering students chose to pursue possible solution paths - not only what strategies they used, but how they managed them. The focus was not on those students who succeeded with a problem immediately, or on those who gave up quickly. Rather, it was on those who did not arrive at a quick solution but were prepared to engage with the task for some time until they either succeeded or chose to give up. The study was not, therefore, concerned with what initially motivated this engagement with a problem. It was intended to compare those persevering students who are ultimately successful with those who give up after persevering for some time, in an attempt to investigate whether, under the specific conditions of the study, there were any different patterns in the use of strategies by those who are successful.

The initial sample consisted of ten boys and ten girls, aged 12 years, in grade 6, their final year of primary school. As well, a sample of ten boys and ten girls, aged 16, studying the top academic level of mathematics in grade 10 was interviewed in order to obtain strategy data on a group of subjects who successfully solved the problems.

The tasks consisted of ten number problems of varying difficulty, all requiring a degree of productive thinking rather than just the recognition and application of an appropriate rule. Care was taken to ensure that subjects were familiar with any necessary knowledge required for successful solution and a calculator was made available to reduce the burden of computation.

Data gathering took the form of interviews with individual subjects in which they were asked to verbalise concurrently with solving the set of problems. Where necessary, they were also asked to reflect on what they had done retrospectively to fill in gaps in the interviewer's understanding of what took place.

Subjects were invited to continue with a problem for as long as they liked and to move backwards and forwards between problems at any stage. Most interviews took place over a series of sessions, each of approximately one hour's duration. As seeking help is regarded as a legitimate strategy for approaching a problem, specific assistance was given when requested. Interviews were recorded and later transcribed for analysis.

Task analysis maps developed by Chick, Watson and Collis (1987) and modified by Collis and Watson (1991) were used to provide overviews of the interview transcripts. These recorded information about the way in which the initial data provided in the problem were used, the concepts, processes and strategies used in attempting to solve the problem and the structure of responses, both correct and incorrect, made at various stages in the solution of the problem. Four stages in the problem solving process were considered. These are shown with the symbols used for their representation:

(i) ▲ the use of cues. This includes cues from the problem which indicated what the subject was required to find. It also includes data which were given or which the subject needed to imply in order to attempt a response,

(ii) ● concepts and/or processes used by the subject to operate on selected data,

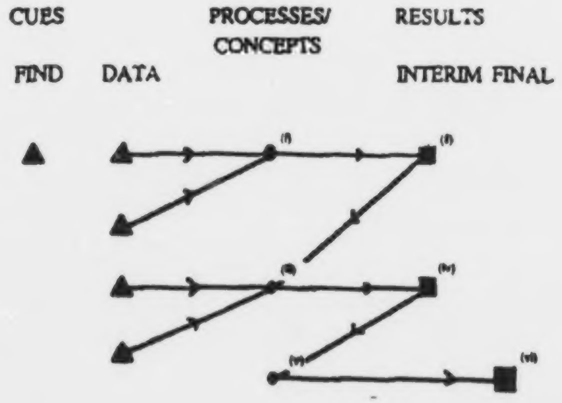
(iii) ■ interim responses which were made as the result of applying a particular process. The symbol] was added if the interim response led to a 'dead end', requiring the subject to return to the cues and start again,

(iv) the final response made before the subject either accepted the (correct or incorrect) answer or abandoned the question. Three final response symbols were used:

■ correct solution
□ incorrect solution with which subject was satisfied,
]] abandoned question.

Figure 1 gives an indication of how a map can be interpreted. The student selects data to which a process (i) is applied. This leads to an interim result (ii). More data are needed for the next piece of processing to occur (iii). This leads to a further interim result (iv). The student then checks this result against the initial cues (v) and is satisfied with a correct solution (vi). Lines were used to indicate single or groups of cues used for an interim response. The process or concept used to obtain this response was noted. Because the maps for the actual problems solutions were complex and lengthy, no examples have been included here, but all followed this basic format. One example of such a map has been included in Appendix 1.

Figure 1: Simple example of a task analysis map



From observation of the maps of subjects who were ultimately successful, it became apparent that these children were more inclined than others to be flexible in their use of strategies. In order to investigate this phenomenon, analysis was made of strategies which subjects employed after having reached a "dead end", that is explored a particular strategy until they could go no further with it.

A descriptive analysis using the constant comparative method (Glaser and Strauss, 1967) indicated nine problem solving strategies within two categories:

(i) repeating the same approach as in the previous strategy and:

  (a) using the same set of data (cues) but bringing in additional cues in order to complete the data required for successful solution,

  (b) using the same data but introducing more cues than previously, as in (a), but not having the complete set of data required for successful solution,

  (c) using exactly the same data, including instances where subjects altered numbers but still used the same ideas,

  (d) using a different set of data,

(ii) changing to a different approach and:

  (a) using the same data but bringing in additional cues in order to complete the data required for successful solution,

  (b) using the same data and introducing more cues than previously, as in (a), but not having the complete set of data required for successful solution,

  (c) using exactly the same data,

  (d) changing to a different set of data,

  (e) using a different set of data which involved returning to an approach which had been used and abandoned previously.

At this stage of the investigation it was considered that sufficient information about strategy use could be obtained by refining the nine strategies into four categories. This refinement is consistent with the model of qualitative analysis discussed by Glaser and Strauss (1967):

(i) using the same approach as in the previous attempt and using the same set of data (cues) ,
(ii) using the same approach as previously but a different set of data,
(iii) changing to a different approach from that used in the previous attempt and using the same set of data,
(iv) changing to a different approach and using a different set of data.

Maps illustrating these four strategies are included in Appendix 2. Descriptive analyses suggested that the most useful strategy used by successful "perseverers" was the "different approach/same data" one, which was often the final strategy which led to success. The "different approach/different data" strategy was also used, but to a lesser degree, by many of the successful students. The "perseverers" who eventually gave up were more inclined to use the "same approach/same data" strategy repeatedly. A closer examination was made of the way the "different approach/same data" strategy was used in relation to the others, to investigate whether successful problem solvers used it systematically and whether it was used at a crucial stage in the problem-solving process, presumably near or at the end.

Observation of the patterns of strategy use revealed that a commonly used sequence of strategies was that presented in Figure 2. Step (ii), refinement, was included because successful students commonly repeated the "same approach/same data" strategy approximately three times to refine an approach before trying something different.

Figure 2: Strategy sequence commonly used
by successful problem solvers

(i) select first strategy
(ii) (if necessary) refine strategy, using
'same approach/same data
strategy 1-3 times
(iii) try a different approach
('different approach/same data'or 'different
approach/different data')
(iv) repeat from (i) if necessary

IMPLICATIONS FOR THE CLASSROOM

The next stage of the investigation was to explore whether the quality of children's perseverance time could be enhanced by training them to use this model. A small scale training programme was conducted with six 13-year-old grade 7 children. These children worked individually in a clinical interview situation with the experimenter on a series of four of the number problems used in the earlier part of the study. On the first problem none of the children could use the model without prompting. However, all were able to use it independently by the third or fourth problem. These patterns in the development of subjects' use of the model suggest that it is feasible to "train" children to use it. Clearly a long-term study would be required to determine the effects of such a training programme on children's ability to use the model to increase successful perseverance over a period of time.

Currently the model is being trialled in group and individual situations with groups ranging from primary to tertiary. Analysis is indicating that it is regarded by those who have used it as a useful tool. The procedure which is followed for implementing the model is indicated in Figure 3.

Figure 3: Recommended procedure for introducing
problem solving model

(i) Give the student a preliminary problem to solve without guidance. Observe whether the student instinctively used the model.
(ii) Introduce the student to the model and demonstrate using the example provided.
(iii) Ask the student to repeat the first problem while you guide him/her to use the model, i.e. prompt the student to change to a different approach after a maximum of three repetitions of the previous approach
(iv) Give two more problems, monitoring the strategy pattern and reminding students, when necessary, to follow the model.
(v) Give a fourth problem and ask the student to try to follow the model, changing approach when appropriate, without any prompting from you.

The presence of the teacher is essential in the early stages of introducing the model, as intervention at the appropriate time to sugest changing to a new strategy is crucial. This can be done on a small group basis. It is also important for the teacher to be equipped with a range of different approaches to the problem solution, as in the early stages problem solvers who are willing to switch approaches are not necessarily able to think of alternatives. Interestingly, as they become more confident in using the model they also seem to become more confident and adept at finding their own alternative strategies. The teacher must be wary of telling the student how to solve the problem, but rather suggest when it might be appropriate to change approach and, if necessary, provide hints that will lead to the selection of a new approach. This gives the teacher a more positive role in offering encouragement, rather than simply telling the student to 'try again'. As problem solvers begin to use the model more automatically, teacher intervention becomes less necessary. At this stage it can be useful to have pairs of students working together to solve the problem, which encourages them to decide together when they should try something different or pursue an idea for a bit longer.

More time is needed to fully explore the long term effects of the problem solving model in the classroom setting. However there is already some evidence to suggest that it can be incorporated easily and effectively into the problem solving programme in the manner outlined above .

REFERENCES

Bransford, John, Hasselbring, Ted, Barron, Brigid, Kulewicz, Stan, Littlefield, Joan and Goin, Laura (1989). Uses of macro-contexts to facilitate mathematical thinking; in Charles, Randall I. and Silver, Edward A. (Eds), The Teaching and Assessing of Mathematical Problem Solving, National Council of Teachers of Mathematics, USA, pp.125-147.

Carpenter, Thomas P. (1989). Teaching as problem solving; in Charles, Randall I. and Silver, Edward A. (Eds), The Teaching and Assessing of Mathematical Problem Solving, National Council of Teachers of Mathematics, USA, pp.187-202

Chick, Helen L., Watson, Jane M. and Collis, Kevin F. (1987). Using the SOLO taxonomy for error analysis in mathematics. Paper presented at the Mathematics Research Group of Australasia Conference, Townsville.

Cockcroft, W.H. (Ed.) (1982). Mathematics Counts; Report of the Committee of Inquiry into the Teaching of Mathematics in Schools, Her Majesty's Stationery Office, London.

Collis, Kevin F. and Watson, Jane M. (1991). A mapping procedure for analysing responses in mathematics; Journal of Structural Learning., 11,1,65-87

Glaser, Barney G. and Strauss, Anselm L. (1967). The Discovery of Grounded Theory: Strategies for Qualitative Research, Chicago, Aldine.

Greeno, J.G. (1976). Indefinite goals on well-structured problems, Psychological Review, 83,6,pp.479-491.

Greeno, J.G. (1980). Trends in the theory of knowledge for problem solving; in Tuma, D.T. and Reif, F. (Eds.), Problem Solving and Education: Issues in Teaching and Research, Lawrence Erlbaum, N.J.

Klowpowitz, Herb (1982). The feeling of knowing when one has solved a problem, in Lockhead, Jack & Clement, John (Eds), Cognitive Process Instruction, Franklin Institute Press, Philadelphia, 305-7.

Larkin, J., McDermott, J., Simon, D. and Simon, H. (1980a). Expert and novice performances in solving physics problems, Science,208, pp.1335-1342.

Larkin, J., McDermott, J., Simon, D. and Simon, H. (1980b).
Models of competence in solving physics problems, Cognitive Science, 4, pp.317-345.

Lester, Frank K. (1985). Methodological considerations in research on mathematical problem-solving instruction; in Silver, E.A. (Ed.), Teaching and Learning Mathematical Problem Solving: Multiple Research Perspectives, Lawrence Erlbaum, Hillsdale, N.J., pp.41-69.

Lester, Frank K. Jr. and Garofalo, Joe (1987). The influence of affects, beliefs and metacognition on problem solving behaviour: some tentative speculations; Eric Document Reproduction Service No. ED 281758.

McLeod, Douglas B. (1985). Affective issues in research on teaching mathematical problem solving; in Silver, E.A. (Ed.), Teaching and Learning Mathematical Problem Solving: Multiple Research Perspectives, Lawrence Erlbaum, Hillsdale, N.J., pp.267-279.

McLeod, Douglas B. (1988). Affective issues in mathematical problem solving: some theoretical considerations; Journal for Research in Mathematics Education; 19, 2, pp.134-141.

Marshall, Sandra P. (1989). Assessing problem solving: a short-term remedy and a long-term solution; in Charles, Randall I. and Silver, Edward A. (Eds), The Teaching and Assessing of Mathematical Problem Solving, National Council of Teachers of Mathematics, USA, pp.159-177.

Mayer, Richard E. (1985). Implications of cognitive psychology for instruction in mathematical problem solving; in Silver, E.A. (Ed.), Teaching and Learning Mathematical Problem Solving: Multiple Research Perspectives, Lawrence Erlbaum, Hillsdale, N.J., pp.123-138.

National Council of Teachers of Mathematics (1980). An Agenda for Action: Recommendations for School Mathematics of the 1980s, NCTM, USA.

NCTM (National Council of Teachers of Mathematics Commission on Standards for School Mathematics, 1989). Curriculum and Evaluation Standards for School Mathematics, U.S.A.

Nelson-Le Gall, Sharon and Scott-Jones, Diane (1983). Teachers' and young children's perceptions of task persistence. Paper presented at the Annual Meeting of the American Educational Research Association, Montreal.

Newell, A. and Simon, H.A. (1972). Human Problem Solving, Prentice-Hall, USA.

Owen, Elizabeth and Sweller, John (1985). What do students learn while solving mathematics problems? Journal of Educational Psychology, 77,3,272-282.

Polya, G. (1957). How to Solve it (2nd Ed.), Princeton University Press, N.J.

Resnick, Lauren B. (1989), Treating mathematics as an ill-structured discipline; in Charles, Randall I. and Silver, Edward A. (Eds), The Teaching and Assessing of Mathematical Problem Solving, National Council of Teachers of Mathematics, USA, pp.32-60.

Schoenfeld, Alan H. (1985a). Mathematical Problem Solving, Academic Press, London, 1985.

Schoenfeld, Alan H. (1985b). Metacognitive and epistemological issues in mathematical understanding; in Silver, E.A. (Ed.), Teaching and Learning Mathematical Problem Solving: Multiple Research Perspectives, Lawrence Erlbaum, Hillsdale, N.J., pp.361-379.

Scott, Mary E. (1988). Learning strategies can help; Teaching Exceptional Children, Spring, 30-34.

Silver, E.A. (1985). Research on teaching mathematical problem solving: some underrepresented themes and needed directions; in Silver, E.A. (Ed.), Teaching and Learning Mathematical Problem Solving: Multiple Research Perspectives, Lawrence Erlbaum, Hillsdale, N.J., pp.247-266.

Silver, Edward A. and Kilpatrick, Jeremy (1989). Testing mathematical problem solving; in Charles, Randall I. and Silver, Edward A. (Eds), The Teaching and Assessing of Mathematical Problem Solving, National Council of Teachers of Mathematics, USA, pp.178-186.

Simon, D.P. and Simon, H.A. (1978). Individual differences in solving physics problems; in Siegler, R.S. (Ed.), Children's Thinking: What Develops?, Lawrence Erlbaum, N.J.

Sowder, Larry K. (1985). Cognitive psychology and mathematical problem solving: a discussion of Mayer's paper; in Silver, E.A. (Ed.), Teaching and Learning Mathematical Problem Solving: Multiple Research Perspectives, Lawrence Erlbaum, Hillsdale, N.J., pp.139-146.

Stanic, George M.A. and Kilpatrick, Jeremy (1989). Historical perspectives on problem solving in the mathematics curriculum; in Charles, Randall I. and Silver, Edward A. (Eds), The Teaching and Assessing of Mathematical Problem Solving, National Council of Teachers of Mathematics, USA, pp.1-22.

Taplin, Margaret L. (1992). An Investigation of Perseverance in Number Problem Solving, Unpublished Ph.D. Thesis, University of Tasmania.

Thompson, Alba B. (1989). Learning to teach mathematical problem solving: changes in teachers' conceptions and beliefs; in Charles, Randall I. and Silver, Edward A. (Eds), The Teaching and Assessing of Mathematical Problem Solving, National Council of Teachers of Mathematics, USA, pp.232-243.

Wertime, Richard (1982). Students' problems and "courage spans", in Lockhead, Jack and Clement, John (Eds.) Cognitive Process Instruction, Philadelphia, Franklin Institute Press.

CONCEPTS AND CONTEXTS IN LEARNING MATHEMATICS

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Modern primary mathematics curricula explicitly aim to develop children's understanding of basic mathematical concepts, in contrast to the more traditional rote learning of procedures. However there has been continuing concern that many children still do not achieve an acceptable level of understanding of these mathematical concepts and most seem to have difficulty generalising their understanding to relevant contexts.

An investigation of the mathematical understanding of twelve students during their transition from primary to secondary school suggests that there may be little educational value in basing curricula on a presumed common learning hierarchy of concepts within mathematics which can be transferred to the learners' minds through a controlled sequence of learning experiences. The conceptual understanding that students were able to demonstrate (though quite limited) revealed that their mathematical conceptualisations often consisted of unique interpretations associated with personally perceived contexts, rather than common and clearly recognisable generalisations.

For many years the development and modification of primary mathematics curricula and teaching methodologies have been explicitly aimed at facilitating children's understanding of basic mathematical concepts (Ellerton & Clements 1988). In Britain, the Cockcroft Report (Cockcroft 1982), which has also had wide influence in Australia, particularly drew attention to the need for a detailed and careful approach to teaching mathematics in the early years "in order that children may develop confidence and understanding" (p83).

In much of the related documentation, the term concept is used in such a way as to imply that there are identifiable entities within mathematics itself which can be transferred to the learner's mind in a fixed learning sequence appropriate to various stages of development. If properly taught and understood these concepts then are presumed to become accessible to the learner as part of a total mental mathematical construct, the characteristics of which should be similar among successful learners of mathematics.

> The child's level of understanding of a concept depends upon the quality, extent and connectedness of verbal (written and oral), concrete, pictorial and symbolic forms of representation.(Western Australian Ministry of Education 1989 p23)

Clearly the intention has been to emphasise the teaching of mathematics in a meaningful way with the child actively making sense of the ideas presented in contrast to the more traditional approach where rote learning of procedures constituted a major focus of classroom practice. However, the frustration for many teachers who seriously attempt to teach conceptual understanding is that children often appear to have great difficulty, not only in developing an understanding of these concepts initially, but in maintaining their understanding during later learning stages and, more particularly, in generalising the concepts to what seem obviously related applications and contexts (Hart 1989; Swan 1990).

The consistency of these difficulties in learning mathematics suggests the need to question the assumption that it is possible to identify a hierarchy of 'concepts' within the discipline which is appropriate to the learning sequence of all individuals. Indeed Ernest (1991) and Sierpienska (1990) argue strongly against this notion and support the proposition that conceptual understanding in mathematics arises out of the personalised and idiosyncratic sense and meaning the learners assign to the contexts of use to which they are exposed, suggesting that a well defined common learning route through sequentially ordered mathematical 'concepts' of increasing difficulty cannot be pre-determined. Kameenui and Griffin (1989) also question the educational appropriateness of focusing on mathematical 'concepts' rather than the unique contextual associations they believe each individual student intuitively develops during early learning experiences.

Clarke's (1989) findings provide some research support for these ideas. His intensive study of the mathematical behaviour of ten students during their transition from primary to secondary school revealed that the mathematical behaviour of the students was highly idiosyncratic and clearly involved complex interactions between students and their perceptions of the learning environment as well as important social factors. All students were in the same classroom during Year Seven and Year Eight and therefore were exposed to the same mathematical experiences at school, and yet each had developed a unique perspective and exhibited widely differing characteristics in their approach to understanding mathematics. Clarke noted that any explanation of these differences must accommodate "the essential individuality of the learning process" (Clarke 1989 p335).

This paper further explores the individuality of students' mathematical conceptualisations with a particular focus on students' ability to contextualise the basic number 'concepts' introduced early in primary school and its relationship to understanding later more abstract ideas. Interview data obtained from students at the end of their primary schooling have been examined and aspects of the students' conceptualisations of number operations are reported and discussed. The implications for primary curricula design are also considered.

SOURCE OF THE DATA

The data reported in this paper were obtained during interviews carried out as part of an investigation into the mathematical understanding and attitudes of students during their transition from primary to secondary schooling (Tomazos 1991). At the end of 1990 all 216 Year Seven students at nine metropolitan primary schools in Western Australia were surveyed using a written questionnaire to determine their attitudes to the main subject areas. From the data obtained, a number of students were identified as having exhibited very strong feelings, both negative and positive, specifically towards mathematics. Of these students six girls and six boys were chosen, with three of each gender showing very positive responses and three of each gender showing very negative responses to mathematics.

The twelve students were interviewed for approximately forty minutes on two occasions, once at the end of Year Seven primary school and once midway through Year Eight, their first year of secondary schooling. Students were questioned about their perceptions of and attitudes towards mathematics and their responses were sought (using a clinical interview approach) to a range of mathematical items designed to examine the students' competence with and understanding of fundamental number operations and relationships. The purpose of the original study was to investigate possible relationships among students' gender, their perceptions and understanding of mathematics,and their affective responses to it.

Because the research focus had been on finding similarities in mathematical behaviours, much interesting data which clearly supported Clarke's (1989) findings that students' mathematical behaviour was highly idiosyncratic had to be set aside at the time. It is largely these data which are now reported and discussed in this paper.

REPORT AND DISCUSSION

Children are generally introduced to the 'basic number concepts' during the first years of schooling through whole number counting activities and the simple combining and separating of sets, extending to multiplication (usually as repeated addition), then division (often in the context of sharing). At some point in these first few years children are exposed to the symbolic representations of these concepts and are usually asked to practise computations using the symbols alone, initially with one digit whole numbers, then (in line with the hierarchical structure of the syllabus) with increasingly larger and more complex numbers at each year level, as new 'concepts' (eg 'place value', 'decimals' etc) are introduced.

Almost all of the items presented in the interviews required application of various number concepts specified in the Western Australian primary mathematics syllabus. Some of the items were presented to the students as decontextualised symbolic representations of operations and students were asked to mentally calculate the items if possible, explain their processes and describe an appropriate context. Ten items included large or more complex numbers (see Appendix A) but would be quite easy to calculate using simple informal mental strategies based on conceptual and contextual understanding.

However, most students actually relied solely on using the standard written algorithmic procedures as their preferred mental method, which involved a very complicated mental process of visualising the numbers, re-arranging them vertically and then mentally performing the steps involved. While only 24% of the attempted items were completed using mental strategies based on conceptual knowledge, the success rate for items completed in this way (68%) was consistently higher than for items in which a complex mental 'written' procedure was attempted (success rate 55%).

The difficulty here is that when children do manipulate numbers successfully using the standard procedures, it is impossible to make valid inferences about their conceptualisations of the numbers or the operations involved, although assumptions are often made in the classroom. It would be very easy to assume, for example, that if students can 'do' *15.01 + 14.99* successfully, they must have a conceptual understanding of decimal numbers and that their conceptualisations of these are similar to all other successful students. Certainly this assumption could not be supported by the students in this study. While the overall success rate for this item was 80%, none of the students were adequately able to explain the meaning of *0.5* or the difference between *0.09* and *0.1* (presented elsewhere in the interviews) even though by Year Eight most students could say that the former was the same as a half and *0.1* was larger than *0.09*. The actual explanations offered revealed a wide range of conceptualisations and personalised perceptions of decimal fractions, none of which could be interpreted as a complete, or even partially complete mental concept for decimals, and differing sufficiently from one another to add support to Ernest's (1991) beliefs about the way in which conceptual understanding develops in mathematics.

Ernest (1991 p241) defines the term *concept* in two ways, firstly in its narrow sense as a simple unitary mental object which can be thought of as a single item or idea, which would only involve simple acts of discrimination. For instance understanding a concept for 'water' in this sense would require the recognition of objects which are "water" and objects which are not. A second broader use of the term 'concept' involves a much more complex mental structure which consists of a number of the simpler concepts as well as the relationships between them. A concept for 'water' in this broader sense might involve a wide range of mental connections and interrelationships with related conceptual structures associated with 'rain', 'ice', 'life', 'chemistry', and 'fire', to name but a few, and would arise out of the personalised contexts, experiences and purposes associated with water up to that point in a person's life. While many of the mental constructs for 'water' may well be shared with others, it is difficult to conceive of a person learning or understanding this complex personalised conceptualisation of water as a complete mental entity of meaning, developed by means of a conceptual pathway common to all learners.

And yet, while most mathematical 'concepts' discussed in the educational literature associated with curricula materials clearly appear to fall within this second broader definition of the term, it is assumed in these curricula that it is not only possible to identify within mathematics suitable discrete conceptual 'chunks' which can be conveyed in their entirety to the learner, but that there is a recognised sequential itinerary of learning stages appropriate for all students into which these conceptual 'chunks' may be slotted. In contrast to this Ernest (1991) claims that "a learner's use of a concept must necessarily be within some context, so the concept is linked to its contexts of use" (p240) and that the learner's "grasp of a concept grows according to the range of contexts of use that are mastered"(p241) during a learning process in which the learner idiosyncratically perceives and attends to personally chosen significant events and relationships. Thus the idea of two learners understanding the same concept can only be contemplated in terms of the similarities of their behaviour in response to a particular mathematical situation or context. Because of the highly personalised nature of their conceptual structures, another seemingly similar mathematical context may well elicit entirely different and quite divergent behaviour from the two learners.

Certainly the students interviewed demonstrated that context could be a powerful determinant of success when standard computational procedures were forgotten. Penny (12:3), for example, had previously mentally calculated *1501 + 1499* using a standard procedure but could see no way of dealing with *15.01 + 14.99*, when the item was presented soon after.

P: Oh, no... I hate decimals.
I: Is there an easy way you can think about them?
P: No (groan) I can't do decimals, would it be...? (pause, shaking head)

Her response changed when she was asked if it would be easier to handle if she thought of the item as money; fifteen dollars and one cent add fourteen dollars, ninety nine.

P: It is, I suppose so... [pause] Yeah!....
I: Okay, well how would you go about that?
P: I'd take the one cent and add it to that to make it one dollar, and then add fifteen and fourteen and one, which is thirty dollars.

The importance of an appropriate context for making the mathematics accessible suggested that students' conceptualisations of the basic operations could be usefully examined by looking at the contexts students provided for some of the simple whole number items included in the interview shedules. These consisted of all combinations of the numerals 'five' and 'ten' using the four operations, (ie *10 + 5, 5 + 10, 10 – 5, 5 – 10, 10* x *5, 5* x *10, 10 ÷5*, and *5 ÷10*). The items were presented one at a time in the above order and students were asked to give an answer, then provide an appropriate 'real life' context for each. .

Of particular interest were the students' responses to two of these items, 5 – 10 and 5 ÷ 10, both of which an hierarchical approach to curriculum development would exclude from the primary syllabus on the basis that they involve difficult and abstract mathematical concepts which could not be understood until later stages when students have acquired the prerequisite concepts. As the students in the study were at the stage when it is assumed they are ready to understand these concepts, it was thought important to investigate how the students' conceptualisation of subtraction and division would be extended to deal with the items, and the role context played in the students' ability to make mathematical sense of the them.

The items *5* x *10* and *5 + 10* were often described by the students as being the "same" as the previous items (*10* x *5* and *10 + 5*) "just the other way around" suggesting students understood commutative properties, but these properties were also assumed for division and subtraction. as demonstrated by responses to the items *5 ÷10* and *5 – 10*. Only when encouraged to articulate a context which reflected the "other way around" did some students realise there were inconsistencies. For example.

Nigel (12:7) Year Eight
I: Okay, what about ten take five?

N: Oh, if you have ten smarties and then you give me five, then you've got five and I've got five.
I: What about five take away ten?
N: That's the same, just done around the other way.
I: Is it? So it would work out the same way, would it?
N: Yeah, so if I've got, no he's got, um, I can't really work that one out. Um, no, it's just the same but you can't tell a story about that one.
I: Have you got an answer for it?
N: Yeah, it's five.
I: It's still five?
N: Yes.

This can be related to the learning difficulties which Kameenui and Griffin (1989 p578) suggest arise because students are likely to intuitively associate newly introduced concepts with those previously taught and will often continue to operate on the basis of earlier conceptualisations for as long as possible. A teacher often assumes evidence for continued conceptual growth is demonstrated when students are successfully applying a learned 'concept' within the framework of the controlled contexts which characterise a hierarchical learning sequence of concepts. The student, in fact, may be relying only on those personalised associations and interpretations that are derived from the contexts in which a concept is first introduced and may well be able to continue to operate successfully on that basis throughout a number of steps of presumed increasing difficulty without significantly adding to his or her developing conceptual framework. The problem reveals itself when a new application is introduced which, according to the syllabus, should require a small conceptual step but, for the child, often demands an entirely new framework of contextual associations.

There is some evidence for this in the case of students who could, in fact, accurately articulate a mathematical interpretation of *5 ÷ 10*, but who were unable to supply an answer nor a suitable context without prompting, even though the relevant fraction concept should be available to them at this stage of their schooling. They seemed, in fact, to be restricted by the contextual associations they had just provided for *10 ÷ 5*, along with their preferred contextual conceptualisation for division. For example, Penny (12:9) during the Year Eight interview, conceptualised the initial division item as representing a quotition (or grouping) situation. Even with prompting, she seemed unable to move from this conceptualisation to partition (or sharing) which would be needed to make contextual sense of 5 ÷ 10.

Penny (12:9) Year Eight
P: (10 ÷ 5) You have ten people and you have to work out how many groups of five you could have.
I: Okay, what about five divided by ten?
P: You can't do that either. (referring to the earlier item *5 – 10*)
I: Not even if I said that was five dollars?
P: And that was ten dollars?
I: Well, was it? I mean would that make sense?
P: Yeah, but I remember last time that you said five dollars and ten cents wouldn't be right because they are different.
I: What about five dollars and ten people?
P: Yeah, I suppose you could really. (pause) I don't know. (shaking head)
I: You're still not comfortable with it, are you?
P: No.

Note that Penny, even when prompted with "five dollars" as a starting point, maintained her initial conceptualisation of division by suggesting groups of ten dollars would be needed. She indicated by her response that she may have even considered the more feasible possibility of separating five dollars into groups of 10 cents, but rejected this because of a remembered mathematical 'restriction', but was not able to re-conceptualise division as a sharing situation. What is particularly interesting in Penny's case is that seven months earlier, during the Year Seven interview, she was able (with a little more prompting) to interpret the *5 ÷ 10*item appropriately.

Penny (12:2) Year Seven
P: (10 ÷ 5) Yeah, if you've got ten dollars and you want to know, if you, you want to buy something like, Christmas presents, for two dollars each, then you can work out how many presents you gotta buy.
I: But you know the five first, so is that how many presents that you want to buy?
P: Yeah, well maybe if you've got ten dollars and you want to buy five presents, and you've got to work out how much money you want to spend on each present.

Note here that the initial context given described 10 ÷ 2(not *10 ÷ 5* and was conceptualised as a quotition (grouping) situation. When Penny restated the situation to accurately reflect *10 ÷5*, the identical context was used (ie $10 to purchase 5 presents at $2 each) but by choosing to keep each component constant, she was actually forced to describe the situation as partition (sharing), although this seems not to be her preferred choice of a context for division. This re-contextualisation of *10 ÷5* did seem to provide a basis upon which she could more easily conceptualise *5 ÷10* when prompted with a similar partition situation, even though she initially rejected the possibility that the item could be answered or that a suitable 'story' could be given.

P: Five divided by ten you can't do!
I: Do you think your teacher or anyone could do it? (P: No)
I: What about a story....is there no way to think about that one? (P: No!)
I: What about if you ask yourself the same question....what about if it was, five dollars and you wanted to buy ten presents, is that possible to think about?
P: Mm yeah, I think so.
I: So how much would you spend on each.
P: Fifty cents.
I: What fraction of a dollar is fifty cents?
P: Half.
I: So do you still think there's no answer, or is it possible.
P: No, it's possible.

Naomi also provided an example where the student's personal conceptualisation of division and the context that was initially provided determined success with the unfamiliar item. During the Year Seven interview she initially struggled to provide a clear context or show whether she viewed *10 ÷ 5* as reflecting quotition or partition. Even though she clearly knew how to deal with the item numerically, she had difficulty contextualising it and finally saw it as ten dollars divided by five dollars would be two dollars, a conceptualisation of division that was restricted to whole numbers unrelated to a useful context. When encouraged to explain her context, she said a little impatiently, "Oh, its just ten dollars, 10 divided by 5 is 2!" Her response to *5 ÷ 10* reflected this difficulty in conceptualising an appropriate division context and clearly showed that she believed that the dividend must be larger than the divisor. She claimed it was impossible because "ten can't go into it". When asked if her teacher would have an answer, she said no and added that "she'd say it's not a sum, not a real sum".

By Year Eight Naomi had developed a conceptualisation of division which then enabled her to adequately deal with both items. Her context for 10 ÷ 5 was stated as a sharing situation ($10 divided between five people), so that when asked to consider *5 ÷ 10* she repeated the context appropriately adapted, but then was not immediately able to adapt her conceptualisation of division as requiring a larger dividend. The context itself then provided the means for her to re conceptualise her understanding of division.

Naomi (12:7) Year Eight

N:	Um. If you had, like, five dollars and then there are ten people you would have to try and divide that.
I:	Would you be able to?
P:	Afraid not.
I:	Think about it. Think about really doing it.
P:	Oh, that would be point five.

The responses of both Naomi and Penny together with numerous other examples discovered in the interview data, clearly support the notion put forward by Ernest (1991) that conceptual understanding in mathematics develops as an idiosyncratic, contextually situated process in which each student's conceptualisations consist of quite unique mental structures.

Sierpienska's (1990 p27) analysis of the process which may be involved in understanding a mathematical concept is also clearly supported by the data. She postulates that the learner extracts mathematical meaning through successive 'acts of understanding' which seek out relationships between the initial 'sense' conveyed by a situation or communication and its perceived 'references' (those other events, situations, words etc to which the initial event may 'refer'). While the initial sense conveyed by a mathematical situation may be consistent among learners, the acts of understanding a student carries out to deal with it are likely to involve references which are extremely personal and may vary for an item such as *15.01 + 14.99* from the visualisation of a real context through to the memory of a previously encountered symbolic representation perceived of as requiring a similar response.

It is clear, for example, that Penny and Naomi made similar sense out of the items presented in that each recognised the requirement for a division operation and could provide a mathematically correct response for 10 ÷ 5. Likewise the initial sense made of 5 ÷ 10 was similar for both students in that it was considered mathematically impossible to solve. However, the references perceived by each student and the acts of understanding they exposed in response to prompting, seemed to differ quite markedly according to their individual perceptions of division in a real context. It seems clear, therefore, that their conceptualisations of division can not usefully be thought of as resembling a common 'concept'. Each may be able to "do" the operation involved, but the route by which their present understanding was attained and the future direction of their conceptualisations are likely to be quite divergent.

In a classroom situation, the diversity of the references each child uses to move from this initial sense to an acceptable answer may well be invisible to the teacher and are likely only to be uncovered in children whose consistently inappropriate responses invite closer scrutiny. These children are then described as having misconceptions, misunderstandings or being in some way deviant in the development of their conceptual structures, the characteristics of which are believed to be quite different from the assumed conceptual homogeneity of their successful counterparts. Whereas, according to both Ernest and Sierpienska, and supported by the data described in this paper, the development of each child's conceptualisation of mathematical ideas is quite idiosyncratic in nature, as is the mental processes involved in operating mathematically, and these cannot easily be accommodated within the current hierarchical structure of most primary mathematics curricula.

CONCLUSIONS AND IMPLICATIONS

In this paper an attempt has been made to justify the argument that the development of conceptual understanding in mathematics is a very personal and complex mental process that cannot be separated from the learners' perceptions of the contexts in which mathematical ideas are experienced, nor can the character or course of these mathematical conceptualisations be standardised across learners.

The hierarchical structure of a typical primary mathematics curriculum is seen to be inconsistent with this approach in that its design seems centrally focused on the identification of fundamental mathematical 'concepts' and the dissection of these in such a way as to simplify their transfer to the learners' minds in a controlled sequence of learning activities. Contexts of use in such an hierarchical learning structure seem to be very narrowly and superficially incorporated primarily as a means of exemplifying or illuminating concepts for the learner, but also to provide practice in the application of specific concepts after they have been 'understood'. To follow such a curriculum a teacher would likely be pre-occupied with controlling the difficulty level of each mathematical learning experience in order to simplify the sequential acquisition of these concepts.

The responses of the students interviewed support the viewpoint that this curricula organisation actually encourages classroom teachers to provide an impoverished mathematical learning environment. The range and quality of the references and associations available to the students during the development of their personalised conceptualisations of mathematical ideas seem to have been severely restricted, narrowing their mathematical repertoire to specific fragmented contexts of use without access to any of the broader conceptual structures which might have allowed the students to perceive appropriate references and relationships associated with the particular range of mathematical situations presented.

An alternative primary curricula framework could more profitably focus on providing students with opportunities to begin engaging in a wide range of mathematical encounters in real contexts with real purposes at a very early age. The types of contexts provided should be dictated by the interests and life experiences of the students involved, as well as the potential richness of the mathematical ideas which *may* be exposed by the situation. In such a rich environment students can be encouraged to enter into genuine mathematical discourse at a very early age, not only without fear of confusion, but purposefully exposing the ambiguities and complexities inherent in mathematising in real contexts. Students then have access to a fertile mathematical environment in which they are expected to make personal sense of their experiences and gradually approximate adult mathematical behaviour, parallel in fact to the whole language approach to literacy learning which is itself based on children's natural 'problem solving' approach to learning.

Because such a mathematics curriculum would be centrally concerned with real contexts for real purposes, the conceptualisation process the learners engage in would be very similar to that used by mathematicians and experienced problem solvers when approaching new ideas, and the teacher could expect to be the students' collaborator in this process of solving problems. Problem solving in this sense is not conceptualised as an activity which can be separated from other mathematical processes, but rather is inherent in mathematical activity *per se*.

LIST OF REFERENCES

Clarke, D.J. (1989) *Mathematical behaviour and the transition from primary to secondary school.* Unpublished doctoral dissertation, Monash University, Clayton, Victoria, Australia.

Cockcroft, W.H. (1982) *Mathematics Counts Report of the Committee of Inquiry into the Teaching of Mathematics in Schools,* London: Her Majesty's Stationery Office.

Ellerton, N.F. & Clements, M.A. (1988) Reshaping school mathematics in Australia 1788-1988. *Australian Journal of Education,* 32(3), 387-405.

Ernest, P. (1991), *The Philosophy of Mathematics,* London: The Falmer Press.

Hart, K. (1989) Chapter 4, fractions: equivalence and addition, in D.C. Johnson, (Ed.), *Children's Mathematical Frameworks 8 - 13: A study of Classroom Teaching,* Berkshire: The NFER-Nelson Publishing Company, 46-75.

Kameenui, E.J. & Griffin, C.G. (1989) The national crisis in verbal problem solving in mathematics: A proposal for examining the role of basal mathematics programs. *The Elementary School Journal,* 89(3), 575 - 593.

Sierpienska, A. (1990), Some remarks on understanding in mathematics. *For the Learning of Mathematics*, 10(3), 24 - 36

Swan, M. (1990) Becoming numerate: developing conceptual structures. In S. Willis (Ed.), *Being Numerate: What Counts?* Melbourne: The Australian Council for Educational Research Ltd. 44-71.

Tomazos, D.J. (1991) *Girls' and Boys' Affective Responses to and Understanding of Mathematics during Transition from Primary to Secondary Schooling.* Unpublished honours dissertation, Murdoch University, W.A.

Western Australian Ministry of Education (1989) *Learning Mathematics Handbook Pre-Primary to Stage Seven Mathematics Syllabus,* Perth: Curriculum Programmes Branch, Ministry of Education.

Appendix A

List of items involving operations and large or complex numbers from the interview schedule.

1. 1499 + 1501
2. 3000 – 1499
3. 15.01 + 14.99
4. 15.01 + 1.499
5. 1.49 + 1 1/2
6. 250 x 200
7. 25 x 0.2
8. 2.5 x 200
9. 2.5 x 0.2
10. 5/8 + 7/8

SEX DIFFERENCE IN ACHIEVEMENT IN VCE MATHEMATICS

COLLEEN VALE

In 1991 all Victorian year 12 students undertook the new Victorian Certificate of Education Mathematics Study designed by the Victorian Curriculum and Assessment Board. This paper presents the results of a study into sex difference in achievement in the new VCE Mathematics study in Victoria. An important goal of the study designers was to encourage more equal participation in senior secondary mathematics by females and males and to include assessment of mathematical skills previously not assessed in a year 12 course in Victoria. These new tasks could conceivably change the degree and direction of sex difference in achievement in senior secondary mathematics.

THE VCE MATHEMATICS STUDY

In 1991 year 12 students were able to choose from six mathematics blocks: Space and Number (SN), Change and Approximation (CA), Reasoning and Data (RD), Extensions Space and Number (ESN), Extensions Change and Approximation (ECA) and Extensions Reasoning and Data (ERD), though they were limited to taking only one Extensions block. The mathematics study is defined in terms of areas of study, work requirements and, for year 12 units (level 3 and 4), common assessment tasks (CATs). Each CAT addressed the content of the respective block.

There were four CATs for each block in 1991: CAT1, an investigative project based on a set theme; CAT2, a challenging problem selected from four set problems completed in a specified period of time; CAT3, facts and skills task, a multiple choice test involving 49 questions and CAT4, an analysis task, a test composed of structured questions designed to assess interpretative and analytical skills (VCAB, 1990, p. 53.)

LITERATURE REVIEW

A recent study into sex differences in mathematical achievement found that the difference in favour of males is very small and reducing over time (Friedman, 1989), however sex differences in favour of males are more likely to occur in senior secondary schooling (Friedman, 1989 and Hyde, et al., 1990.) Willis (1989) reviewed the studies of year 12 mathematics results and concluded that "girls do very well in mathematics examinations at the end of year 12 (p.9.)" In South Australia girls had higher percentages of pass rates and higher grades for two mathematics subjects and in Western Australia significant differences have favoured girls. Willis argues that these results may be because "girls enrol in mathematics courses which are easier than their ability would indicate (p.9.)" In Victoria, for HSC group 1, though there has not been a consistent pattern of sex difference in favour of males, males have consistently been more highly represented in the top grade levels than females (Leder, 1980) and have been found to perform better on examination questions which are unfamiliar, require interpretation, and are imbedded in a real world context (Pattison, Holton and Gordon, 1986.) Fennema (1980) argued that observed sex difference in mathematical achievement in the senior secondary levels in favour of males could be explained by the difference in the amount of mathematics studied. This argument was confirmed by Hyde et al (1990.)

Sentry and Money (1990a and 1990b) reported on the results of the trial of the common assessment tasks conducted in 1989 by VCAB using nine non-randomly selected schools. There were 623 students in the trial of which 41% were females, however their participation was not evenly distributed across the five blocks in the trial. They found that for each CAT, there was a difference in the mean score which favoured males. The lowest mean difference was CAT1: Investigative project and the highest mean difference was for CAT2: Challenging problem. The difference in the proportion of males and females achieving the highest grades was also most marked for CAT2: Challenging problem.

METHODOLOGY

This study of the 1991 population of year 12 VCE students involved an analysis of the grades of the all students completing CATs for mathematics. The analysis included comparisons between males and females overall and for each of the six mathematics blocks at the year 12 level, for each CAT and for total mathematics score.

Data for this study was provided by the Victorian Curriculum and Assessment Board (VCAB). For each 1991 VCE student they provided an anonymous identification number, sex, mathematics block(s) taken and grade for each CAT for each mathematics block taken. VCAB used a system of 12 grades for assessment: A+, A, B+,..., E as well as UG (un-graded) and NA (not assessed). For this analysis numerical equivalents were used: A+=10, A=9,....E=1, UG=0 and NA=0.

RESULTS

Almost 39,000 year 12 students enrolled in Mathematics in 1991. About the same number of females (18,122) and males (18,630), but a higher proportion of males studied mathematics. This higher proportion was for mathematics overall (72% compared to 60%), for students studying two mathematics blocks (22.5% compared to 9.1%) and for students taking each mathematics option. The proportion of females and males studying only one mathematics was almost equal (51.1% compared to 49.8%.) Males also outnumbered females in each mathematics block with the exception of CA. RD and SN were the most popular blocks for both females and males and ESN the least popular. Participation in ERD was negligible.

The total mathematics score for a student was calculated by adding together the student's CAT scores for the mathematics block taken, hence they could score between 0 and 40 for mathematics. All students who commenced the year are included in the analysis. Scores for students who took two mathematics blocks were averaged.

The mean total scores are recorded in table 1 and show that there was no significant difference between the total mathematics score for females and males, though the mean is slightly higher for females. There was also no significant difference in the total mathematics score between females and males for SN and ECA, but females performed significantly better than males in ESN and CA whilst males performed significantly better than females in RD.

Table 1 shows that the much larger proportion of males than females taking two mathematics blocks (22% compared to 9%) had the effect of increasing the achievement of males relative to females. For students taking only one mathematics block females have scored significantly better than males overall and for each block. For students taking two mathematics blocks there is no significant difference in mean total scores between females and males except in RD where males performed significantly better (at the 5% level) than females and in CA where females performed significantly better than males (at the 1% level.)

Table 1

Mean scores for total mathematics score$^{(a)}$ overall and for each block and by the number of blocks taken.

block	female			male			t test	
	N	M	SD	N	M	SD	t-value	p$^{(e)}$
all$^{(b)}$	18,122	17.53	9.5	18,630	17.46	10.4	0.70	0.48
1$^{(c)}$	15,377	16.27	9.2	12,814	14.37	9.6	**16.92**	**0.00**
2$^{(d)}$	2,739	24.61	8.1	5,802	24.29	8.8	1.61	0.11
SN	5,702	14.33	8.8	5,747	14.09	10.0	1.38	0.14
ESN	1,676	10.94	7.2	1,788	9.41	7.6	**6.04**	**0.00**
CA	2,722	24.20	9.1	2,461	22.69	10.0	**5.70**	**0.00**
ECA	3,230	23.91	8.5	5,653	23.70	9.2	1.06	0.28
RD	7,512	18.86	9.1	8,798	20.33	10.1	**-9.73**	**0.00**

Note: (a) Total mathematics score for a student is calculated by adding together the score for each CAT. The range therefore is from 0 to 40. (b) The results for all students. (c) The results for students taking only one mathematics block in year 12. (d) The results for students taking two mathematics blocks in year 12. (e) Significant differences in mean scores are in bold type.

Table 1 also shows that overall and in each block the standard deviation for total scores of males is greater than females. This indicates a greater spread of scores for males than females. More males than females achieved the highest total score, 40. The ratio of females to males achieving a total score of 40 is 1:1.3 overall and 1:3.5 for ECA, 1:3 for RD and 1:2.5 for SN. The lower ratio for all students is probably because total scores for students taking two blocks have been averaged.

Table 2 below shows the mean scores for each CAT for females and males overall and according to the number of blocks taken. It shows that overall females have scored significantly better than males on CAT1: Investigative Project and CAT2: Challenging Problem and that there is a significant difference in mean scores in favour of males for both CAT3: Facts and Skills Task and for CAT4: Analysis Task. For students doing only one block females score significantly better than males in each CAT and for students taking two blocks there is no significant difference in mean scores between females and males for CAT3, but a significant difference in favour of females for CAT1 and CAT2 and in favour of males for CAT4. Therefore male superiority for CAT3 and in part for CAT4, can be explained by the amount of mathematics studied.

Table 2

Comparison of mean scores for each CAT.

CAT	female			male			t test	
	N	M	SD	N	M	SD	t-value	p
CAT1								
all	18,122	4.70	2.7	18,360	4.32	2.8	**12.93**	**0.00**
1	15,377	4.44	2.7	12,814	3.66	2.7	**24.20**	**0.00**
2	2,739	6.14	2.2	5,802	5.80	2.4	**6.13**	**0.00**
CAT2								
all	18,122	4.65	2.8	18,630	4.43	3.0	**7.13**	**0.00**
1	15,377	4.34	2.7	12,814	3.64	2.8	**21.17**	**0.00**
2	2,739	6.40	2.2	5,802	6.20	2.5	**3.65**	**0.00**
CAT3								
all	18,122	4.51	2.8	18,630	4.67	2.9	**-5.38**	**0.00**
1	15,377	4.15	2.7	12,814	3.84	2.7	**9.38**	**0.00**
2	2,739	6.53	2.3	5,802	6.49	2.4	0.70	0.49
CAT4								
all	18,122	3.68	2.6	18,630	4.03	2.9	**-12.1**	**0.00**
1	15,377	3.35	2.5	12,814	3.23	2.7	**3.74**	**0.00**
2	2,739	5.55	2.4	5,802	5.80	2.6	**-4.34**	**0.00**

For students taking only one mathematics block there was a significantly higher proportion of 0 scores awarded to male students for each CAT. Males may not have submitted work for assessment in greater proportions since students could still be awarded satisfactory for the completion of work requirements. This may in part explain the superior performance of females among students taking one block.

Table 3

The number of females and males achieving an A^+ or an average greater than A for each CAT.

CAT	CAT	female		male		c^2 test	
		N	%	N	%	c^2	*p*
CAT1	all	478	1.6	518	2.8	0.71	0.40
	1	311	2.0	154	1.2	**29.02**	**0.00**
	2	166	6.1	364	6.2	0.15	0.70
CAT2	all	499	2.8	711	3.8	**31.69**	**0.00**
	1	293	1.9	165	1.3	**16.69**	**0.00**
	2	205	7.5	546	9.4	**8.61**	**0.00**
CAT3	all	524	2.9	774	4.3	**43.02**	**0.00**
	1	294	1.9	209	1.6	3.15	0.08
	2	229	8.4	562	9.7	3.89	0.05
CAT4	all	295	1.6	604	3.2	**100.3**	**0.00**
	1	170	1.1	182	1.4	**5.62**	**0.02**
	2	124	4.5	420	7.2	**22.94**	**0.00**

Note: Discrepancies in the number for all, that is the total for 1 and 2 can be explained by the small number of students taking 3 blocks of mathematics in year 12.

The proportions of females and males achieving the top grade, A^+ or an average greater than A, for each CAT is shown in table 3. It shows that a higher proportion of males were awarded the top grade for each CAT, overall and for students taking two blocks. This was significant for CAT2 and CAT4. However, for students taking one block a significantly higher proportion of females achieved the top grade for CAT1 and CAT2, but males still dominated CAT4 and there was no difference for CAT3.

DISCUSSION

This is the first Australian study to establish such a consistent pattern of higher average achievement in mathematics by females. However, males still remain the most likely sex to achieve the best results. This study has also established that the amount of mathematics studied at year 12 accounts for some but not all of the sex difference in achievement for those components which favour males. It is likely that the sex differences in favour of females, especially for students taking only one block, can be expalined by participation rates, that is, that females are more likely to choose to study one mathematics block rather than two. Evidence for this explanation is found in the significantly higher proportion of females who were represented in the top group of scores for students doing only one block and a higher proportion of males scoring 0. This finding supports the view of Willis (1989.)

It is also likely that the nature of the assessment tasks have suited females. Females have performed better than males on the "take home" CATs but there are other factors which may have contributed to their relative success such as better interpretative, research and communcation skills. Annice et al. (1988) and Cheung (1991) observed that a sex difference in favour of males is more likely for routine problems than for non-routine problems. This study support those findings in terms of the results for CAT2 and CAT4. A complementary study of 1991 VCE students who gained entry to university by Vale (1992) examines other reasons such as the context of problems in CAT2 and sex differences in importance attached to CATs.

Reasoning and Data (RD) was the odd one out. This is possibly due to the pattern of course taking by females and or some peculiar characteristics of the curriculum and teaching of the course. Perhaps the more able females doing only one mathematics chose CA or ECA rather than RD, but this doesn't explain the sex difference in achievement for students taking two mathematics and indicates a superiority for males in achievement for the particular content areas of the course: probability, statistics and logic. However the correlation rates between CATs were lowest for RD indicating an inconsistency of performance by both females and males across the CATs for RD. The teaching of the course may have been problematical, either because of inexperienced staff, or lack of familiarity with the content by the staff. It is therefore possible that females have been disadvantaged relative to males (Vale, 1992.)

Since the introduction of the VCE in 1991, modifications and changes have occurred to both the structure and arrangement of content in the mathematics offerings and the assessment tasks. CAT2 has been dropped and Reasoning and Data has also been dropped but the content of Reasoning and Data has been included in other offerings within a new structure of blocks. The dropping of CAT2 is unlikely to have an effect on sex difference in achievement in total score for mathematics, since it is presumed that females will continue to perform better than males on CAT1 and CAT3 for students doing the same number of mathematics courses at year 12. The inclusion of Reasoning and Data content in other courses may have the effect of closing the gap between females and males in the reason for the sex difference was to do with content areas rather than participation rates or pedagogy.

In conclusion, this study of 1991 VCE Mathematics results has found that the amount of mathematics studied in year 12 is a crucial factor when testing for sex difference in achievement in mathematics. The pattern of participation for both females and males has been shown to influence the results of sex difference tests both of the mean scores and the distribution of scores. The study has also found that females are more likely to perform better than males on investigative projects, non-routine problem solving and multiple choice tests of basic skills and that males are mores likely to perform better than females on routine analysis problems. It is likely that the nature and conditions of the assessment tasks have been important factors in the sex differences in achievement. It remains to be shown whether the superiority of females shown in this study will be maintained over time and with changes and modifications to the VCE Mathematics Study.

REFERENCES

Annice, C., Atkins, W., Peterson, D., Pollard, G. and Taylor, P. (1988), Gender differences in the Australian Mathematics Competition, in J. Pegg (ed), *Mathematical Interfaces*, Australian Association of Mathematics Teachers, 100-105.

Cheung, K.C. (1991), Gender difference in the junior secondary (grade 7) mathematics curriculum in Hong Kong, *Educational Studies in Mathematics*, **20**, 97-103.

Fennema, E. (1980), Sex-related differences in mathematics achievement: Where and why, in Fox, L.H. et al. (eds), *Women and the Mathematical Mystique*, John Hopkins University Press, Baltimore, 76-93.

Friedman, L. (1989), Mathematics and the gender gap: A meta-analysis of recent studies on sex difference in mathematical tasks, *Review of Educational Research*, **59**(2), 185-213.

Hyde, J., Fennema, E. and Lamon, S. (1990), Gender differences in mathematics performance: A meta-analysis, *Psychological Bulletin*,**107**(2), 139-155.

Leder, G. (1980), Sex differences in achievement and participation in mathematics - a selective review, *Improving Maths for Girls*, Report of a conference held at Raywood Inservice Centre, 30 May - 1 June, 1980, Adelaide college of the Arts and Education.

Pattison, P., Holton, D. and Gordon, R. (1986), Describing sex differences in mathematics, unpublished paper, Department of Psychology, University of Melbourne, Victoria.

Sentry K., and Money, R.(1990a), Common assessment tasks in VCE mathematics, Milton, K. and McCann, H. (eds), *Mathematical Turning Points - Strategies for the 1990s, Thirteenth Biennial Conference Proceedings*, University of Tasmania, Hobart, Australian Association of Mathematics Teachers, **2**, 478-493.

Sentry, K. and Money, R., (1990b), VCE mathematics: The 1989 trials of common assessment tasks, Paper presented at the 1990 MERGA Conference, July 1990, University of Tasmania, Hobart, Tasmania.

Vale, C. (1992), Sex difference in achievement in year 12 mathematics and non-routine problem solving, unpublished thesis submitted for M.Ed., University of Melbourne.

Victorian Curriculum and Assessment Board (1990), *VCE Mathematics Study Design*, Melbourne, Victoria.

Willis, S. (1989), *'Real Girls Don't Do Maths', Gender and the Construction of Privilege*, Deakin University Press, Geelong, Victoria.

ALGEBRA: BEYOND MANIPULATING SYMBOLS

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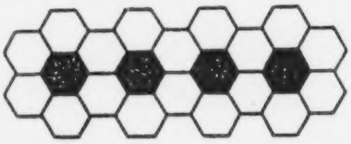
Over the last two decades a vast number of research projects have identified areas of students' misunderstandings in the algebraic domain. It appears that the main focus of this research has been on developing an understanding of variables, the translation of word problems, and on "doing" algebra - the manipulation of symbols. Although these aspects are important to algebra, Booth (1989) and Kieran (1989) believe that a critical aspect is understanding just what the algebraic statement represents, both visually and symbolically. When linking and using various representations, spatial skills and higher order thinking skills play key roles, and are therefore crucial to the acquisition of algebraic understanding. This paper illustrates the importance of these skills to the algebraic domain, reviews the literature pertaining to these skills within the algebraic domain, and identifies the research implications drawn from this literature.

RESEARCH DONE ON ALGEBRA

Research in the algebraic domain has identified many areas of students' misconceptions. Children appear to have difficulty with the variable concept (Booth, 1988; Kuchemann, 1981; Usiskin, 1988), the visual syntax of algebra (Bennett, 1988; Chalouh and Herscovics, 1988; Kirshner, 1989), the concatenation of algebraic expressions (Chalouh and Herscovics, 1988), the changing nature of the equal sign (Davis, 1989 & Kieran, 1989), and the manipulation of symbols (Wheeler, 1989). This research reflects the most prevalent approach historically used for introducing algebra to the beginning student, linking algebra to arithmetic. For this approach the focus seems to be on the acquisition of routine skills and procedures or algorithmic methods (Wheeler 1989), with a heavy reliance on the manipulation of symbols (Davis, 1989). Thus the focus of past research has been narrow and limited with the majority of projects being mainly concerned with the algebraic symbol and its manipulation.

A more recent approach for teaching algebra uses patterning from which algebraic expressions are generated (Bennet 1988). This approach entails introducing algebra by looking at patterns, creating tables, describing the pattern, and "short handing" these descriptions into algebra. The following example typifies this approach.



The council with to create flower beds and surround them with hexagonal paving slabs according to the pattern shown above. Find a formula that the council can use to decide the number of slabs needed for any number of flower beds.

In order to reach an appropriate solution students are required to proceed through a number of steps. They need to see the pattern in terms of its component parts (one flower bed, two flower beds etc.), identify the generalities within these parts (e.g. each new flower bed requires four extra tiles), and extend the pattern using these generalities (for every flower bed there are four tiles plus the two at the beginning). These generalities then need to be transformed into a suitable formula. Success for this problem appears to involve more than an ability to manipulate symbols. Success not only requires sound visual skills (seeing the pattern, dissecting the pattern into its component parts) but also an array of specific thinking skills (mentally rotating the pattern, logical reasoning, an ability to complete the pattern etc.).

Ursini (1991) found that this approach is fraught with difficulties. He reported that, in their attempts to generalise, children did not link the two proposed representations, could not express a generalisation, and disregarded all patterns when trying to generalise. Even though the approach attempts to link algebraic symbols to visual patterns students tend to use procedural means for reaching a solution e.g. "Well we were taught that you draw a table and work it out...you use trial and error...first powers, then square numbers, then triangle numbers...". Stacey (1988) found most students, when looking for patterns from tables tended to use the strategy of guess and check, concentrating on repeated addition, multiplication and simple ratio relationships. The reflex of checking the formula against the given (usually visual) data is not present (Lee and Wheeler, 1989). Yet Booth (1989) and Kieran (1989) believe that a critical aspect in algebra is understanding just what the algebraic statement represents, both visually and symbolically. Endeavouring to link visual and symbolic representations is important and yet it seems that the majority of students are failing to achieve this goal. The role of thinking skills and spatial skills in attaining this aim needs to be explored. Some questions that need to be addressed are:-

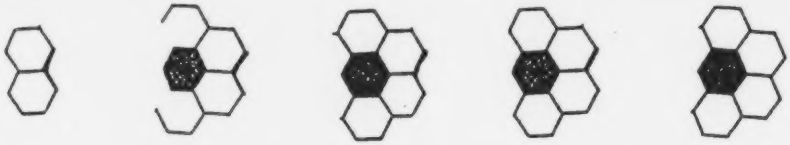
How important is it for students to make these links in the development of algebraic understanding?

Why are students failing to make these links?

What role do visual skills play in forging these connections?

What thinking skills enhance the formation of these links?

The importance of the establishment of these links can be drawn from the literature pertaining to expert vs novice algebraists. Blais (1988) believes the major difference between the novice and expert algebraist is that the expert perceives "essence". In the case of algebraic simplification the expert perceives visual forms of the representation along with informal English. Blais (1988, p 626) states that "since novices are usually symbolically illiterate, communication restricted to the use of formal symbolic expressions has the effect of concealing essence from them." Lesh, Post, and Behr (1987) believe that in order to understand an idea one must first recognise the idea embedded in a variety of representational systems, flexibly manipulate the idea within these systems, and accurately translate the idea from one system to another. Good problem solvers tend to be sufficiently flexible in their use of a variety of relevant representation systems, and instinctively switch to the most convenient representation to emphasise the solution process (Jones, Palinscar, Ogle, & Carr, 1987). However, Lesh et al (1987) maintain that students rarely seem to get things into a single coherent representation of the problem so that they can process the information. Thus is it not only important to develop a variety of representations for a problem but also to understand how these representations interact with each other. Students not only need to attain algebraic expressions from visual patterns but also to interpret these expressions in terms of the visual data. For the "flower bed" problem students need to see the following pattern representing the expression 2 + 4n.



Further they need to be able to relate expressions such as 2(n+1) + 2n to the pattern. Thus the role visual skills play in forging these connections needs to be explored.

In the literature, visualisation appears to consist of two distinct interpretations. Zimmermann and Cunningham (1991) define visualisation as the process of producing and using geometrical or graphical representations of mathematical concepts, principles and problems. This contrasts with the psychological studies which tend to focus on the subject's ability to form and manipulate mental images. The literature on visualisation identifies students with differing visual abilities. Krutetskii (1976) claims that individuals can be classified into three main groups with respect to visual ability. These are "geometric type, analytic type,and harmonic type" (Krutetskii, 1976) or more loosely termed "visualisers, verbalisers and mixers" (Clements, 1982). Individuals are categorised according to the processes they use in problem solving. Visualisers habitually employ imagery or pictorial notations, verbalisers tend to use verbal codes rather than visual images or pictorial notations, and mixers tend to use a mixture of both linguistic and spatial strategies. The preponderance of students appear to be verbalisers (Clements, 1982).

Ben-Haim (1983), and Fennema and Sherman (1977) found visual ability shows about the same magnitude of correlation with mathematical achievement as the verbal component. By contrast, Krutetskii (1976), Presmeg (1986), and Lean and Clements (1981) conclude that there is a tendency for students who prefer to process mathematical information by verbal logical means to outperform more visual students on both mathematical and spatial tests. They felt that these findings reflect the types of mathematics presented in the majority of classrooms, and in fact students may be highly successful in learning school mathematics without needing to resort to visual thinking. Two main reasons were proffered to explain this phenomena. Firstly, in most curriculum, thinking visually is not deemed important. Presmeg (1986) maintains that teachers and the curriculum often present visual reasoning as an introduction, auxiliary argument, or an accessory. Secondly, visual methods are time consuming and the time constraints of testing procedures militate against extensive use of visual methods (Clement, 1982; Eisenberg & Dreyfus, 1991; Presmeg, 1986). Hence the types of mathematics being taught and tested in schools appears to favour the verbalisers and therefore it is not surprising to find many of these being high achievers under the present curriculum constraints.

Presmeg (1986) reports that when a topic is first taught, a visual presentation aids visualisers' understanding but practice of the procedure or formula leads to habituation rendering the use of imagery unnecessary. Eisenberg and Dreyfus (1986) support this finding and suggest that for many students visual thinking and analytical thinking seem to be dichotomous modes, with the analytical mode being overwhelmingly stronger. Thus, for these students, being able to visualise may not be a crucial requisite for success. Eisenberg and Dreyfus (1991) claim thinking visually makes higher cognitive demands than thinking algorithmically and thus wherever possible students tend to choose a symbolic framework for processing mathematical information. They found that even if students were forced to visually process they still preferred to think symbolically. Lean and Clements' (1981) research supports this finding. They maintain that in some situations visualisation can have a detrimental effect on abstract conceptualisation as visualisation can cause memory "overload" and result in slow retrieval during recall. Though visualisation is considered to be a difficult process, it plays a key role when linking and using various representation in problem solving (Schonberger, 1981). Even though this role has been acknowledged the amount of research carried out in this area has been minimal. Of particular interest to this research is the role it plays in pattern generalisation and interpretation, and more specifically how children with differing spatial abilities approach and solve pattern generating problems.

Any transformation of information is considered to involve higher-order thinking skills. The literature reflects difficulties encountered in defining the term, "thinking". Many researchers describe thinking in terms of the features it encompasses. Resnick (1987) identifies some of the key features of higher order thinking as nonalgorithmic and complex, involving interpretation, uncertainty, applying multiple criteria, and imposing meaning. It is effortful and depends on self-regulation. Linking problem representations seems to invoke particular thinking skills. The literature identifies some specific thinking skills that enhance this process. It seems that developing flexibility in mode of representation involves developing an array of mathematical thinking processes. Some of the key processes crucial to flexibility are thought to be spatial thinking including a facility with mental rotation, logical and analogical reasoning, classifying and hypothesising, and an ability to complete pattern and generalise (Lipman, 1985). Thus the interaction between these specific thinking skills and visual ability, and the role they play in enhancing the formation of links between various representations, needs to be explored. This research project endeavours to address some of these issues. Of particular interest is how visual ability and thinking skills, such as logical reasoning, analogical reasoning, pattern completion and mental rotation, enhance and support the elaboration of simple algebraic expressions, and whether differing visual abilities reflect differing capabilities in this linking and elaborating process. Some of the specific aims of the research are as follows:-

1. to identify students' dominant mode or thinking i.e. visual, verbal or mixers.
2. to investigate any relationships between students' dominant thinking mode and transfer between knowledge representations in linear algebraic situations i.e.

(a) do visual thinkers prefer transfer from visual representation to symbolic representation?
(b) do symbolic thinkers prefer transfer from symbolic representation to visual representation?

3. to identify possible relationships between student's general thinking skills and their competence in transferring from one form of algebraic representation to the other.
4. to investigate whether these higher-order thinking skills differ for the two dominant thinking modes (visual and symbolic).

REFERENCES

Ben-Haim, D. (1983). Spatial visualisation sex and grade level differenced in grades five through eight. In R. Herskowitz (Ed.), Proceedings of the 7th international conference for the psychology of mathematics education. Bennett, A. (1988). Visual thinking and number relationships. Mathematics Teacher, 81(4), 267-272..

Blais, D. (1988). Constructivism - a theoretical revolution for algebra. Mathematics Teacher, 81(5), 624-631.

Booth, L. (1988). Children's difficulties in beginning algebra in A. F. Coxford, & A. P. Shulte (Eds.), The ideas of algebra K-12 Virginia: The National Council of Teachers of Mathematics, Inc.

Chalouh, L., & Herscovics, N. (1983). The problem of concatenation in early algebra. In J. Bergeron, & N. Herscovics (Eds.), Proceedings of the fifth annual meeting of the North American Chapter of the international group for the psychology of mathematics education: Montreal.

Clements, M. (1982). Visual imagery and school mathematics. For the Learning of Mathematics, 2(3), 33-38.

Davis, R. (1989). Three ways of improving cognitive studies in algebra. In S. Wagner, & C. Kieran (Eds.), Research Issues in the Learning and Teaching of Algebra. Virginia: Lawrence Erlbaum Associates.

Eisenberg, T., & Dreyfus T. (1991). On the reluctance to visualise in mathematics. In W. Zimmermann, & S. Cunningham (Eds.), Visualisation in teaching and learning mathematics. Mathematical association of America.

Eisenberg, T., & Dreyfus T. (1986). On visual versus analytical thinking in mathematics. In L. Burton, & C. Hoyles (Eds.), Proceeding of the 10th Psychology of Mathematics Education Conference, 153-158.

Fennema, E. and Sherman, J. (1977). Sex related differences in mathematics achievement, spatial visualisation and affective factors. American Educational Research Journal, 14, 51-71.

Jones, B.F., Palincsar, A. S., Ogle, D. S., & Carr, E. G. (1987). Learning and thinking. In B.F. Jones, A. S. Palincsar, D. S. Ogle, & E.G. Carr (Eds.) Strategic teaching and learning: Cognitive instruction in the content areas. (pp. 3-32).Elmhurst: North Central Regional Educational Laboratory.

Kieran, C. (1989). The early learning of Algebra: A structural perspective. In S. Wagner, & C. Kieran (Eds.), Research Issues in the Learning and Teaching of Algebra. Virginia: Lawrence Erlbaum Associates.

Kirshner, D. (1989). The visual syntax of algebra. Journal for Research in Mathematics Education, 20(3), 274-287.

Krutetskii, V. A. (1976). The psychology of mathematical abilities in schoolchildren. Chicago: The University of Chicago Press.

Kuchemann, W. (1981). Algebra. In Hart, K. (Ed.),Children's understanding of mathematics :11-16. London: John Murray.

Lean, G., & Clements, M. (1984). Spatial ability, visual imagery and mathematical performance. Educational Studies in Mathematics, 12, 267-299.

Lee, L., & Wheeler, D. (1989). The arithmetic connection. Educational Studies in Mathematics, 20(1), 41-45.

Lesh, R., Post, T., & Behr, M. (1987). Representations and translation among representations in mathematics learning and problem solving. In C. Janvier (Ed.), Problems of representation in the teaching and learning of mathematics. Hillsdale, New Jersey: Lawrence Erlbaum Associates.

Lipman, M. (1985). Thinking skills fostered by philosophy for children. In J. W. Segal, S. F. Chipman, & R. Glaser (Eds.), Thinking and learning skills. vol 1: Relating instruction to research. Hillsdale, New Jersey: Lawrence Erlbaum.

Presmeg, N. (1986). Visualisation in high school mathematics. For the Learning of Mathematics, 6(3), 42-46.

Schonberger, A. (1981). Gender differences in solving mathematics problems among two-year college students in a developmental algebra class and related factors. Paper presented at the midyear meeting of the American Education Research Association Special Interest Group on Woman in Education.

Stacey, K. (1988). A framework for the analysis of mathematical behaviour. Journal of Mathematical Behaviour, 6, 353-360.

Resnick, L. B. (1987) Education and learning to think. Washington D.C.: National Academy Press.

Usiskin, Z. (1988). Conceptions of school algebra and uses of variables. In A. F. Coxford, & A. P. Shulte (Eds.), The ideas of algebra K-12 Virginia: The National Council of Teachers of Mathematics, Inc.

Ursini, S. (1991). First steps in generalization processes in algebra. In F. Furinghetti (Ed.), Proceedings of the 15th Psychology of Mathematics Education Conference, 3,316-323.

Wheeler, D. (1989). Contexts for research on the teaching and learning of algebra. In S. Wagner, & Kieran, C. (Eds.), Research Issues in the Learning and Teaching of Algebra. Virginia: Lawrence Erlbaum Associates.

Zimmermann, W., & Cunningham S. (1991). Visualisation in teaching and learning mathematics. Mathematical association of America.

INITIAL CONSIDERATIONS CONCERNING THE UNDERSTANDING OF PROBABILISTIC AND STATISTICAL CONCEPTS IN AUSTRALIAN STUDENTS

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Following the questions raised by Watson (1992) concerning research in probability and statistics education in Australia in the 1990s, this paper reports on the initial trialling of items with 64 Grade 9 students. The analysis supports the belief that misconceptions observed in other countries also are present in Australia. Further, the application of a developmental cognitive model offers promise for classifying responses to items and structuring remediation procedures.

The rationale for a growing interest in research in probability and statistics education is given in Watson (1992). Briefly, topics associated with chance and data constitute 20 percent of the content of A National Statement on Mathematics for Australian Schools (Australian Education Council, 1991). As the newest areas in the mathematics curriculum, chance and data are least likely to be well understood by teachers at all levels. They are also the areas about which the least is known of students' understandings and misconceptions. As state curricula are amended to take account of the National Statement and professional development is planned to assist teachers, it is important to provide as much support as possible based on analyses of students' and teachers' understanding of these topics.

Watson (1992) also briefly reviews the history of overseas research into probabilistic and statistical understandings and offers some starting points for Australian research. These include studies of student beliefs about probability (e.g., Fischbein, Nello, & Marino, 1991; Garfield & delMas, 1991), conceptions of randomness (e.g., Konold, Lohmeier, Pollatsek, Well, Falk, & Lipson, 1991), sampling in relation to sample size and representativeness (e.g., Gal, Rothchild, & Wagner, 1989), the relationship between two variables (e.g., Konold, 1991), the applicability of measures of central tendency (e.g., Gal, Rothchild, & Wagner, 1990), and the understanding of conditional probability (e.g., Pollatsek, Well, Konold, & Hardman, 1987).

A theoretical basis for the analysis of student understanding could come from various sources. A developmental model, such as that suggested and extended by Biggs and Collis (1982, 1989, 1991) has proved useful in other areas of mathematics (e.g., volume measurement (Campbell, Watson, & Collis, in press) and fractions (Watson, Campbell, & Collis, in press)) and parallels the model suggested by Shaughnessy (1992). Two aspects of the Biggs and Collis SOLO (Structure of Observed Learning Outcome) Taxonomy will be exemplified in the analyses presented in this discussion. First is the unistructural-multistructural- relational (UMR) cyclic structure which operates within the modes of functioning. Of particular interest here are the concrete symbolic mode where most school learning occurs and the ikonic mode which commences functioning earlier and is associated with feelings, intuitions and imaging. Second is the multimodal aspect of functioning which recognises various ways in which interactions take place between modes, in particular the ikonic and concrete symbolic, as both continue to develop throughout childhood and adolescence.

The remainder of the discussion will focus on the initial sampling of student responses to explore conceptions suspected to be of interest and on the usefulness of the SOLO model in analysing results.

ITEMS

Items were selected and adapted from those reported by Fischbein and Gazit (1984) on intuitive probability (six items), Fischbein et al., (1991) on outcomes and sample spaces (three items), Reading and Pegg (1992) on sample selection (two items), and Garfield and delMas (1991) on conceptions of probability (three items). Following the interest of Mokros and Russell (1992), an item asked for an explanation of the term 'average'. Finally a problem solving situation was devised to explore student understanding of criteria which could be used to compare two data sets. Although administered differently the final section was based on ideas suggested by Wagner and Gal (1991) and Garfield (1992). The 12 items to be summarized in this paper are presented in full in Watson (1993); three are shown in Figure 1.

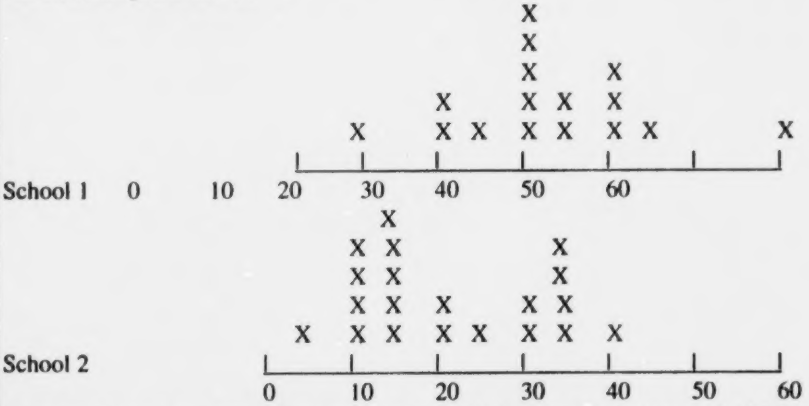
THE SAMPLE

Because the purposes of this preliminary study were to assess the items chosen, to explore possible misconceptions, to assess the potential of the theoretical developmental model, and to suggest strategies for a larger scale study, convenience samples were used. They consisted of two Grade 9 mathematics classes from Hobart, Tasmania, suburbs, totalling 64 students. They were 13 or 14 years of age and the sample was evenly split between females and males; gender differences are not explored here. Students completed the written questionnaire during class time. One of the classes had earlier in the year studied a unit of probability while the other had studied one on statistics. Although these would not have lasted more than a few weeks, it was expected that the sample would present some good responses to the items.

MISCONCEPTIONS

The 12 items used for this analysis led to an examination of the seven misconceptions listed in Table 1. The number of items associated with each are noted and the percentage of students exhibiting the misconception for the items is given. The percentages are estimates based on the number of responses to each item, ranging from 64 to 32.

Item A **Two maths classes, one from your school and one from the nearby high school, have had a competition to see which is better at quick recall of maths facts. The scores of the two classes are shown on the two line plots below.**



How would you analyse these data to show which school is better and which school would probably be your school?

Item B **Consider rolling two dice. Is it more likely to obtain**

(a) a 5 with one die and a 6 with the other?, or

(b) 6 with both dice?, or

(c) is the probability the same in both cases?

Item C **Again consider rolling two dice. Which is more probable?**

(a) To obtain the same number with both dice., or

(b) To obtain different numbers on the dice., or

(c) is the probability of each event the same?

Figure 1 Three items used in preliminary study

Hence non responses are ignored and this means the percentages are lower bound estimates for this sample. The percentages in Table 1 are also lower bounds in the sense that not all other answers gave correct, or well-expressed, responses - they just did not show the misconception indicated in the right hand column.

No students believed in lucky numbers and very few agreed with superstitions associated with putting a certain foot into a room first or with winning a chance game due to being older. Approximately a quarter of students displayed no understanding of average and a similar fraction used the total of scores to compare two groups despite differing sample sizes. Between 20% and 50% of students had difficulty using proportion to make judgements in relation to samples and populations. Nearly half had difficulty with the concept of randomness in relation to lottery numbers and/or sequences of binomial trials. Finally up to 85% of students had problems with the sample space when two dice are tossed.

Construct	Number of Items	Percentage difficulty for item
Luck (lucky numbers)	1	0%
Superstition (right foot, age)	2	5%, 2%
Average (no basic understanding)	1	24%
Compare different sized groups (using total)	1(A)	28%
Proportion (samples and populations)	3	26%, 20%, 56%
Randomness (lottery, births)	3	46%, 47%
Sample spaces (dice)	2(B,C)	83%, 35%

Table 1. Percentages displaying basic misconceptions in Grade 9.

ANALYSIS AND DISCUSSION WITH COGNITIVE MODEL

There are two ways that the SOLO framework can be used initially in a study such as this. First the items themselves can be classified by the level of response required to obtain a correct answer. Second responses can be categorised by the level actually exhibited by the student.

Consider first the constructs listed in Table 1. The first two, labelled Luck and Superstition, relate to intuitions which are associated with the ikonic mode of functioning. As the mode which begins functioning earlier, it would be expected that by Grade 9, students functioning well in the concrete symbolic mode would be expected to have rejected these intuitive level concepts as adequate responses. This would appear to have been the case for this sample of students. The other five constructs listed in Table 1 are related directly to operations of the concrete symbolic mode as they rely mainly on school-based instruction. They are listed in order of increasing difficulty as judged by the percentages showing the misconceptions. This seems to be a good starting point for suggesting a hierarchy of difficulty, as well as for indicating the functioning level required for a correct concept to be displayed. It is too early, however, to put much weight on such an ordering, particularly in the light of differing formats for the items.

When moving to the responses themselves, it is possible that ikonic and concrete symbolic functioning will be demonstrated on the same item, regardless of what is hypothesized to be required for a 'correct' response. In looking at the responses reported, from a cognitive perspective, two aspects are significant. One is the developmental structure of the response and the other is evidence of multimodal functioning. In the latter case, in this study, this means the interaction between intuitive/ikonic and concrete symbolic functioning.

For the purposes of this study the following SOLO notation is used to classify different levels of response for each item within the concrete symbolic mode:

U - uses one piece of relevant information to respond;

M - uses more than one piece of relevant information: these pieces of information to be of a similar type;

R - relates more than one piece of relevant information, of different types, in a coherent fashion.

In previous research on understanding of volume measurement (Campbell, et al., in press) and fractions (Watson, et al., in press), it was found that two UMR cycles operated in problems involving these constructs. The first cycle, labelled U_1-M_1-R_1, led to the correct understanding of the concept, while a second, labelled U_2-M_2-R_2, resulted when the concepts were applied in problem solving contexts of varying difficulty.

An application of this model is shown below in relation to the item concerning 'average'.

Response: *'Normal'*

This is classified as unistructural, U_1, as it shows a lack of consideration for the mathematical aspects of the construct, while selecting a descriptive common sense aspect.

Response: *'Average means the middle number'*

This appears multistructural, M_1, in that, in the context of the arithmetic mean, it implicitly considers a set of numbers and makes a judgement about a single number describing it. (Note the respondent was not aware of the median concept.)

Response: *'Average is when you add up your total digits and then divide it by the number of digits you have,'* and *'Each time you did something and it had answers that were a little bit apart you could find the average witch [sic.] would get it down to one number'*

These would appear to exhibit a relational, R_1, understanding of the arithmetic mean as average.

Moving to item A in Figure 1, the question arises as to the application of this average concept to a given problem. Because a comparison must be made, it would be expected that a relational, R_2, response is required. The following response qualifies as an R_2 for this concept:

If I were to show which school was better I would find out the averages of both schools. The higher the average determines which school is better.

The nature of the question precluded U_2 & M_2 responses in this item which indicates that we will need to redevelop the item to get the full range of responses. A U2 response would be expected to be similar to an R1 response in that the student would be able to calculate the 'average' (arithmetic mean) when asked to do so with data in a suitably obvious form. An M2 response would involve, say, the calculation of several means from given data and then a separate judgement being made for example on the largest or smallest. The R2 example given requires the student to make the decision that the arithmetic mean is the appropriate construct to use, calculate them for each group and then compare to reach a conclusion; in other words it requires an overview of the problem which is not required at U2 or M2 response levels.

Another example of functioning in the second UMR cycle is found in items B and C in Figure 1. As was found on an earlier item not discussed here, students had a good grasp of the sample space associated with a single die and could distinguish impossible, possible and certain events. This is representative of R_1 thinking in relation to the concept of sample space in this context. Items B and C, however, require an R_2 level of functioning in the concrete symbolic mode as they involve an understanding of the sample space in relation to throwing dice extended to the more complex situations where two dice are involved. The U2 level of response, represented by the following:

'You have 1 in 6 chances of getting a five and it is the same with a six, so they are the same'

and

'because there is a 50/50 chance of either'

These show an R_1 concept and justify the same probability for throwing a 5 and a 6, as two 6's, despite using two dice. An early R_2 response,

'Because you can get a 5 or 6 on either dice,'

justifies the higher probability for a 5 and a 6 by applying the concept over the extended sample space of the pairs of numbers involved.

For item C, where more correct answers were obtained, the justifications for 'the same' probability were still at the U_2 level:

'Because rolling dice is random. Any no could come up.'

'This is because the dice is fair and any sequence should be just as likely as any other.'

The following was judged M_2 because, although it has the right idea, it does not tie the concepts together in a coherent fashion:

'Because the dice has many different numbers but only one of each.'

The following response to the item seems to have related the concepts together satisfactorily to achieve the correct answer at the the R_2 level:

'Because there are more pairs of different numbers than there are pairs of the same number. So the chance of getting a double is smaller.'

As mentioned earlier, ikonic functioning may or may not take place in conjunction with concrete symbolic functioning. In the examples presented here the ikonic responses generally relate to beliefs and intuition, not to imaging. These responses are most commonly found in explanations where the students appear unable to reason effectively in the concrete symbolic mode. Hence for item B,

'Nobody knows what will come up on the dice - its [sic.] just fate'

and

'It occurs by chance only'

would be considered ikonic (IK) responses as would,

'It is all to do with luck',

in item C.

A response such as,

'I think it is a chance thing and you would have better odds getting random numbers',

on an item about a straight sequence of numbers in a lottery, shows lack of appreciation of chance in the context but has some idea of what odds means. It appears to be an IK-M_1 response, struggling with the transition from intuition to an understanding of the association of randomness and chance. Sometimes, however, there seems to be an interaction of intuitive (IK) and concrete symbolic reasoning at a higher level. The following discussion, for example, associated with the belief that entering a room with one's right foot first will enable a good performance, shows an appreciation of superstitions but the student makes the decision on the basis of concrete symbolic reasoning.

I don't agree with this [as it is not logical]. *But if Joseph believes that putting his right foot forward [helps] then that [is] what he should [do] because it is something which he believes in...'*

The response appears to show a high level of ikonic interaction with concrete symbolic reasoning and could be categorised as IK-U_2 in this context.

Although it has been possible to explore responses from those collected in the initial sampling of students which seem to exemplify the UMR cycle and multimodal functioning, it is clear that a great deal more work needs to go into item development to ensure that the items are flexible enough to allow for a full range of responses both within and between modes of functioning. It may be useful to devise items similar to those designed for the Collis-Romberg (1992) Problem Solving Profiles; such items would have the additional advantage that they can be used later to identify achievement levels to assist teachers in planning instruction and remediation.

ACKNOWLEDGMENT

The authors would like to acknowledge the assistance of Fiona Shalley in recording and collating the data in this study.

REFERENCES

Australian Education Council. (1991). A National Statement on Mathematics for Australian Schools. Canberra: Author.

Biggs, J.B. & Collis, K.F. (1982). Evaluating the quality of learning: The SOLO Taxonomy. New York: Academic Press.

Biggs, J.B. & Collis, K.F. (1989). Towards a model of school-based curriculum development and assessment using the SOLO Taxonomy. Australian Journal of Education, 33, 151-163.

Biggs, J.B. & Collis, K.F. (1991). Multimodal learning and the quality of intelligent behaviour. In H. Rowe (Ed.), Intelligence: Reconceptualisation and measurement. Hillsdale, N.J.: Lawrence Erlbaum.

Campbell, K.J., Watson, J.M. & Collis, K.F. (in press). Volume measurement and intellectual development. Journal of Structural Learning and Intelligent Systems.

Collis, K.F., & Romberg, T.A. (1992). Collis-Romberg Mathematical Problem solving Profiles. Hawthorne, Vic.: Australian Council for Educational Research.

Fischbein, E., & Gazit, A. (1984). Does the teaching of probability improve probabilistic intuitions? Educational Studies in Mathematics, 15, 1-24.

Fischbein, E., Nello, M.S., & Marino, M.S. (1991). Factors affecting probabilistic judgements in children and adolescents. Educational Studies in Mathematics, 22, 523-549.

Gal, I., Rothchild, K., & Wagner, D.A. (1989, April). Which group is better?: The development of statistical reasoning in elementary school children. Paper presented at the Society for Research in Child Development meeting, Kansas City, MO.

Gal, I., Rothchild, K., & Wagner, D.A. (1990, April). Statistical concepts and statistical reasoning in school children: Convergence or divergence. Paper presented at the American Educational Research Association conference, Boston.

Garfield, J.B. (1992). Personal communication.

Garfield, J.B., & delMas, R. (1991). Students' conceptions of probability. In D. Vere-Jones (Ed.), Proceedings of the third international conference on teaching statistics. Volume 1: School and general issues (pp. 340-349). Voorburg: International Statistical Institute.

Konold, C. (1991). Chance Plus: A computer-based curriculum for probability and statistics. (Annual Review: Year 2, NSF Grant No. MDR-8954626). Amherst, MA: Scientific Reasoning Research Institute.

Konold, C., Lohmeier, J., Pollatsek, A., Well, A., Falk, R., & Lipson, A. (1991). Novices views on randomness. In R.G. Underhill (Ed.), Proceedings of the thirteenth annual meeting, North American Chapter of the International Group for the Psychology of Mathematics Education, Volume 1 (pp. 167-173). Blacksburg, VA: VPI & SU.

Mokros, J.R., & Russell, S.J. (1992). Children's concepts of average and representativeness (Working Paper 4-92). Cambridge, MA: TERC.

Pollatsek, A., Well, A.D., Konold, C., & Hardiman, P. (1987). Understanding conditional probabilities. Organizational Behavior and Human Decision Processes, 40, 255-269.

Reading, C. & Pegg, J. (1992). Personal communication.

Shaughnessy, J.M. (1992). Research in probability and statistics: Reflections and directions. In D.A. Grouws (Ed.), Handbook of research on mathematics teaching and learning (pp. 465-494). New York: MacMillan.

Wagner, D.A., & Gal, I. (1991). Project STARC: Acquisition of statistical reasoning in children. (Annual Report: Year 1, NSF Grant No. MDR90-50006). Philadelphia, PA: Literacy Research Center, University of Pennsylvania.

Watson, J.M. (1992) What research is needed in probability and statistics education in Australia in the 1990s? In B. Southwell, B. Perry, & K. Owens (Eds.), Proceedings of the Fifteenth Annual Conference of the Mathematics Education Research Group of Australasia (pp. 556-567). Kingswood, NSW: MERGA.

Watson, J.M. (1993). Initial considerations concerning the understanding of probabilistic and statistical concepts in Australian students: Working Paper. Unpublished manuscript, University of Tasmania, School of Education, Hobart.

Watson, J.M., Campbell, K.J., & Collis, K.F. (in press). Multimodal functioning in understanding fractions. Journal of Mathematical Behavior.

DEVELOPMENT OF POSITIVE ATTITUDES TO MATHEMATICS: THE PERSPECTIVE OF PRE-SERVICE TEACHERS

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This paper reports on the further development of an instrument designed to measure self-concept in mathematics and the attitude of pre-service teachers toward the teaching of mathematics. It presents the results of the questionnaire's application to teacher education students across various programs at the University of Western Sydney, Nepean. The findings of the study revealed that the students' attitude towards teaching mathematics improved as they progressed through the programs, but that their self-concept in mathematics remained unchanged by the experience. In addition there were clear and significant differences in attitude to teaching mathematics and self-concept in mathematics based on gender and program of study. Males compared to females were found to display consistently higher levels of self-concept and more positive attitudes. Secondary students also displayed significantly higher scores on both constructs than did primary students who in turn scored consistently higher than early childhood students.

Attitudes and Pre-service Teachers

> *"An important aim of mathematics education is to develop in students positive attitudes towards mathematics and their involvement in it......The notion of having a positive attitude towards mathematics encompasses both liking mathematics and feeling good about one's own capacity to deal with situations in which mathematics is involved." (Australian Education Council, 1991, p.31)*

This extract from A National Statement on Mathematics for Australian Schools reflects the widespread belief among educators in the importance of the development of positive attitudes in the teaching and learning of mathematics.

Although definitions of attitude vary, they generally include the ideas that attitudes are learnt, manifest themselves in one's response to the object or situation concerned, and can be evaluated as being either positive or negative (Leder, 1992). One such definition states...." Attitude is a mental and neural state of readiness, organised through experience, exerting a directive or dynamic influence upon the individual's response to all objects and situations with which it is related " (Allport in Kulm, 1980, p.356). When exploring the attitudes of pre-service teachers toward mathematics it is necessary to not only to consider their attitude towards the subject itself, but also their attitude regarding the teaching of mathematics.

The attitudes of pre-service teachers are of particular importance because of their potential influence on pupils. Although the research evidence is certainly not conclusive it has been sufficient to suggest that positive teacher attitudes contribute to the formation of positive pupil attitudes (Aiken, 1976; Sullivan, 1987). There is an additional concern that female teachers, who constitute the majority of primary school teachers, may be perpetuating negative attitudes with the girls in their classes (Kelly & Tomhave, 1985; Relich, Conroy, & Webber, 1991). Some studies have indicated that teacher attitudes towards a subject and the teaching of that subject influence the instructional techniques they employ and that these, in turn, may have an effect on pupil attitudes (Carpenter & Lubinski, 1990).

One argument presented in support of the need for positive attitudes is that such attitudes can enhance achievement in mathematics at primary, secondary and tertiary level (Dungan & Thurlow, 1989). Most studies on the relationship between attitude and achievement have revealed a low but significant correlation (Aiken, 1976; Kulm, 1980). However, the nature and direction of this relationship is yet to be unravelled (Kulm, 1980; Suydam, 1984).

Unfortunately large proportions of pre-service teachers have been found to hold negative attitudes towards mathematics and mathematics teaching (Becker, 1986; Sullivan, 1987). Some studies have revealed links between attitude to mathematics, the choice or avoidance of mathematical studies, self concept and attitudes towards the teaching of mathematics. It appears that students (both male and female) with low self-concepts in mathematics are less likely to pursue mathematical studies. Not surprisingly then, research has shown that most pre-service teachers who exhibit negative attitude towards mathematics have not chosen to study mathematics in their final years of high school (Aiken, 1976; Relich, Conroy, & Webber, 1991; Sullivan, 1987).

The potential of teacher training courses to change the negative attitudes of pre-service teachers towards mathematics must also be considered. Sullivan (1987) found that almost half of the students entering a teacher training course possessed negative attitudes towards mathematics. He states..."The course improved their attitudes overall, but those who started with negative attitudes still had the most negative attitudes at the end." (Sullivan, 1987, p.1). He concluded that if these initial attitudes are so significant, teacher education courses may need to establish entry criteria based on the mathematics background of the applicants.

Several studies suggest that beginning teachers are especially concerned about methodology as well as content and therefore training courses should cater for this (Aiken, 1976; Blunden & De La Rue, 1990; Mansfield, 1981; Sovchik et al., 1981; Watson, 1987). Relatively high correlations have been found between mathematical achievement, the enjoyment of mathematics and the perception of mathematics being useful, which also carries implications for the design of teacher education courses (Watson, 1987). If the attitudes of pre-service teachers are to be improved, there first needs to be a reliable instrument with which to measure levels of attitudes and perhaps to identify groups of students with special needs (Aiken, 1976; Nisbet, 1991; Watson, 1987).

Measuring Attitudes

In recent years, researchers of 'attitude' have acknowledged that attitude appears to be a multi dimensional construct and therefore requires a multi dimensional measure (Leder, 1992). However, there is some dispute as to which components of attitude, such as anxiety, enjoyment, self-concept and belief in the usefulness or value of mathematics, provide the best indicators of attitude.

One component that has received much attention is that of 'mathematics anxiety'. There is little doubt as to the existence of 'mathematics anxiety' and a number of instruments for measuring levels of anxiety have been developed and implemented (Richard & Suinn, 1972). However, Sovchik, Meconi and Steiner (1981) suggest that the construct 'mathematics anxiety' is not as well defined and measurable as assumed by some mathematics researchers. There is some doubt as to whether anxiety is in fact a separate construct. It may just be a reflection of some deeper attitude (Wood, 1988). For example, Bandura (1982) points out a person may judge that a particular action will result in a particular outcome (expectancy judgement), but may doubt his/her ability to successfully perform the action (efficacy judgement), which in turn may be related to existing anxiety or the production of anxiety. Anxiety's relationships to factors such as enjoyment, general attitude towards mathematics and performance are unclear. Wigfield and Meece (1988) concluded that mathematics anxiety and perception of mathematics ability are conceptually distinct, though related, constructs. Others argue that self-concept is a better measure of how people feel about themselves as mathematicians and as teachers of mathematics, and that self-concept has an influence on the formation of attitudes (Relich, Conroy, & Webber, 1991; Relich & Way, 1992). Studies have also found a consistently high positive relationship between self-concept and mathematics achievement (Hackett & Benz, 1989).

Instruments designed for determining the attitudes of pre-service teachers towards mathematics and the teaching of mathematics have been scarce. Nisbet's (1991) instrument consisted of the Fennema and Sherman (1976) 'Mathematics Attitude Scales' plus his own parallel scales constructed to cover the 'Attitudes to Teaching Mathematics' aspect. Neither of these scales included items regarding self-concept. Relich and Way (1992) supplemented Nisbet's instrument with self-concept items developed by Marsh (1988) to form a large composite questionnaire, referred to as the "Teaching Mathematics Questionnaire". The results of this study (Relich & Way, 1992) supported the contention that attitude towards mathematics and the teaching of mathematics can best be defined through feelings about the teaching of mathematics and through self-concept of one's abilities to cope with mathematical tasks. Anxiety did not emerge as a separate variable, but was highly correlated with attitude and therefore subsumed into the attitudinal profiles.

The major purpose of this research was to further validate the relevance of this questionnaire in accurately gauging attitude towards mathematics among pre-service teachers, and to determine how attitude differs based on characteristics such as program of study, year of study, level of previous study of mathematics in high school and tertiary institutions, age and gender. The questionnaire focuses on "attitude to teaching mathematics" as measured by a composite scale (see Relich & Way, 1992) and "self-concept in mathematics". Through a cross-sectional analysis of student data at various levels of study, we can determine whether attitude and self-concept change as pre-service teachers progress through their course and whether changes in attitude are concomitant with changes in self-concept.

METHOD

Subjects

The questionnaire "Teaching Mathematics" developed by Relich and Way (1992) was used to gather information from all students in Education at a NSW university. A total of 564 students participated across a variety of programs with a substantial majority (85%) being female. The subjects represented all the years of study within the programs and were asked to participate on a voluntary basis in completing the forms. They represent approximately 70.5% of the population of students within the Faculty enrolled in these programs. Over 81% studied mathematics in high school at the 2 unit level or less. Those who undertook study at the three or four unit level were represented in all the programs but constituted the majority (67%) of secondary students.

Instrumentation

The "Teaching Mathematics Questionnaire" consists of 20 items, 11 related to specific attitudes on the teaching of mathematics and nine mathematical self-concept items. The development of this instrument, an adaptation and amalgamation of various currently available instruments, is discussed in detail in Relich and Way (1992). The 20 items required Likert scale responses on an eight point continuum from definitely false to definitely true. For those statements which referred to practice teaching experiences, participants had an option of indicating that the question was not applicable. The purpose of this category was twofold. First to cater for students in their first year who may not have had such an experience and second to gauge what proportion of students may not view themselves as teachers of mathematics.

A test of reliability using Cronbach alpha suggests that the two sub-scales are highly reliable with coefficients of .92 for the attitude scale and .88 for the self-concept scale.

Procedure

The administration of the questionnaire took place during normal lecture periods when a majority of students within a particular year and program were all expected to attend at the same time. Students were briefly instructed about the purpose of the questionnaire and given information about how to complete it on a voluntary basis. They were however encouraged to participate because the researchers wish to conduct a longitudinal study. While no student can be directly identified in order to ensure anonymity, each student was able to provide an identification number known only to the participant. No more than ten to fifteen minutes of class time were required to complete the task.

Regression analysis was used to examine the results and to determine what set of independent variables were the best predictors of attitude to the teaching of mathematics and self-concept in mathematics. We also used t-tests and oneway analysis of variance to determine if there were significant differences in response patterns based on gender, program and year of study variables. These techniques were used in preference to factorial analysis of variance because the "not applicable to me" response reduced much of the available data from the early childhood and first year students across all programs resulting in either empty cells or cells with very small numbers.

RESULTS

An initial correlation analysis indicates that there are some significant relationships among attitude, self-concept and the set of independent variables. In particular attitude and self-concept are significantly (r=.62, p<.001) correlated and share over 36% variance. Attitude (r=.12, p<.005) and self-concept (r=.10, p<.05) are significantly related to gender, indicating that males are significantly more likely to display positive attitudes to teaching mathematics and register higher self-concept in mathematics than females. There is a similar positive and significant relationship between level of study of high school mathematics and attitude (r=.15, p<.001) and self-concept (r=.32, p<.001) suggesting that higher levels of study at this level are associated with more positive attitudes and higher self-concepts. Interestingly, it is significantly more likely that males study high school mathematics at a higher level than females (r=.09, p<.05). There is also an interesting relationship between age and the two dependent variables. While older students are significantly (r=.15, p< .001) more likely to have positive attitudes, younger students display significantly (r=-.10, p<.05) higher levels of self-concept. On an encouraging note, there is a positive and significant (r=.35, p<.001) relationship between the year of study and attitude, with those in the latter years of study exhibiting more positive attitudes. For self-concept, however, no such relationship exists suggesting little change on this construct as students progress through their programs of study.

Stepwise regression analysis was used to determine which set of independent variables were the best predictors of attitude and self-concept. Gender, level of study of high school mathematics and year of study all contributed significantly (multiple R=.30, p<.001) to the prediction of attitude to teaching mathematics. For self-concept, however, level of study of high school mathematics provided the most significant (r=.34, p<.001) relationship which was not significantly improved by the inclusion of any other variable.

Significant gender differences in attitude (t=3.94, p<.001) and self-concept (t=3.68, p<.001) were verified through the use of t-tests. In addition significantly (p<.05) more positive attitudes and higher self-concepts were found among secondary (m=77.8, s.d.=8.5; m=43.5, s.d.=8.2) students when compared to primary (m=69.2, s.d.=13.0; m=36.7, s.d.=9.8) students who in turn had significantly (p<.05) more positive attitudes than early childhood (m=59.1, s.d.=10.3; m=33.4, s.d.=8.2) students. A further oneway analysis based on the year of study confirmed the significant (p<.05) attitudinal changes from the second (m=67.0, s.d.=13.9) year of study to the third (m=70.4, s.d.=11.3) year of study. No similar changes occurred for self-concept profiles.

These results strongly suggest that while attitude to the teaching of mathematics may be ameliorated through participation in a teacher training programs which focus on the affective state of the student, self-concept seems far more resistant to change. Those who enrol in teacher training programs with high levels of study in mathematics at the high school level, generally tend to have more positive attitudes and better self-concepts. This type of student profile is also much more likely to be exhibited by the males compared to females.

DISCUSSION

The application of this questionnaire to such a large sample and the consequent findings further validates its relevance for teacher trainers in mathematics. It was clear from the results that attitude and self-concept while highly related are separate constructs. Although there were clear attitudinal changes towards the teaching of mathematics occurring in the latter years of study, it is clear that self-concept is stable and not easily malleable. The evidence suggests that pre-service teachers do change their attitudes about their ability to teach mathematics but unfortunately do not perceive themselves as better mathematicians as a consequence. This raises the question as to whether such changes are enduring and even whether such changes are superficial and inconsequential to their overall ability to deal effectively with mathematics lessons. Perhaps the other side of the argument is that as educators we should be grateful for any positive changes that we can engender and that expecting to change self-concept is not a realistic goal. Nevertheless, there is something inherently attractive to a teacher in attempting to change deeply entrenched views about personal ability and the possible consequent positive effect that this may have on the enthusiasm of a future teacher for mathematics.

All the evidence continues to point to differences in attitude and self-perceptions in mathematical ability between males and females. Our evidence is clear that these differences are the baggage that students bring with them from their study at the high school level. Those who studied mathematics at higher levels are more likely to be advantaged and invariably such students are more likely to be male than female. Given the high proportion of females to males, especially at the primary and early childhood levels, we need to continue to seek ways in which to encourage girls to attempt higher levels of study in mathematics. This of course assumes that the level of study has an effect on subsequent levels of self-concept but this is not entirely clear. It is also possible that those with greater self-concept seek out the more challenging mathematics subjects. Whatever the explanation may be, these are issues which we need to continue to study and address if we hope to deliver quality programs which will result in teachers at all levels who are positive not only about their ability teach mathematics but also about their own levels of skill and ability as mathematicians.

REFERENCES

Aiken, L. R. (1976). Update on attitudes and other affective variables in learning mathematics. Review of Educational Research, 46, 293-311.

Australian Education Council and Curriculum Corporation. (1991). A National Statement on Mathematics for Australian Schools. Carlton, Victoria. Curriculum Corporation.

Bandura, A. (1982). Self-efficacy mechanisms in human agency. American Psychologist, 37,122-147.

Becker, J. R. (1986). Mathematics attitudes of elementary education majors. Arithmetic Teacher, 1, 50-51.

Blunden, R., & De La Rue, P. (1990). A survey of teacher concerns across one year of teacher training. Journal of Teaching Practice, 10, 1-16.

Carpenter, T., & Lubinski, C. (1990). Teachers' Attributions and beliefs about girls, boys, and mathematics. Educational Studies in Mathematics, 21, 55-69.

Fennema, E., & Sherman, J. (1976). Fennema-Sherman mathematics attitude scales: Instruments designed to measure attitudes toward the learning of mathematics by females and males. Psychological Documents (Ms No. 1225). Washington, DC: American Psychology Association.

Hackett, G., & Benz, N. E. (1989). An exploration of the mathematics self-efficacy/mathematics performance correspondence. Journal for Research in Mathematics Education, 20, 261-273.

Kulm, G. (1980). Research on mathematics attitude. In R. J. Shumway (Ed.), Research in Mathematics Education (pp. 356-387). Reston, V. A.: National Council of Teachers of Mathematics.

Leder, G. C. (1992). Measuring attitudes to mathematics. Fifteenth Annual Conference of the Mathematics Association of Australasia: Conference Proceedings.. Richmond, N.S.W.

Mansfield, H. (1981). A report on pre-service mathematics education for primary school teachers in Australia. Australian Mathematics Teachers, 36, 12-15.

Marsh, H. W. (1988). Self description questionnaire: A theoretical and empirical basis for the measurement of multiple dimensions of pre-adolescent self-concept: A test manual and research monograph. The Psychological Corporation, San Antonio, Texas.

Marsh, H. W. (1990). A multi dimensional, hierarchical model of self-concept: Theoretical and empirical justification. Educational Psychology Review, 2, 77-173.

Nisbet, S. (1991). A new instrument to measure pre-service primary teachers' attitudes to teaching mathematics. Mathematics Education Research Journal, 3, 34-56.

Relich, J., Conroy, J., & Webber, V. (1991). Gender differences in self-concept: Male and female mathematics teachers as role models. A Report for the Department of School Education.

Relich, J., & Way, J. (1992). Pre-service primary teachers' attitudes to teaching mathematics: A reappraisal of a recently developed instrument. Fifteenth Annual Conference of the Mathematics Association of Australasia: Conference Proceedings. Richmond, N.S.W.

Sullivan, P. (1987). The impact of a pre-service mathematics education on beginning primary teachers. Research in Mathematics Education in Australia, August, 1-9.

Sovchik, R., Meconi, L. J., & Steiner, E. (1981). Mathematics anxiety of pre-service elementary mathematics methods students. School Science and Mathematics, 81, 643-648.

Suydam, M. N. (1984) Attitudes toward mathematics. Arithmetic Teacher, 32(3), 8-11

Watson, J. (1987). The attitudes of pre-service primary teaches toward mathematics: Some observations. Research in Mathematics Education in Australia, August, 48-56.

Wigfield, A., & Meece, J. L. (1988). Math anxiety in elementary and secondary school students. Journal of Educational Psychology, 80, 210-216.

Wood, E. F. (1988). Math anxiety and elementary teachers: What does research tell us ? For the Learning of Mathematics, 8(1), 8-13.

IS A PASS GOOD ENOUGH IN TERTIARY STATISTICS?

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Research in mathematics education and cognition suggests that knowledge, its acquisition, and its transfer are complex processes. In this study, end of semester totals are recorded for a large group of students in a traditional introductory subject in statistics at the tertiary level. A group of eight students are selected for observation while solving four statistical problems during the semester. This paper illustrates the capacity of students to pass an exam, yet highlights their limited understanding of statistical concepts, their narrow mechanical focus, their poor study habits and the lack of statistical transfer in atypical problems. Alternatives to the traditional approach are suggested.

The last decade has seen increased interest in statistics education. In particular, three international conferences on the teaching of statistics (see Davidson & Swift, 1987; Grey, Holmes, Barnett & Constable, 1983; Vere-Jones, 1991) have addressed some major issues involved in the teaching of statistics, namely: why should statistics be taught, what should be taught, to whom, by whom, and (most important of all) how.

With respect to introductory tertiary courses in statistics, there is some concern (see Grimmer, 1990; Moore, 1988; Morley, 1992; Taffe, 1991) that such courses often leave students dissatisfied, and fail to encourage enrolment in further statistics subjects. Despite this concern, a glance through students' records at any university would reveal that all but a small percentage of the huge numbers of students studying introductory subjects in statistics pass the subject. This raises questions about the usefulness of the knowledge acquired, and the type of statistical knowledge that is assessed.

Research into the nature of knowledge and the process of its acquisition emphasises the complexities involved. There is some agreement in the literature (e.g., Anderson, 1982; Long, 1989) that mathematical knowledge consists basically of two components: knowledge of content (facts, concepts, skills, procedures, strategies) and knowledge about how learning takes place (specific purpose procedures and higher order thinking strategies such as planning, analysing, monitoring, evaluating, modifying and predicting). Long (1989) suggests that for successful problem solving, both components should be used. To be used meaningfully, knowledge must be organised internally as patterns which are continually updated and modified. These patterns form the foundations for understanding. Research by Lawson (1984) and Perkins and Salomon (1987) imply that it is the knowledge of these higher order skills that allows students to solve novel problems. Thus it may be that current teaching practices in statistics do not encourage the second type of knowledge.

This paper focuses on one aspect of a larger study that explored the relationship between attitude and performance in a compulsory introductory subject in statistics. The purpose of the paper was to: (i) assess student understanding of statistical concepts in both typical text book as well as novel problems; and (ii) discuss the implications arising from the results.

METHOD

At the beginning of the first lecture of a compulsory inferential statistics subject, a large group of students, most of whom were enrolled in accounting, management or accounting/law degrees, filled in a questionnaire. This was designed by Wise (1985) to assess quantitatively the attitudes students hold towards statistics. A total mark ranging from 29 to 145 was tallied on this Likert scale of 29 items. Scores from 40 to 88 were classified as low attitudes: scores from 105 to 145 were classified as high attitudes. Eight students were selected for individual observation while solving statistical problems, one student for each possible combination of attitude (low, high), age (less than 25 years, at least 25 years) and gender.

After the final exam, the end of the semester results were recorded for 456 students, those who had not withdrawn from the subject. End of semester results were an aggregate of exercises which were worked and assessed in various ways throughout the semester. Most were the typical text book exercise, in which the question was clearly defined, an hypothesis test was required and numerical information was supplied.

Students had minimal statistical experience prior to this subject. It was understood that they had some knowledge of basic concepts such as mean, standard deviation and sampling techniques, but these were revised in the first week. Teaching and learning were "traditional", in that lectures were teacher-centred and exercise-driven. Topics in the course ranged from data collection, estimation and hypothesis testing, to multiple regression and nonparametric tests - a heavy workload. To assist students in doing hypothesis tests, a six-step solution had been emphasised throughout the semester. These steps involved: stating the hypothesis, significance level, test statistic (formula) and decision rule, performing the calculations, and reaching a conclusion.

The eight students were observed and audio-taped individually as they worked through four statistical problems (see Appendix 1), using a calculator when necessary. They were asked to talk aloud while they worked. The term "problems" is used here in the broad sense, to refer to typical statistical exercises as well as novel ones. The first two problems were worked during week 9 of a 14 week semester. This week was selected because it was just after the midsemester test. Had students prepared well for their exam, their statistical knowledge would then be at a premium. The midsemester test was multiple choice, and consisted mainly of exercises similar to those in the text. The remaining two problems were worked in week 14, just two weeks prior to the final exam.

The attributes of the four problems (see Appendix 1) are summarised in Table 1 below.

Table 1 - Summary of the Problems

Attributes	Problems			
	1	2	3	4
Similar to test book exercises	Yes	No	Yes	No
Open-ended	No	Yes	Yes	Yes
Hypothesis test expected	1-sample (mean)	2-sample (means)	corr/ regr	corr/ regr

Problem 1 was a typical text book exercise. Problem 3 was also typical, except for the fact that what was required was not clearly defined. Interpretation was left to the student. As such, it was "open-ended". Problems 2 and 4 were similarly open-ended, but they lacked numerical information.

RESULTS

In the study group, approximately eighty-six percent gained a pass or better. The average was 62, just below a credit award. In the sample of eight students, six passed (four passes, two distinctions) and two failed. The problem solving attempts by the latter group are summarised below for each problem. "P" refers to a student who passed the subject, "F" refers to a student who failed.

Problem 1 was recognised by all eight students as being similar to problems that had been worked before. They immediately summarised the given information with its statistical notation. All students recognised and stated that this question involved hypothesis testing, and set out to perform the steps highlighted for such problems in lectures and tutorials. Null and alternate hypotheses were given correctly in five cases (3P, 2F). Only three students (3P) used the correct test statistic (formula). Hereafter the responses differed. At one extreme, one student (1P) became confused, stating "I'm thinking of five different ways" and continued no further.

At the other extreme, another (1P) proceeded thoughtfully through the six-step procedure, explaining his choices, retracing his steps when his expected answer was not satisfied, then following through again to a final conclusion. Between these extremes, all other students (4P, 2F) were able to work through the exercise and make a conclusion of rejection of the null hypothesis. Four of these (3P, 1F) went beyond this rejection and actually answered the question. Only one (1P) looked at the numerical information for guidance in interpretation.

A number of different approaches were suggested to solve Problem 2. Even the student (1P) who could not complete the previous problem was fluent in explaining her strategy, but she stated that "This [problem] is great with no numbers...You can use judgement. That may be more beneficial than averages and stats." One notable finding in this problem was that only four students (3P, 1F) suggested the need for a statistical test, two of whom nominated a t-test, the appropriate test. By contrast, the other four students based their answers on qualitative judgement.

No student could offer a complete solution for Problem 3. One student (1P) could volunteer no information whatsoever. Another (1P) could only suggest (incorrect) hypotheses and a test statistic. Another (1P) tried to apply the chi-square test, the one topic he had revised, for 10 minutes, without success. The remaining three pass students (3P) wrote hypotheses related to correlation or regression analysis (demonstrating some knowledge of the problem type) and explained the direction of the relationship. All concluded that they had not revised this section yet, and could go no further.

Only one student (1P) attempted to state hypotheses for Problem 4, but like his attempt at Problem 3, his use of the chi-square test was incorrect, and his efforts ended in confusion. The student (1P) who could provide no information in the previous (quantitative) problem stated that "probably it's something to do with regression and correlation. It's the only chapter I've not looked at", but she could go no further. Another student (1P) could not even start, and suggested trying this exercise after the exam. The other three pass students (3P) explained the expected relationship between fuel consumption and weight, and stated immediately that this problem was concerned with regression analysis and correlation.

DISCUSSION AND CONCLUSION

While the examination results were pleasing, the individual performances emphasised problems in the understanding of statistical concepts.

Firstly, with a typical text book example, student attention was focused on the mechanics of the solution, and not on the analysis of the real situation. For example, in Problem 1, most students could use the six-step method to obtain a conclusion, yet only half could interpret this result, and only three could use the correct statistical technique. Hence, the emphasis on hand calculations and "getting the mathematics done" appeared to limit understanding, and lessen the ability to evaluate. The statistical knowledge acquired was likely to be content knowledge. The second type of knowledge had not been acquired.

Secondly, there were indications of poor study habits. For example, the text book question attempted late in the semester (Problem 3) could not be completed by anyone in the sample. Only half could suggest the topic to which this question related. Given the comparatively better performance in the earlier text book example (worked just after the midsemester test) one can only conclude that students postponed study of this subject until just before assessment. Such a strategy is likely to encourage rote learning and cramming as coping strategies, with little time for the development of deep understanding. As Keefe (1988) notes, learning this way requires constant practice and special cues for recall: the distinguishing features of the information are easily lost. Hence, the long-term benefits derived from studying this way would be minimal.

Thirdly, there was limited evidence of the transfer of statistical knowledge to novel problems. For example, only half the students suggested using statistical techniques to solve Problems 2 and 4. For the remaining students, there was no knowledge transfer from the current statistical subject they were studying, nor from the topics they had studied for the midsemester exam. Given their difficulties in selecting statistical techniques in Problems 1 and 3, this result is not surprising. A sound statistical knowledge foundation did not exist in the first place. Hence the higher order skills which Perkins and Salomon (1987) implied would facilitate transfer would be underdeveloped. Factors such as over-emphasis on mechanical calculations, poor study habits and rote learning (mentioned previously) would be intertwined in explaining this underdevelopment.

Now to the question: Is a pass good enough in statistics?

This study highlighted the rudimentary understanding students have of statistics, regardless of whether or not they pass; it illustrated the sketchy knowledge of key concepts just prior to the exam, and the limited skills in knowledge transfer.

Clearly it is time to consider the goals of introductory statistics courses. If the aims are to acquire the mechanical skills of statistical analysis, then this study demonstrates that students can achieve this through last minute cramming. However, research literature suggests the benefits may not be long-lasting. If the goals are in fact broader, and involve understanding, acquisition of higher order thinking skills, appreciation of the relevance and utility of statistics, then the traditional method of presentation and emphasis is not the way to achieve these.

In an era of mass tertiary education, there is a need to change the teaching and learning emphasis in statistics. New strategies are needed. For example, greater use of the computer for all statistical problems, including exam problems, would de-emphasise hand calculations, freeing time for reflection and experimentation (see Lock, 1987). This would increase understanding, and perhaps even lessen the dissatisfaction with introductory statistics subjects. The statistical data presented in the various media could be analysed and interpreted to ensure contextual relevance, thus facilitating transfer (see Kapadia & Andersson, 1987). Even if this approach were no more effective in terms of pass rates, the opportunity to see statistics in context would be advantageous. Students' perceptions about statistics may be enhanced and that may be critical to their decisions about its future usage.

REFERENCES

Anderson, J.R. (1982). Acquisition of cognitive skill. Psychological Review, 89, 369-406.

Davidson, R. & Swift, J. (Eds.). (1987). Proceedings of the second international conference on teaching statistics. Victoria, BC: University of Victoria.

Grey, D.R., Holmes, P., Barnett, V. & Constable, G.M. (Eds.). (1983). Proceedings of the first international conference on teaching statistics, (vols. 1-2). Sheffield: University of Sheffield.

Grimmer, M. (1990). A learner-centred approach to teaching statistics. Auslink, 13-14.

Kapadia, R. & Andersson, G. (1987). Statistics explained: Basic concepts and methods. Chichester, West Sussex: Ellis Horwood Ltd.

Keefe, J.W. (1988). Profiling & utilizing learning style. Reston, VA: National Association of Secondary School Principals.

Lawson, M.J. (1984). Being executive about metacognition. In J.R. Kirby (Ed.). Cognitive strategies and educational performance (pp. 89-109). Newcastle, NSW: Dept of Education.

Lock, R. (1987). The role of computers in statistics instruction. Proceedings of the second conference on the teaching of statistics, (pp.87-95). Oneonta, New York: State University of New York.

Long, E. (1989, November). The role of domain-specific knowledge in classroom problem-solving. Paper presented at the AARE conference, Adelaide.

Moore, D.S. (1988). Should mathematicians teach statistics? The College Mathematics Journal, 19(1), 3-7.

Morley, C.L. (1992, July). Rediscovering the management statistics curriculum. Paper presented at the ANZAM conference, Sydney.

Perkins, D.N. & Salomon, G. (1987). Transfer and teaching thinking. In D.N. Perkins, J. Lochhead & J. Bishop (Eds.). Thinking: The second international conference. Hillsdale, NJ: Lawrence Brothers Assoc.

Taffe, J. (1991). Panel Session: Teaching statistics - mathematical or practical model. In R. Davidson & J. Swift (Eds.). Proceedings of the second international conference on teaching statistics (pp. 332-336). Victoria, B.C., Canada: University of Victoria.

Vere-Jones, D. (1991). Proceedings of the third international conference on teaching statistics (vol. 2). Voorburg, Holland: International Statistical Institute.

Wise, S.L. (1985). The development and validation of a scale measuring attitudes towards statistics. Educational and Psychological Measurement, 45, 401-405.

APPENDIX 1

Problem 1: A particular brand of wine contains on average 750 mls per bottle. To check the quality control, a sample of 36 bottles was randomly chosen from a consignment of wine, and the contents of each bottle were measured. Each bottle was found to contain an average of 746 mls with a standard deviation of 2 mls. At the 5% level of significance, does this sample indicate that the wine level is now different from 750 mls?

Problem 2: The manager of a firm is recruiting students from your University, who, like yourself, have some background in statistics and have completed this subject. He wishes to find out whether students enrolled in an Accounting degree are as good at statistics as those enrolled in a Management degree. How would you suggest he investigate this?

Problem 3: Peter Gordon is frustrated because he believes that his low student evaluations are due to his large classes (e.g., he can't learn the students' names, and there is less discussion). He samples 10 classes in the School of Business in order to test his hypothesis.

Class Size X	Class Rating of Instructor Y
10	4
20	4
20	5
30	4
50	2
30	3
40	4
40	3
40	2
50	3

a) Write down the hypothesis warranted by the data.
b) Investigate the problem and draw the conclusion warranted.

Problem 4: The advertising agency for a car manufacturer is interested in the answer to the question: Does the weight of a car relate to its fuel consumption? If you were in the employ of the agency, and, given that you have now completed this statistics subject, how would you investigate this question?

STUDENTS' IDEAS ABOUT MATHEMATICS

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University of Technology, Sydney

This study examined students' attitudes to and ideas about mathematics at the beginning of a mathematics or engineering degree at university. The instrument used was one of Schoenfeld (1989) considerably modified for use with university students. Seventy-four students took part in the study.

Several interesting results were noted. Both male and female students attributed their success in mathematics to intrinsic values such as hard work. There was a significant difference between the ideas of students who had studied 3 unit mathematics and those who had studied 4 unit (the highest level in NSW). Students who had studied 3 unit described school mathematics as "just memorising". However 86% of the students described mathematics as "interesting" and gave that as a reason for doing mathematics.

The most interesting part of the study was the answers to 10 open-ended questions. Here, this group of students demonstrated a mature understanding of the links between areas of mathematics, of everyday applications of what they had studied at school and were at ease with discussing their feelings about mathematics and learning mathematics. The mathematics they had studied so far was widely perceived as useful in the development of personal skills - teaching clear and logical thinking, fostering persistence, patience and discipline.

This study examined students' attitudes to and ideas about mathematics at the beginning of a mathematics or engineering degree at university. The instrument used was one of Schoenfeld (1989) considerably modified for use with university students. Seventy-four students took part in the study.

The aim of the study was to provide a basis for comparison of students' ideas of mathematics at the beginning and end of their degree and aim to encourage them to reflect on their previous learning experiences. The study will not be complete until 1995 however results of the initial survey are of interest and are presented here.

These results will form part of the information used for the design of future curriculum. Parallel studies of language used in lectures by academic staff and of student responses to language-based assessment in mathematics will contribute to curriculum change.

METHOD

Subjects

The subjects were 74 first-year university students at the University of Technology, Sydney. These were made up of 46 students from the Bachelor of Science degree in mathematics (64% of cohort) and 28 from a tutorial group of 30 from the Bachelor of Engineering (Electrical).

The subjects were predominantly a young group straight out of school with 75% aged 17-19 and 19% aged 20-24. There were 49 male and 25 female students. There are more males because of the inclusion of Engineering students - in the BSc (mathematics) the numbers of males and females are approximately equal. 78% had completed 3 or 4 unit mathematics in the NSW Higher School Certificate.

The Instrument

The instrument is based on one designed by Schoenfeld (1989). Many of the questions were changed to language more suitable for older Australian students. Other questions did not apply in the Australian context and were deleted. Some questions were added to meet our needs (see appendix for instrument).

Procedures

The surveys were distributed in the first weeks of semester just after the students arrived at university. It is therefore a survey of their ideas regarding school mathematics rather than a survey of ideas regarding university mathematics.

All students, 102 in total, comprising 72 from the BSc (Mathematics) and 30 from BE (Electrical) were given the opportunity to participate and invited to take the survey home and complete it at their leisure. The surveys were anonymous. 74 surveys were returned. From the detailed answers given to the open-ended questions, students took considerable time and care with their answers.

DISCUSSION OF RESULTS

Attributions of success or failure

This group of students attributed their success in mathematics to intrinsic factors - primarily to hard work (97%) and to a lesser extent, natural ability (70%). The idea of teacher bias was strongly rejected. Eighteen percent felt that luck played some role but in general this idea was also rejected.

Doing badly in maths was also strongly attributed to intrinsic factors - not studying hard enough (92%) and careless mistakes (90%). Interestingly lack of ability was not seen as a significant factor in explaining lack of success (11%).

It appears that the students felt that they had some natural ability in mathematics which can explain some of their success, but are aware that hard work and attention to accuracy are the major determinants of their performance. This certainly sounds like an ideal group to do a degree in mathematics or engineering!

The attributions of success did not vary significantly between mathematics and engineering students nor was there a significant difference between male and female students. The results on sex differences (or lack of them) replicate Schoenfeld's (1989) results. In fact, no sex differences were noted on any question so results have been quoted as totals.

Perceptions of School Mathematics

Of the 74 students, 57 had studied either 3 or 4 unit mathematics for the 1991 NSW Higher School Certificate, two had studied 2 unit and fourteen had not sat for the 1991 HSC (mature aged or overseas students). Overall, three-quarters of the students felt that school maths was "mostly facts and procedures that have to be memorised", although a similar proportion also felt that school maths was "thought provoking".

When given the chance to expand on the idea of memorising in the open-ended questions, students seemed to be aware of the advantages and disadvantages of memorising and so this perception of memorising must be tempered with the fact that these students have recently completed the Higher School Certificate exam where speed is essential.

There was a significant difference between 3 and 4 unit students ($p < 0.02$), with 3 unit students more inclined to classify school maths as just memorising.

Views on Mathematics and English

There was a strong feeling that natural ability plays an important role in mathematics and English. 86% felt it was true that some people are good at maths and some are not and 82% felt the same about

English. Further, almost two-thirds of students felt there was no connection between being good at English and being good at mathematics.

Most students (86%) believed that in mathematics something is either right or wrong, whereas this is not considered true for English (14%). This differed from Schoenfeld's (1989) results where he found that students found the right/wrong distinction sharper in English classes. He attributed this to the classroom practice where part marks are given for incorrect mathematics answers but English grammatical and spelling errors are emphasised by `ugly red marks.'

There were several questions comparing features of the teaching of mathematics and English. Students considered it desirable that both English and mathematics teachers show students a variety of ways to look at the same question (86% and 96% respectively) but whereas showing you the exact way to answer test questions was not considered important for English teachers (21%) nearly half of the students (44%) felt that it had some importance for Mathematics teachers.

Perceptions of Mathematics

There were some encouraging results in this section of the questionnaire. Most students (71%) felt that everything important is not already known by mathematicians, that in mathematics you can be creative and discover things by yourself (86%) and that it is not true that maths problems can be done correctly in only one way (92%). This contrasts with students perceptions of school mathematics which most considered was mostly facts and procedures that have to be memorised.

Reasons for Learning Mathematics

The strongest reasons for learning mathematics were wanting to do well in the course, because it will be useful in getting a job, and because it is required for the student's program - all practical considerations. However of equal importance was that mathematics is interesting, with 86% quoting this as a reason for learning mathematics, and no student disagreeing strongly.

Open Ended Questions

Many respondents to the survey gave surprisingly detailed comments showing that they had given considerable thought to their answers and that they appreciated the opportunity to voice their opinions.

(Question: Do you think mathematicians work alone on problems or together? Which do you think is better and why?) There was a strong preference for mathematicians to work at least part of the time in groups, to share ideas and gain different perspectives. *(Question: Are the different mathematics courses you have taken related to each other in any way? If so, how?)* Most students perceived the different mathematics courses (algebra, geometry, trigonometry) as related, with particular mention being made of algebra as a link between courses, and of basic problem solving techniques as common to all courses. *(Question: Why do you think so much time in geometry classes is spent learning to write proofs?)* Learning to write proofs in geometry was seen as important in aiding understanding of the theory, in improving analytical skills, and in learning that you must justify your answers. This last attribute was seen as fundamental in real-life problems particularly in engineering.

(Question: How much of you ability to do proofs shows up when you take maths tests?) Students varied their assessment of how much of their ability to do mathematics showed up in maths tests, with opinions evenly split between "most," "some" and "not much". One student questioned the meaning of "doing maths" - "What is meant by doing maths? If doing mathematics means pressing the keys of the calculators, speed of writing and time of thinking, and then paper work free of `silly' mistakes then during any maths test I realise I am unable to do maths."

(Question: What can you do if you get stuck while doing a maths problem?) Various strategies for coping when you get stuck while doing a maths problem were suggested, ranging from "keep on working" to

"panic". The most common strategies were to leave the question and come back later, approach it from a different angle, or seek for help from notes, friends, tutors, lecturers. There may have been some confusion over the interpretation of this question as to whether the "getting stuck" was under exam conditions or not.

(Question: In what way, if any, is the maths you have studied useful?) Responses to the question on the usefulness of mathematics fell into two categories - external (relevance) and internal (personal development). Mathematics was seen as relevant to problem solving in work, particularly engineering and computing and to everyday life. "I use maths all the time - working out my economy rate when bowling in cricket, working out at which angle to hit the ball when playing snooker, counting my money, working out my weight index so that I don't get too fat etc". And somewhat more succinctly "it means the puzzles are easier". Mathematics studied so far was also widely perceived as useful in the development of personal skills - teaching clear and logical thinking, fostering persistence, patience and discipline.

(Question: Do you think students can discover mathematics on their own, or does mathematics have to be shown to them? Please explain.) Despite most students indicating in the questionnaire (Q24) that in maths you can be creative, many revealed a lack of confidence when asked whether students can discover mathematics on their own. The most common response was that students need to be shown the basic rules, but can discover maths on their own, given motivation, dedication and guidance along the way. Some students felt that self-discovery of mathematics was only for the very smart, while a number felt it was a waste of time. "After all, it took mathematicians years to work out some of the stuff". "Teachers are more successful at teaching new ideas rather than trying to follow it in the text book".

(Question: If you understand the material, how long should it take to solve a typical homework problem? What is a reasonable amount of time to work on a problem before you know it is impossible?) Although most students felt that a typical homework problem should take less than ten minutes, opinions varied widely as to what is a reasonable amount of time to work on a problem before you know it is impossible. A few students are apparently able to assess whether a question is impossible within two to three minutes, most will work for between ten minutes and one hour; and a few very determined students are prepared to work indefinitely. "I don't stop working on something until someone says it is impossible and then I usually try to prove them wrong." "A problem is impossible when you have exhausted all avenues available".

(Question: How can you know whether you understand something in maths? What do you do to measure (test) yourself?) Again a number of methods for knowing whether you understand something in maths were suggested. Most popular was simply when you get your problems correct, or can solve them without reference to textbook or notes. Also mentioned was being able to follow the theory and apply it in different situations; when you actually start enjoying the work; and being able to explain it to others. "If I can explain it to someone else I know that I understand it myself". Most students tested themselves with extra questions.

(Question: How important is memorising in learning mathematics? If anything else is important, please explain how.) Finally there was strong agreement that while memorising formulas is important in that it can save you time, understanding is vital. "Memorising becomes less important as understanding is achieved". "Memorising (formulae) has its role but understanding how to use them is the key".

IMPLICATIONS FOR TEACHING

There are several implications for teaching from this study. The strong preference for mathematicians to work at least part of the time in groups suggests greater use of group activities in mathematics learning and assessment. Being able to explain the work to others as an aid to personal understanding suggests part of the assessment of mathematics students could be the presentation of short seminars to class groups. The fact that many students expressed a preference for teachers rather than books reinforces the importance of backup services such as mathematics study centres, peer teaching or after school homework sessions. Students are looking for personal contact in their mathematics learning.

There are also implications for the teaching of university mathematics. Students are willing to work hard, they can make connections between branches of mathematics, most display a mature ability to reflect on their learning. This is a good basis on which to start a university career. We need to ensure that the overloaded curricula and the types of assessment procedures we use are changed so that students are not forced into memorising mode. Our survey has shown that students are willing to articulate their reflections on school mathematics and I believe this can be translated into reflection on the conceptual difficulties in early university calculus as identified in Tall (1992) provided students are given an opportunity. Students in this survey showed a lack of confidence in their ability to discover mathematics on their own - again care must be taken to give students confidence in their abilities.

As one of my students said to me last week "You don't want me to just learn this, you want me to think!"

CONCLUSION

Several interesting results were noted. Both male and female students attributed their success in mathematics to intrinsic values such as hard work. There was a significant difference between the ideas of students who had studied 3 unit mathematics and those who had studied 4 unit (the highest level in NSW). Students who had studied 3 unit described school mathematics as "just memorising". However most of the students described mathematics as "interesting" and gave that as a reason for doing mathematics. The mathematics they had studied so far was widely perceived as useful in the development of personal skills - teaching clear and logical thinking, fostering persistence, patience and discipline.

There were difficulties with students conceptions of `impossible' with most students believing that a problem was impossible if they spent ten minutes to an hour working on it. This was a longer time span than that shown in Schoenfeld's study and may represent the older age group of the students surveyed. One would expect the idea of a problem being `impossible' to change over their course of study at university.

The students involved in this study were enrolling in either a mathematics degree or an engineering degree, both of which clearly require a high level of competence in mathematics. However many other degrees, such as Business and Psychology, include either mathematical or statistical subjects. The students enrolling in these courses have a more varied background in mathematical experience and a comparison of their attitudes to and ideas about mathematics with those of our original survey group will also prove interesting.

REFERENCES

Schoenfeld, A.H. 1989 *Explorations of Students' Mathematical Beliefs and Behaviour*. Journal of Research in Mathematics Education, 20,4,338-355.

Tall, D. 1992 *The Transition to Advanced Mathematical Thinking: Functions, Limits, Infinity and Proof.* In Handbook of Research on Mathematics Teaching and Learning. (Ed. Grouws, D.A.) Macmillan, New York, 495-511.

MATHEMATICS IN THE LOWER PRIMARY YEARS: A RESEARCH-BASED PERSPECTIVE ON CURRICULA AND TEACHING PRACTICE

BOB WRIGHT
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Drawing on current research the author explicates twelve assertions about current curricula and teaching practice in lower primary mathematics. Topics discussed include: the under-challenging curriculum, differences in children's knowledge, the need for compensatory programs, curriculum constraints on teachers, 'anti-interventionism' and discovery learning, verbal versus written arithmetic, the role of problem solving, and the need to better understand how children learn mathematics.

Paper not available for publication

CHILDREN'S PROGRESS IN AN EXPERIMENTAL MATHS RECOVERY PROGRAM: AN INITIAL REPORT

BOB WRIGHT, GARRY STANGER, MARGARET COWPER AND ROSALIE DYSON
University of New England — Northern Rivers

In each of six schools four prenumerical first graders were taught four times per week for eight weeks as part of an experimental Maths Recovery Program. Their progress is compared with that of 13 prenumerical counterparts using Steffe's five learning stages in the construction of the number sequence. The Maths Recovery children progressed notably more than the counterparts thus providing a positive indication of the viability of the program.

Paper not available for publication

LEARNING MATHEMATICS: A GENDERED PRACTICE

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In this paper I employ classroom transcripts to develop Bourdieu's writings with respect to the construction of gendered life in the mathematics classroom and the disempowering consequences of many practices for a significant number of girls within mathematics education. Using Bourdieu's concepts of habitus, practice and dispositions, the construction of a gendered mathematical habitus will be proposed. Bourdieu (1977) describes habitus as being a "matrix of perceptions, appreciations and actions." Bourdieu does not employ gender as a primary category of capital where capital is central to the construction of social space. Feminist writers, such as McCall (1992) have been able to reapply Bourdieu's notions of cultural capital to incorporate a gendered habitus. Through a gendered habitus, students learn to take on certain dispositions which will be influential in their success in schooling and their later positions within the social structure. For females, a gendered habitus is paramount to the construction of their marginality in mathematics.

By the time students enter school, they have been socialised into the world as a result of many experiences. These experiences provide a habitus which will influence how perceptions and behaviours will be interpreted and enacted. Bourdieu (1992, p.) extends this so that the habitus gives "disproportionate weight to early experiences." The primary socialisation of students will impact upon how they their worlds at school will be constructed, so that social differences, such as gender, class and ethnicity, since a different habitus has been internalised. The dispositions which they bring with them into the school situation will be different for different individuals and social groups. In this way, the habitus becomes a mediating device between the social structures and practices and individual action and perception. Through the insertion into certain gendered practices, a gendered habitus is constructed, and through this habitus, children come to see and act in the world in certain gendered ways. The ways in which gender is organised and internalised can be related to two crucial aspects of Bourdieu's writings - habitus and capital.

Bourdieu's notion of *embodied* cultural capital is particularly important in developing an understanding of the distribution of gendered power since he argues that cultural capital can exist in three forms, embodied as long-lasting dispositions of the body and mind; objectified as cultural goods (such as books, instruments, machines) and in an institutationalised state in such things as educational qualifications (Bourdieu, 1983, p. 243). In this way, Bourdieu is arguing that certain dispositions are forms of cultural capital. McCall (1992) develops Bourdieu's writings to question whether women can possess gendered capital in a form that will be recognised and rewarded . She argues, "The dichotomous action of gender acts to constrain and subordinate the meaning of women's activity, whatever the so-called capital "(McCall, 1992, p. 846).

MATHEMATICS HABITUS

The insertion of young students into the mathematics classrooms exposes them to practices which will construct a habitus for their school mathematics learning. This habitus, when linked to other habitus, provides a lens for viewing and acting in mathematics classrooms. The mathematics habitus will be constrained by a number of influences and practices which will include:

The conditions of the field (objectivism, utilitarianism, psychologism, normativity)
Certain linguistic codes
Teacher and student goals
Culturally valued knowledge, skills, abilities, morals, ethics and values
Specific language, examinations, streaming, pedagogies, etc.

Through the construction of practices and participating in the practices, they will come to construct a mathematical habitus which will impact upon the way they perceive and act in the mathematics classroom. As part of the production of a mathematical habitus, students will be exposed to gendered practices which influence the construction of a gendered mathematical habitus. A gendered mathematics habitus is possible because of other gendered practices outside mathematics practices.

ORGANISING PRINCIPLES OF THE FIELD

It is through the practices associated with the field of school mathematics that participants accept the non-random failure of a significant number of students from marginalised social groups. The conditions of objectivism, utilitarianism, psychologism and normativity mask the political agenda of school mathematics. The macro practices associated with mathematics education perpetuate the belief that failure in the subject is essentially a function of the individual. Apart from the dominance of the belief in the objective nature of mathematics, there is the dominance of psychology in the field as well. The monolithic presence of psychology in the field helps to reify the complex nature of the beast while masking its socially constructed roots. Psychological research has aided in defining mathematics as a complex subject which needs to be organised in ways to reflect the increasing complexity of the nature as is apparent in the van Hiele levels of Geometric thought, and taxonomies such as SOLO. Psychological research has also introduced ideas about learning styles which help to reify the ideas that people from different socio-cultural backgrounds and contexts think in ways which may be incongruous with the rational and logical thinking associated with success in mathematics. It would appear that the supposed inability of many social groups to be successful in their study of mathematics is a reflection of some internal characteristics. In this way, habitus is constituted by practices which relate to social regulation of the population.

THE SITE

The site of the study was a grade one classroom where the children were learning about shapes. The school has a small population of 130 and is located in a middle-class suburb of a major provincial city. There is a disproportionate number of boys to girls (16:8) so that it would be reasonable to expect more male talk than female talk. The lesson is being taken by a (male) "specialist teacher" who is responsible for the applied maths curriculum for the grade. The children are very familiar with the teacher who also takes them for physical education and art. Extracts from the hour lesson will be used to illustrate certain practices which may be influential in students learning gendered dispositions, and hence, habitus. Similar broad themes are found in other sites, so that this particular lesson should not be seen as a reflection of the individual teacher. Such practices are shared practices found across many sites and form part of teaching practice.

The Lesson

The teacher is introducing formal names to a variety of shapes. He introduces the topic formally on the chalk board and then sets the children to work with concrete materials. This is done through a variety of formats. He sets the classroom up with a series of work stations which contain different activities for the students. The students rotate around the stations after a given amount of time.

"BEING MACHO AND FEMININE"

The practices within the classroom contain certain messages that the children are likely to perceive as natural and hence incorporate them into their habitus. In the following phase of the lesson the teacher is introducing formals names of shapes. He attempts to use the shapes in ways that he perceived to be drawing on the children's experiences, which are gendered. In the following extract, the teacher is acting a way whereby the boys are likely to see the dispositions of being "macho", sporting (because Leigh is good at football, isn't he) and mateship (Luke, my best mate) as part of the natural way of being a boy.

When the teacher attempts to include the girls in the discussion of shape, he adopts a practice which focuses on the aesthetics of possessions (as in diamonds) and appealing to the idea of desiring things as being part of being a girl (girls really love having this shape). Girls are the receivers in relationship and the bearers of manners and tact, i.e social etiquette. (If you gave a girl a triangle, she would probably say thank you ...).

B Um they're not a circle but they're a bit like ..got a a side
T Show us
.boy goes to board and points to the sides on the oval)
T Well, who can tell us what that shape is? and I'll give you a clue that you play football on it.
laughter
B I know
chatter with lots of "football ovals" being called out[1]
T Well I did say I'd give a clue
Leigh An oval
T That's very good, that's the one. Well apart from that, because of course Leigh is good at football isn't he? Are there any other ones?
B There's a piece of orange that looks like the shape of a semi-circle
T That's very good. Can you go and point to that one? ..A semi-circle
b Up there
T Oh very good, very good, like on one of those oranges that are cut up there. Um OK that's s semi-circle. What shape is a semi-circle?
Rachel A half a circle
T Good girl, good girl. And one more.
G See that banana thing up there, it's spinning around, it's like a circle shape on it but I don't know what's it called
Bs We've already had that
G I don't know what's it shape is.
B It's like a semi-circle

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[1] The protocol has been adopted in these transcripts whereby children who can be identified have been done so through the use of names unknown children have been identified according to the gender B: Boy, G:Girl, and where the gender is not known C:Child has been used. All names used are pseudonyms to protect the identity of the students, teacher and the school.

T It's like a semicircle isn't it? In fact we'll have to name out our own shape for that one. Right now, you're all rotten kids because you didn't pick the one that I wanted you to pick

Cs I know

T Turn around this way. My wife told me about this one...girls really love having this shape

C I know

Cs Oh no

Cs I know I know

chattering and giggling

T What's a girl's best friend? Luke do you know what shape a girl likes?

Luke Yes

T What is it?

Luke A triangle

T No! If you gave a girl a triangle, she'd probably say thank you, but she wouldn't say Luke! That's what I always wanted!

giggles

G A heart

T excellent, but not that one

Cs Ohhh

giggles

T Bad luck but not that one. You couldn't work out what one it was.

C I know which colour it is

T The one that I wanted was a diamond. every time I go to the jeweller's shop, she goes "I want a diamond."

While there is a strong emphasis on naming shapes in this segment, the introduction of a slightly irregular shape becomes a slight problem. The girl does not know the name of the shape (somewhere between a semicircle and a crescent) and is open about her lack of knowledge. She recognises that it is not a semicircle, even though a group of boys seem quite content that a "semicircle" is an adequate signifier for the shape, and to a certain degree, the teacher reinforces their assertions with "It's like a semicircle, isn't it?" Part of the habitus of schooling is being able to guess what the teacher has in mind (as is apparent when the teacher says that they "didn't pick the one that I wanted you to pick"). Whereas there has been a strong emphasis on correct signification, the naming of this shape suddenly becomes quite arbitrary, and clearly not part of the game that the teacher has in mind. The teacher dismisses the girl's shape since it is not part of the game he has in mind, and moves on to the next segment of the lesson. As a result, the message that the girl is likely to receive is that naming shapes can be quite arbitrary, whereas for the boy who was not sure of the shape earlier in the segment, it was pointed out that there was a name for that shape (semicircle), suggesting that naming shapes is a precise action. In this way, there were quite different messages being given to the students about the processes involved in naming shapes. Unless the focus was on a gendered analysis, it would be difficult to observe the construction of gender in mathematics. With a non-gendered focus, the teacher could be seen to be using a dominant pedagogical strategy which seeks to draw on the prior experience of the students in order to engage them in the discussion, thus rendering the construction of a gendered habitus as invisible. However, when the focus is on the construction of gender, different practices can be seen to be operating for girls who likely to receive quite different messages about the patterns of signification in mathematics, along with the importance of their contributions to the class discussions.

BOYS HAVE ABILITY, GIRLS JUST TRY

In the following extract where the children have been called to draw diamonds on the chalk board, the teacher appears to be constructing a situation where the messages are ones that the boys are good at maths and have the right skills (reflecting) through overt and covert practices. (David is *extremely* clever, Sean's done a good thing, he did it quickly and then he thought to himself ....). In contrast, Sally was given messages that indicate an inferior position when she was told that her effort "was a lovely try." While her drawing was the same as David's, she was not clever like David, since she had done the wrong thing by drawing the same shape. Sally had not been able to guess the teacher's game since her diamond was pointing in the same direction as David's, which the teacher did not want, although he only mentioned that he wanted the diamond in a different colour. Through this type of practice, girls learn that their work, although the same as boys', does not rate them as clever but rather as "good triers." In this way there is a gendered difference in the construction of ability versus effort in the classroom.

T Who can come up to the chalk board and draw me a diamond?

(David comes up the board and draws a diamond)

T Now as you can see David is extremely clever and had done a great job. Alright, who can draw me a different coloured diamond

B I can

T You've already had one.

(Sally comes up to the board and draws a diamond similar to David's) ..

T Right very good, thanks Sally, that's a lovely try. Now they're all pointing in the same direction, but I wonder where it points. I suppose that one would point up and that one would point down. Who can make it pointing in a different direction?

??

T This little tail,,I don't know what's its called? Sean. Marcus, if you want to have a shot next, don't be rude.

Sean I don't know which way to

giggles

T Now Sean's done a good thing, he did it quickly and then he thought to himself "It doesn't look the best and he better do something about it"

Sean It's hard

T It is pretty difficult to do these aren't they? ..Same as you did it then...right very good.

B Very good

Contradictory messages were being given to the students. When Sean has difficulties drawing on the board, the teacher encourages him by indicating that the task is a difficult one, and guides him through the drawing process. Sean is also encouraged for being "reflective." Yet Sally's drawing was as good as Daniel's, yet received little recognition for it. The interaction with the two boys is more extensive and positive than his interaction with Sally. At one level, the quality of interaction is quite different, but the length of interaction is also considerable. As a result, the feedback that girls would expect would be quite different to that which boys could expect. This practice creates quite different possibilities for the construction of gendered mathematics habitus.

GIRLS AND PASSIVITY

Through the practices of school mathematics, students are positioned in ways which will facilitate the construction of certain dispositions. In the following transcripts, the girls in positioned in ways which would be conducive to the construction of passive dispositions. Girls learn through their interactions with the teacher and their peers that they are not as valued in the classroom as their male peers, and to take a less active and compliant role in the mathematics classroom.

GIRLS ARE IGNORED, BOYS ARE VALUED

In the following extract, the teacher is interacting with a group of children working on a task where they had to make patterns with shapes and then illustrate them. He interacts with the boys, giving them attention, asking them how their work is proceeding etc. etc. During all of this the voice of Gail keeps asking for affirmation of her work, but is continually ignored. When she finally gains his attention, she is not praised in any way, but told to do more of the same. To this she responds that her page is full, he replies that she should help another child, almost as a punishment for finishing too early . In contrast, during the interaction he praises Sean and uses Glen's work as an exemplar to the other children, even though Gail has clearly finished her work. He also praises a group of boys for "lovely' work. Such praise is not forthcoming with the girl's work.

Practices such as these aid in the construction of a disposition suggesting that girls should expect to be ignored, their work has little value and that there is not much praise to be expected when they complete their work. In contrast, boys can expect a significant amount of the teacher's attention, most of which is supportive and encouraging.

T So can you do just at least one pattern that I can see..or we can work out what might come next. You're doing a good job Glen, just keep on going. Can you maybe give Sean a hand, he's looking pretty sad there. Alright

B He doesn't know what to do.

T Alright, what you have to do is to make a pattern like you know on a jumper, right? So what I want you to do is to work out a pattern like a diamond then a square and ten you might have say another diamond. Can you see a pattern happening already? What's the next one we do?

Sean A square

T A square. So just do that, that's it. Do another one, Do another pattern. Alright? OK, well if you were doing a pattern, there's these two and what would be going to go down next?..What goes after that? ...good boy, so that's what you draw. You draw that...

Gail Mr Mason, this is what my pattern looks like

T and then one more and then you've finished. That's one, two three. two more

//

Gail Mr Mason

T Like these guys over here. Like his too. That looks lovely, leave it there. That looks lovely, just keep going on it

Gail Mr Mason, that's my pattern

T OK let's have a look (works with another boy)..Right I'll show you. It goes, square, diamond, rectangle, square, diamond, rectangle, same, alright ? So just draw that. Just do number one. Just draw it here. ..I'll do it for you. Then you go that one then that one and then that one and then that one and then that one, just like that. Alright? Now do another one.

Gail I've finished mine Mr Mason,

T Then do two more
Gail I can't fit any more
T Then help Rachel

In defence of the teacher, it can be argued that he was trying to ignore Gail for calling out, so that he was attempting to extinguish a behaviour (through negative reinforcement) which he considered to be undesirable in the classroom, yet as the following extract indicates, interrupting is also a gendered practice.

GIRLS DON'T INTERRUPT

While the boys could interject with comments, particularly if they are relevant, the practices in the classroom suggest that girls are expected to be more docile, put their hands up and wait to be asked to reply. While it was argued in the previous section that girls can expect to be ignored, when they do try to participate with boys insofar as trying to have some voice in the classroom, they can expect to be admonished. In the following extract the teacher was asking children do sort equilateral triangles from non-equilateral. When a girl interrupts, she is reprimanded and her response is ignored. Yet when a boy calls out, his answer is accepted and even expanded.

T Now some triangles are..look like triangles, which ones look like triangles and which ones don't
Gail These ones
T Get one that looks like a triangle...
B These ones look like a triangle
Gail This one looks like a triangle
T Now most people will pick these ones like these 2 boys did because they've all got equal sides haven't they? But these ones are really triangles too. These ones are called "equilateral triangles". Guess what's equal? Because they're called equilateral triangles
Rachel Because it's got equal sides
T Don't call out Rachel. What's equal about it?
B The sides
T The sides are the same size. The sides are all equal. You have to sort out the equilateral triangles from the other triangles. You want two bunches, one of beautiful equilateral triangles and the other triangles that look like flags. Ok off you go.

It would seem that the girls are likely to be exposed to different practices in the classroom which will construct different habitus for them than their male peers.

From these practices, it would appear that there are many messages that children receive that construct certain gendered dispositions within the individual. These have focussed on the unequal power relationships associated with the student and the teacher, but it is also apparent that when students are working together, there are similar displays of gendered dispositions.

WORKING WITH PEERS

In the working on a task to make patterns with shapes, the boys had commandeered all the rectangular shapes and the girls required some. While the girls use all the "correct" language for asking for co-operation, the boys are resisting joining the game. This interaction indicates that the boys have taken on some of the dispositions of being dominant in the classroom and assuming the role of the more powerful participants in the practice. On the other hand the girls have acquired dispositions of a more subservient and compliant participant, avoiding confrontation with the boys in the classroom, even though the boys have more rectangles but are not willing to share them with the girls.

Gail Hey can I please have some rectangles
Bob No
Gail yes we can
Bob What? .... well we've used them all
Bill Yeah we used them all
Gail Can we please have some rectangles?
Gina They haven't got any
Cate They don't have any
Gail I need one more rectangle
G ?
Gail Just one more rectangle
Bob Then what about if you fly up there

It would appear that transcripts such as these are illuminating in that they are able to shed some light on the practices, albeit discursive, that occur in the classroom which may be influential in the construction of gendered dispositions and habitus in mathematics. The gendered positions that are available to the girls are not as empowering as those available to the boys, and, as such, may be seen to be influential in the distribution of power in the wider society.

RANGE OF DISPOSITIONS AND POSITIONS

From these transcripts, it appears that girls are positioned in mathematical practices in ways that are different to their male peers. Such practices could be influential in the construction of gendered dispositions which will impact upon the later positioning within the social order. For these girls, it is possible that their experiences will construct dispositions which position them as compliant, passive and "subservient" within the classroom whereas the boys are exposed to more empowering practices. For boys, such practices will impact more positively on the dispositions which are likely to develop . This may improve their chances of success, and hence, social positioning within mathematics education and the wider society.

These transcripts are examples of practices in the mathematics classroom but it is noted that they are never fixed and that there are a range of positions and dispositions that are available to students at any time. The positions and dispositions that are available to the female members in mathematics do not have the same capital associated with them as do those offered to their male peers.

HABITUS AND CAPITAL

A view of mathematics education as being apolitical and culture free is a fairly dominant view amongst mathematicians, mathematics educators and the wider society that masks the hidden function of mathematics as a social filter. Such practices belie the social sorting based on the grounds of gender, class and ethnicity that mathematics fulfils in the wider social arena. Through covert means, mathematics is employed to play a major role in the process of cultural and social reproduction. Atweh and Cooper (1991) have argued persuasively in support of the hegemonic nature of the mathematics curriculum. Those students who are excluded from the study of mathematics based on their perceived inability to cope with the content accept their marginalisation as an internal lacking, or dispositional attribute. Conversely, those students who enjoy the success and the increased career options associated with success in maths, come to perceive their status as natural and deserved.

Bourdieu is concerned with understanding the ways in which beliefs and practices are constructed in cultural contexts and the ways in which certain relationships are more powerful than others, along with an understanding of how such relationships come to be seen as natural. Through these "natural" and "taken-for-granted" beliefs the unequal distribution of power, status and economic rewards remain unchallenged. Through the practices at the macro level of society, and the micro level of the classroom, certain "habitus" are internalised to provide the lens with which to view the world. It is at a level that could be referred to as unconscious so that it, for the main, remains unquestioned. Through the practices of schooling, children are induced into socially patterned habitus which is closely aligned to the social patterns of the wider society.

Through analysing classroom interactions, an examination of those practices which construct girls in a gendered way so that they learn to take certain gendered dispositions will be undertaken. These dispositions will influence not only to their learning of mathematics content *per se* but also their social positioning.

REFERENCES

Atweh, B. & Cooper, T. (1991). Hegemony in the mathematics curricula: The effect of gender and social class on the organisation of mathematics teaching for year 9 students. In F. Furinghetti (Ed.) Proceeding of the Fifteenth PME Conference. Assisi, Italy. Vol. 1. pp. 88-95. International Group for the Psychology of Mathematics Education.

Bourdieu, P. (1977). *An Outline of a Theory of Practice.* Cambridge: Cambridge University Press.

Bourdieu, P. (1986) The forms of capital. In J.G. Richardson (Ed.) *Handbook of Theory and Research for the Sociology of Education.* New York:

Bourdieu, P. (1992). *The Logic of Practice.* Polity Press.

McCall, L. (1992).Does gender fit? Bourdieu, feminism, and the concepts of social order. *Theory and Society, 21.* 837-867.